Modeling, Analysis and Optimization of Networks (Part 2: design)

Lecture 2

Alberto Ceselli
alberto.ceselli@unimi.it

Dipartimento di Informatica
Università degli Studi di Milano

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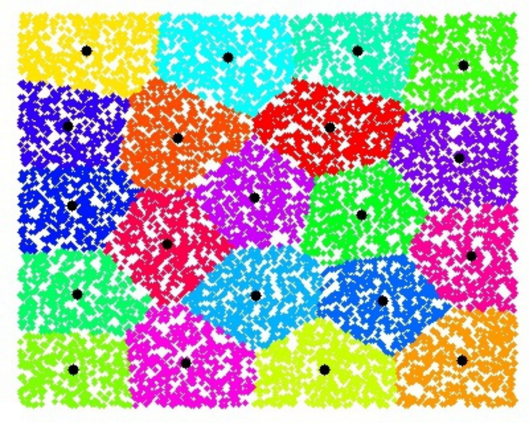
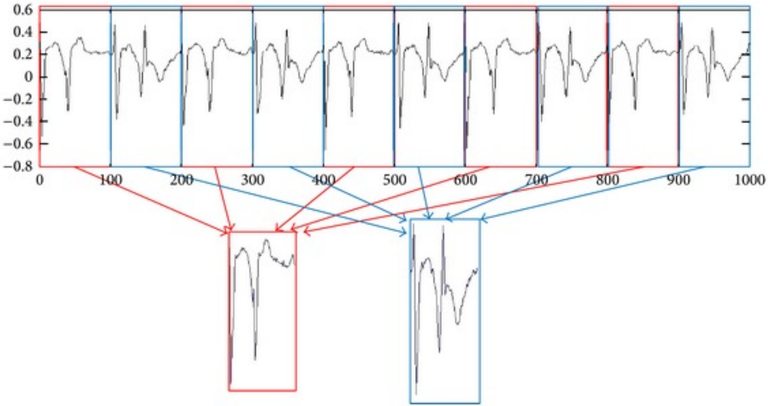
Motivation 1/3

Motivation 2/3



Motivation 3/3



The problem

Given

- a set of nodes,
- a set of links connecting them,
- a set of **service requests**, **one for each node** of the network,
- a set of **devices**, able to provide service, to be installed in the network,

I want to

- decide where to place the service provider devices,
- decide how to satisfy service requests,
- maximizing the quality of service (e.g. minimizing delay time)

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- is the problem single or multi commodity?

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- is there a limit on the *number* of service requests that can be served by the same provider (e.g. devices with limited number of ports)?
- is the problem single or multi commodity?
- how to measure the overall quality of service?

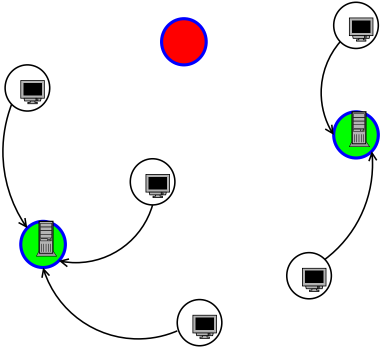
Assumptions

Let us begin with a basic location problem:

1. providers with very large resources (capacity and number of ports are not an issue),
2. possibility of splitting service requests,
3. single commodity,
4. minimize the average *connection cost*.

In this case we talk about a **Uncapacitated Facility Location Problem**.

Visually:



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- Choose if and where to intall the servers (binary variables y_i) and how to connect terminals to servers (variables x_{ij})...

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- Choose if and where to intall the servers (binary variables y_i) and how to connect terminals to servers (variables x_{ij})...
- ...in such a way that each terminal is connected to a server.

Uncapacitated Facility Location Problem (UFLP)

$$\begin{aligned}
\min \quad & \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i \in I} x_{ij} = 1 && \forall j \in J \\
& x_{ij} \leq y_i && \forall i \in I, \forall j \in J \\
& x_{ij} \geq 0 && \forall i \in I, \forall j \in J \\
& y_i \in \{0, 1\} && \forall i \in I
\end{aligned}$$

N.B. Without additional conditions, variables x_{ij} take integer values.

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- this solution provides an upper (primal) bound: might not be optimal, but in general good enough;
- the difference between upper and lower bound gives a *quality estimation* of our solution;
- if the quality is not enough, we resort to branch-and-bound (enumeration).

UFLP: Lagrangean Relaxation

$$\min z = \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

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\begin{aligned}
w(\underline{x}, \underline{y}, \underline{\pi}, \underline{\eta}) = \min \quad & \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \\
& - \sum_{j \in J} \pi_j (\sum_{i \in I} x_{ij} - 1) \\
& - \sum_{i \in I} \sum_{j \in J} \eta_{ij} (y_i - x_{ij}) \\
\text{s.t.} \quad & \\
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$$w(\underline{x}, \underline{y}, \underline{\pi}, \underline{\eta}) = \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \\ - \sum_{j \in J} \pi_j (\sum_{i \in I} x_{ij} - 1) \qquad - \sum_{i \in I} \sum_{j \in J} \eta_{ij} (y_i - x_{ij})$$

$$L(\underline{\pi}, \underline{\eta}) = \min_{\underline{x}, \underline{y}} \; w(\underline{x}, \underline{y}, \underline{\pi}, \underline{\eta})$$

$$\text{s.t. } 0 \leq x_{ij} (\leq 1) \qquad \forall i \in I, \forall j \in J$$

$$y_i \in \{0, 1\} \qquad \forall i \in I$$

$$u_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J$$

$w(\underline{x}, \underline{y}, \underline{\pi}, \underline{\eta})$ is called the *Lagrangean Function* and $\max_{\underline{\pi}, \underline{\eta}} L(\underline{\pi}, \underline{\eta})$ is called the *Lagrangean Dual Problem*.

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- Possible from a theoretical point of view, very hard from a computational point of view.
- We resort to *approximate* solutions for the Lagrangean Dual Problem.

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- The most simple choice is the *gradient* algorithm. Iteratively:
  - fix the multipliers and find a (local) optimum with respect to the remaining variables,
  - compute the gradient of the Lagrangean Dual Function with respect to the multipliers,
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- Our Lagrangean Dual Function is *nondifferentiable* (piecewise linear), hence we use *subgradients* instead of gradients.

Subgradient algorithm for UFLP (1)

- Choose an initial value for the multipliers $\underline{\pi}^0$ e $\underline{\eta}^0$.

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- Choose an initial value for the multipliers $\underline{\pi}^0$ e $\underline{\eta}^0$.
- Iteratively $(k = 1 \ldots)$
  - *Solve the Lagrangean Subproblem* in $\underline{x}$ e $\underline{y}$, getting a solution $(\underline{x}^k, \underline{y}^k)$ of value ω^k.
  - Compute the subgradients:
    - $\nabla h_j^k = 1 - \sum_{i \in I} x_{ij}^k$
    - $\nabla g_{ij}^k = x_{ij}^k - y_i^k$
  - Choose a step length α^k.
  - Update the multipliers:
    - $\pi_j^{k+1} = \pi_j^k - \alpha^k \cdot \nabla h_j^k$
    - $\eta_{ij}^{k+1} = \eta_{ij}^k - \alpha^k \cdot \nabla g_{ij}^k$

Subgradient algorithm for UFLP (2)

A popular choice is

- to set the step length α^k according to the rule:

$$\alpha^k = \rho^k \cdot \frac{\text{UB} - \omega^k}{\sum_{i \in I, j \in J} (\nabla g_{ij}^k)^2 + \sum_{j \in J} (\nabla h_j^k)^2}$$

where UB is an upper bound to the optimal value of $w(\underline{x}, \underline{y}, \underline{\pi}, \underline{\eta})$.

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- Moreover, if for some (i,j) $\eta_{ij}^{k+1} \leq 0$, in order to fulfil constraints on the sign of multipliers $\underline{\eta}$, set $\eta_{ij}^{k+1} = 0$.

Subgradient algorithm for UFLP (3)

- Iterate until
  - $\nabla g_{ij}^k = 0$ and $\nabla h_j^k = 0$ for each $i \in I, j \in J$ (optimum reached), or
  - a *maximum* number of iterations is performed, or
  - the value ω^k is *sufficiently* near to UB, or
  - the value ρ is *sufficiently* small

In the first case, the Lagrangean Dual Problem has been solved to optimality, otherwise we have no guarantee on the quality of the (dual) solution.

Solving the Lagrangean Subproblem (1)

$$\min_{\underline{x},\underline{y}} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} +$$

$$- \sum_{j \in J} \pi_j \left(\sum_{i \in I} x_{ij} - 1\right) - \sum_{i \in I, j \in J} \eta_{ij} (y_i - x_{ij})$$

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Solving the Lagrangean Subproblem (2)

$$\min_{\underline{x},\underline{y}} \sum_{i\in I}(f_i - \sum_{j\in J}\eta_{ij})y_i + \sum_{i\in I}\sum_{j\in J}(c_{ij} - \pi_j + u_{ij})x_{ij} - \sum_{j\in J}\pi_j$$

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- Set $\tilde{f}_i = (f_i - \sum_{j\in J}\eta_{ij})$ and $\tilde{c}_{ij} = (c_{ij} - \pi_j + \eta_{ij})$.
- If $\tilde{f}_i \ge 0$ set $y_i = 0$, otherwise set $y_i = 1$;
- If $\tilde{c}_{ij} \ge 0$ set $x_{ij} = 0$, otherwise set $x_{ij} = 1$

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- In this case, computing a lower bound to the integer problem or computing a lower bound *to its continuous relaxation* is *the same*.
- Result: the lower bound given by Lagrangean Relaxation in *not tighter* than the lower bound given by the continuous relaxation of the problem.
- When the problem is continuous, by solving the LD Problem we obtain *the optimum* of the original problem.

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- If the Lagrangean Subproblem has the integrality property, the corresponding lower bound is not tighter than the continuous relaxation lower bound ...
- ...but might provide useful information for finding *good integer solutions*!

Finding a feasible solution

Let us look at the Lagrangean Subproblem:

$$
\begin{aligned}
\min_{\underline{x},\underline{y}} \; & \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \\
& - \sum_{j \in J} \pi_j \Big(\sum_{i \in I} x_{ij} - 1 \Big) - \sum_{i \in I, j \in J} \eta_{ij} (y_i - x_{ij}) \\
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& u_{ij} \geq 0 && \forall i \in I, \forall j \in J
\end{aligned}
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Finding a feasible solution

The original problem is:

$$
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\text{s.t. } & \sum_{i \in I} x_{ij} = 1 & \forall j \in J \\
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and therefore, the Lagrangean Subproblem solution might violate

- assignment constraints
- (consistency) constraints linking x and y variables

how to build a feasible UFLP solution?

Finding a feasible solution

Some observations:

- once the y variables are fixed, optimizing over x is easy ...
- ... let's keep the values of y variables found in the Lagrangean Subproblem, and re-optimize over x.

(but we could do the other way round as well).

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- Subgradient algorithm is generic (not exploiting the structure of the problem).

The idea of dual ascent

- If the Lagrangean subproblem has the integrality property, the corresponding lower bound is not tighter than that given by the continuous relaxation ...
- ...but might be easier to compute!
- Subgradient algorithm is generic (not exploiting the structure of the problem).
- We can exploit LP theory and problem features to find a good set of multipliers.

Dual UFLP

- Let us consider the continuous relaxation of the inital UFLP formulation:

$$\begin{aligned}
\min \quad & \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} & \\
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\min \; & \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} & \\
\text{s.t.} \; & \sum_{i \in I} x_{ij} = 1 & \forall j \in J \\
& y_i - x_{ij} \geq 0 & \forall i \in I, \forall j \in J \\
& x_{ij} \geq 0 & \forall i \in I, \forall j \in J \\
& y_i \geq 0 & \forall i \in I
\end{aligned}$$

- the corresponding dual problem is:

$$\begin{aligned}
\max \; & \sum_{j \in J} v_j & \\
\text{s.t.} \; & \sum_{j \in J} u_{ij} \leq f_i & \forall i \in I \\
& v_j - u_{ij} \leq c_{ij} & \forall i \in I, \forall j \in J \\
& u_{ij} \geq 0 & \forall i \in I, \forall j \in J
\end{aligned}$$

Dual UFLP

- Dual problem:

$$
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\text{s.t.} \ & \sum_{j \in J} u_{ij} \leq f_i & \forall i \in I \\
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- For each choice of the dual variables v_j, fix u_{ij} to the maximum possible value (keep feasibility and leave objective function unchanged). Hence

$$u_{ij} = \max\{0, v_j - c_{ij}\}$$

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- In this way we obtain a "condensed dual" problem:

$$
\begin{aligned}
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\text{s.t.} \ & \sum_{j \in J} \max\{0, v_j - c_{ij}\} \leq f_i && \forall i \in I
\end{aligned}
$$

DuaLoc

- Condensed dual problem:

$$\max \sum_{j \in J} v_j$$

$$\text{s.t.} \sum_{j \in J} \max\{0, v_j - c_{ij}\} \leq f_i \qquad \forall i \in I$$

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- The DUALOC algorithm (Erlenkotter '78)
  - Inizialize $v_j = \min_{i \in I}\{c_{ij}\}$
  - Inizialize $s_i = f_i - \sum_{j \in J} \max\{0, v_j - c_{ij}\}$
  - Iteratively:
    - For each $j \in J$:
    - let $\Delta_j = \min_{i \in I}\{s_i | v_j - c_{ij} \geq 0\}$
    - decrease s_i of Δ_j for each i with $v_j - c_{ij} \geq 0$; increase v_j of Δ_j
  - Until there are no more changes in the dual solution.

Rebuild a primal solution

- Condensed dual problem:

$$\max \sum_{j \in J} v_j$$

$$\text{s.t.} \sum_{j \in J} \max\{0, v_j - c_{ij}\} \leq f_i \qquad \forall i \in I$$

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- Complementary slackness conditions:

$$y_i(f_i - \sum_{j \in J} \max\{0, v_j - c_{ij}\}) = 0 \forall i \in I$$
$$(y_i - x_{ij}) \max\{0, v_j - c_{ij}\} = 0 \forall i \in I, \forall j \in J$$

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- Second, assign each terminal j to a server in $\text{argmin}_{i \in I | y_i = 1} c_{ij}$
- This condition might violate complementary slackness conditions, but is feasible, and therefore a valid *upper bound*.

Università degli Studi di Milano

Location Lab

Experimenting on Location Modeling variants, Lagrangean Relaxation and Column Generation algorithms in AMPL.