Evaluation Driven Proof-Search in Natural Deduction Calculi for Intuitionistic Propositional Logic

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Motivations

Natural deduction and sequent calculus are dual presentations of Intuitionistic Logic with different features:

- Sequent calculus

  More appropriate for meta-theoretical reasoning and proof-search (cut-elimination).

- Natural Deduction calculus

  Natural and immediate computational interpretation via the Curry-Howard isomorphism (normalization).

Lot of work has been done to extend the Curry-Howard isomorphism to sequent calculi, see the Herbelin's permutation-free sequent calculus and its successive refinements

H. Herbelin. A lambda-calculus structure isomorphic to Gentzen-style sequent calculus structure. CSL, 1994

J.E. Santo. The lambda-calculus and the unity of structural proof theory. Th.Comp.Syst, 2009.

Motivations

Instead, proof-search in natural deduction calculi has been scarcely investigated.

Main references:

- Proof-search procedures based on the intercalation calculus

  W. Sieg and J. Byrnes. Normal natural deduction proofs (in classical logic). Studia Logica, 1998.

  W. Sieg and S. Cittadini. Normal natural deduction proofs (in non-classical logics). LNCS, 2005

  These procedures are highly inefficient and do not provide a clear characterization of the proof-search space.

- Recent work:

  G. Mints and SH. Steinert-Threlkeld. ADC method of proof search for intuitionistic propositional natural deduction. JLC, 2016.

  The authors introduce the ADC method (Analysis and Direct Chaining) to construct natural derivations where there are not I-rules above E-rules.

  Derivations of this kind can be built in polynomial time, but ADC method is incomplete for Intuitionistic Logic.

Our contribution

We reconsider the problem of proof-search in the Natural Deduction calculus for Intuitionistic Propositional Logic (**IPL**).

- We introduce **Nbu**, a variant of the standard Natural Deduction calculus for **IPL**.
- We define a goal-oriented proof-search procedure for **Nbu**, which considerably improves the one based on intercalation calculus.

  The key point is that **Nbu** internalizes some aspects of the proof-search procedure.

This show that goal-oriented proof-search is not a distinctive feature of sequent calculus but can be recovered also in the context of natural deduction.

Natural Deduction calculus

The natural deduction calculus has been introduced to capture logical mathematical reasoning.

> *The formalization of logical deduction, especially as it has been developed by Frege, Russel, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return.*
>
> *I intended, first of all, to set up a formal system which comes as close as possible to actual reasoning. The result was a calculus of natural deduction (NJ for intuitionist, NK for classical predicate logic).*
>
> *[Gentzen, "Investigations into logical deduction", 1934]*

Natural Deduction calculus

- Formulas A, B, $\ldots$ of **IPL** are built starting from a set $\mathcal{V}$ of propositional variables:

$$
\begin{array}{lll}
A, B & ::= & \bot \mid p \mid A \wedge B \mid A \vee B \mid A \to B \qquad p \in \mathcal{V} \\
\neg A & ::= & A \to \bot
\end{array}
$$

- For each logical connective it is defined an introduction rule (I-rule) and an elimination rule (E-rule)
  - I-rule
    How to introduce a compound formula.
    Infer a complex formula from already established components
  - E-rule
    How to de-construct information about a compound formula.
    Specify how components of assumed or established complex formulas can be used as arguments.

Localizing Hypothesis

A derivation of B having open assumptions $A_1,\ldots,\ A_n$ is represented by a proof-tree $\mathcal{D}$ of the form

$$\begin{array}{c} A_1,\ldots,A_n \\ \mathcal{D} \\ B \end{array}$$

In our presentation, it is more convenient to *localize hypothesis* and present derivations in sequent style:

$$\begin{array}{c} \mathcal{D} \\ \Gamma \Rightarrow B \end{array}$$

The context Γ (multiset) contains the assumptions $A_1,\ldots,A_n$ on which B depends.

$\mathbf{NJ}_0$: Natural Deduction calculus for **IPL** in sequent style

Rules of $\mathbf{NJ}_0$

- Axiom rule

$$\frac{}{A, \Gamma \Rightarrow A}\ \mathrm{Id}$$

It represents a single-node derivation (a derivation with open assumptions A, Γ and conclusion A).

- I-rules for $\wedge$, $\vee$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge I \qquad\qquad \frac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ A & B\end{matrix}}{A \wedge B}\ \wedge I$$

$$\frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow A_0 \vee A_1}\ \vee I \qquad k \in \{0, 1\} \qquad \frac{\begin{matrix}\mathcal{D} \\ A_k\end{matrix}}{A_0 \vee A_1}\ \vee I$$

Rules of $\mathbf{NJ}_0$

- I-rules for $\rightarrow$

$$\dfrac{\begin{matrix}[A]\\ \mathcal{D}\\ B\end{matrix}}{A \rightarrow B}\rightarrow I$$

The assumption A of $\mathcal{D}$ can be discharged by the rule application.

We split the rule into $\rightarrow I_1$ and $\rightarrow I_2$:

$$\dfrac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\rightarrow I_1 \qquad \dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\rightarrow I_2$$

A is not discharged — A is discharged

Rules of $\mathbf{NJ}_0$

- E-rules for $\wedge$, $\rightarrow$

$$\frac{\Gamma \Rightarrow A_0 \wedge A_1}{\Gamma \Rightarrow A_k} \wedge E \qquad k \in \{0,1\} \qquad \frac{\begin{array}{c}\mathcal{D} \\ A_0 \wedge A_1\end{array}}{A_k} \wedge E$$

$$\frac{\Gamma \Rightarrow A \rightarrow B \qquad \Gamma \Rightarrow A}{\Gamma \Rightarrow B} \rightarrow E \qquad \frac{\begin{array}{c}\mathcal{D}_1 \\ A \rightarrow B\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ A\end{array}}{B} \rightarrow E$$

modus ponens

Rules of $\mathbf{NJ}_0$

- E-rule for $\vee$

$$\frac{\Gamma \Rightarrow A \vee B \qquad A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{\Gamma \Rightarrow C} \vee E \qquad \frac{\begin{matrix} \\ \mathcal{D}_1 \\ A \vee B \end{matrix} \qquad \begin{matrix} [A] \\ \mathcal{D}_2 \\ C \end{matrix} \qquad \begin{matrix} [B] \\ \mathcal{D}_3 \\ C \end{matrix}}{C} \vee E$$

- Rules for $\bot$

  Falsehood corresponds to a disjunction with no alternatives:
  - no I-rule
  - E-rule has no cases

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow C} \bot E \qquad \frac{\begin{matrix} \mathcal{D} \\ \bot \end{matrix}}{C} \bot E$$

The calculus $\mathbf{NJ}_0$

$$\frac{}{A, \Gamma \Rightarrow A}\ \mathrm{Id} \qquad \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A}\ \bot E$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge I \qquad \frac{\Gamma \Rightarrow A_0 \wedge A_1}{\Gamma \Rightarrow A_k}\ \wedge E \quad k \in \{0,1\}$$

$$\frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow A_0 \vee A_1}\ \vee I \qquad \frac{\Gamma \Rightarrow A \vee B \qquad A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}\ \vee E$$

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \to I_1 \quad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \to I_2 \qquad \frac{\Gamma \Rightarrow A \to B \qquad \Gamma \Rightarrow A}{\Gamma \Rightarrow B}\ \to E$$

Theorem (Completeness of $\mathbf{NJ}_0$)

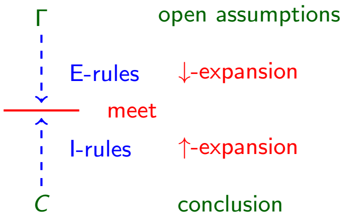
$A \in \mathbf{IPL}$ *iff there exists a derivation of* $\cdot \Rightarrow A$ *in* $\mathbf{NJ}_0$

A naïve proof-search strategy for $\mathbf{NJ}_0$

We perform two expansion steps:

- $\uparrow$-expansion: apply I-rules bottom-up
- $\downarrow$-expansion: apply the E-rules top-down

and meet in the middle.



To get a derivation $\mathcal{D}$, one has to alternate $\uparrow$-expansions and $\downarrow$-expansion phases (intercalation calculus).

By definition, $\mathcal{D}$ is in normal form.

A naïve proof-search strategy for $\mathbf{NJ}_0$

To formalize the strategy, we introduce two kinds of judgments, where sequents are oriented:

- $\Gamma \Rightarrow A\uparrow$

  A has a *normal derivation (nd)* from assumptions Γ
- $\Gamma \Rightarrow A\downarrow$

  A can be *extracted* from the assumptions Γ

$$\mathbf{NJ}_0 \ + \ \text{arrows}\downarrow,\uparrow \ = \ \mathbf{NJ}$$

- Proof-search:

  W. Sieg and J. Byrnes. Normal natural deduction proofs (in classical logic). Studia Logica, 1998.

  W. Sieg and S. Cittadini. Normal natural deduction proofs (in non-classical logics). LNCS, 2005

- Arrow notation:

  F. Pfenning. Automated theorem proving. Lecture notes, 2004.

  R. Dyckhoff, L. Pinto. Cut-elimination and a permutation-free sequent calculus for intuitionistic logic. Studia Logica, 1998.

Rules for Normal Derivations

We rewrite the rules of the calculus by orienting the sequents.
Arrow suggest in which direction a rule must be applied in proof-search.

- $\wedge I$

$$\frac{\Gamma \Rightarrow A\uparrow \qquad \Gamma \Rightarrow B\uparrow}{\Gamma \Rightarrow A \wedge B\uparrow} \wedge I$$

$A \wedge B$ has a nd from Γ if both A and B have a nd from Γ (hence $\wedge I$ must be applied bottom-up).

- $\wedge E$

$$\frac{\Gamma \Rightarrow A \wedge B\downarrow}{\Gamma \Rightarrow A\downarrow} \wedge E \qquad \frac{\Gamma \Rightarrow A \wedge B\downarrow}{\Gamma \Rightarrow B\downarrow} \wedge E$$

If $A \wedge B$ can be extracted from Γ, then both A and B can be extracted from Γ (hence $\wedge E$ must be applied top-down).

Rules for Normal Derivations

- $\vee I$

$$\frac{\Gamma \Rightarrow A\uparrow}{\Gamma \Rightarrow A \vee B\uparrow}\ \vee I \qquad \frac{\Gamma \Rightarrow B\uparrow}{\Gamma \Rightarrow A \vee B\uparrow}\ \vee I$$

$A \vee B$ has a nd from Γ if A or B has a nd from Γ.

- $\vee E$

$$\frac{\Gamma \Rightarrow A \vee B\downarrow \qquad A, \Gamma \Rightarrow C\uparrow \qquad B, \Gamma \Rightarrow C\uparrow}{\Gamma \Rightarrow C\uparrow}\ \vee E$$

C has a nd from Γ if $A \vee B$ can be extracted from Γ and C has a nd from A, Γ and a nd from B, Γ
($\uparrow$-expansion meets $\downarrow$-expansion)

Rules for Normal Derivations

- $\rightarrow I_1$

$$\frac{\Gamma \Rightarrow B\uparrow}{\Gamma \Rightarrow A \rightarrow B\uparrow} \rightarrow I_1$$

$A \rightarrow B$ has a nd from Γ if B has a nd from Γ.

- $\rightarrow I_2$

$$\frac{A, \Gamma \Rightarrow B\uparrow}{\Gamma \Rightarrow A \rightarrow B\uparrow} \rightarrow I_2$$

$A \rightarrow B$ has a nd from Γ if B has a nd from A, Γ.

- $\rightarrow E$

$$\frac{\Gamma \Rightarrow A \rightarrow B\downarrow \qquad \Gamma \Rightarrow A\uparrow}{\Gamma \Rightarrow B\downarrow} \rightarrow E$$

If $A \rightarrow B$ can be extracted from Γ and A has a nd from Γ, then B can be extracted from Γ.

Rules for Normal Derivations

- Axiom Id

$$\frac{}{A, \Gamma \Rightarrow A{\downarrow}}\ \mathrm{Id}$$

A can be extracted from A, Γ *(start of* $\downarrow$*-expansion)*.

- $\bot$ E

$$\frac{\Gamma \Rightarrow \bot{\downarrow}}{\Gamma \Rightarrow A{\uparrow}}\ \bot E$$

If $\bot$ *can be extracted from* Γ*, then* A *has a nd from* Γ.

- Coercion $\downarrow\uparrow$

$$\frac{\Gamma \Rightarrow A{\downarrow}}{\Gamma \Rightarrow A{\uparrow}}\ {\downarrow\uparrow}$$

If A *can be extracted from* Γ*, then* A *has nd from* Γ.

The calculus **NJ**

$$\frac{}{A, \Gamma \Rightarrow A{\downarrow}}\ \mathrm{Id} \qquad \frac{\Gamma \Rightarrow A{\downarrow}}{\Gamma \Rightarrow A{\uparrow}}\ {\downarrow\uparrow} \qquad \frac{\Gamma \Rightarrow \bot{\downarrow}}{\Gamma \Rightarrow A{\uparrow}}\ \bot E$$

$$\frac{\Gamma \Rightarrow A{\uparrow} \qquad \Gamma \Rightarrow B{\uparrow}}{\Gamma \Rightarrow A \wedge B{\uparrow}}\ \wedge I \qquad \frac{\Gamma \Rightarrow A_0 \wedge A_1{\downarrow}}{\Gamma \Rightarrow A_k{\downarrow}}\ \wedge E \quad k \in \{0, 1\}$$

$$\frac{\Gamma \Rightarrow A_k{\uparrow}}{\Gamma \Rightarrow A_0 \vee A_1{\uparrow}}\ \vee I \qquad \frac{\Gamma \Rightarrow A \vee B{\downarrow} \qquad A, \Gamma \Rightarrow C{\uparrow} \qquad B, \Gamma \Rightarrow C{\uparrow}}{\Gamma \Rightarrow C{\uparrow}}\ \vee E$$

$$\frac{\Gamma \Rightarrow B{\uparrow}}{\Gamma \Rightarrow A \to B{\uparrow}}\ {\to} I_1 \qquad \frac{A, \Gamma \Rightarrow B{\uparrow}}{\Gamma \Rightarrow A \to B{\uparrow}}\ {\to} I_2 \qquad \frac{\Gamma \Rightarrow A \to B{\downarrow} \quad \Gamma \Rightarrow A{\uparrow}}{\Gamma \Rightarrow B{\downarrow}}\ {\to} E$$

Theorem (Completeness of NJ)

*$A \in$ **IPL** iff there exists a derivation of $\cdot \Rightarrow A{\uparrow}$ in **NJ** (i.e., a nd of A).*

On normal forms

Derivations in **NJ** are *by definition* in normal form.

For instance, let us consider the following detour which introduces a maximal formula $p \to q$:

$$\dfrac{\dfrac{\dfrac{\vdots}{p, \Gamma \Rightarrow q}}{\Gamma \Rightarrow p \to q} \to I_2 \qquad \dfrac{\vdots}{\Gamma \Rightarrow p}}{\Gamma \Rightarrow q} \to E \quad \leftrightsquigarrow \quad \dfrac{\dfrac{\dfrac{[p], \Gamma}{\vdots}}{\dfrac{q}{p \to q} \to I} \qquad \dfrac{\Gamma}{\dfrac{\vdots}{p}}}{q} \to E$$

This derivation cannot be replicated in **NJ**:

$$\dfrac{\dfrac{\cancel{p, \Gamma \Rightarrow q}}{\Gamma \Rightarrow p \to q\downarrow} \not\to I_2 \qquad \dfrac{\vdots}{\Gamma \Rightarrow p\uparrow}}{\Gamma \Rightarrow q\downarrow} \to E$$

Application of $\to I_2$ is not allowed ($p \to q$ must be *extracted* from Γ).

On normal forms

To build non-normal derivations, we should add to **NJ** the rule

$$\frac{\Gamma \Rightarrow A\uparrow}{\Gamma \Rightarrow A\downarrow} \upharpoonleft\!\downarrow \quad \text{converse of coercion}$$

Example of use of $\upharpoonleft\!\downarrow$ to introduce a maximal $p \to q$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ p, \Gamma \Rightarrow q\uparrow\end{matrix}}{\Gamma \Rightarrow p \to q\uparrow} \to I_2}{\Gamma \Rightarrow p \to q\downarrow} \upharpoonleft\!\downarrow \qquad \begin{matrix}\vdots \\ \Gamma \Rightarrow p\uparrow\end{matrix}}{\Gamma \Rightarrow q\downarrow} \to E$$

- The presence of coercion rule $\downarrow\!\upharpoonleft$, and the lack of $\upharpoonleft\!\downarrow$, is crucial to "coerce" derivations in normal form.

A proof-search example

Let us search for a nd of

$$A \to q_1 \qquad\qquad A \;=\; p_1 \,\wedge\, (p_1 \vee p_2 \to q_1 \wedge q_2)$$

Proof-search starts from the ↑-sequent

$$\cdot \Rightarrow A \to q_1 \uparrow$$

- ↑-expansion

$$\dfrac{A \Rightarrow q_1 \uparrow}{\cdot \Rightarrow A \to q_1 \uparrow} \to I_2$$

We need a ↓-expansion step to extract q_1 from A:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A \Rightarrow q_1 \downarrow\end{matrix}}{A \Rightarrow q_1 \uparrow}\ \downarrow\uparrow}{\cdot \Rightarrow A \to q_1 \uparrow} \to I_2$$

A proof-search example

To extract q_1 from A, we can build the following proof-tree with root-sequent $A \Rightarrow q_1{\downarrow}$:

- $\downarrow$-expansion

$$A \;=\; p_1 \wedge (p_1 \vee p_2 \to q_1 \wedge q_2)$$

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A{\downarrow}}\,\mathrm{Id}}{A \Rightarrow p_1 \vee p_2 \to q_1 \wedge q_2{\downarrow}}\,\wedge E \qquad A \Rightarrow p_1 \vee p_2{\uparrow}}{\dfrac{A \Rightarrow q_1 \wedge q_2{\downarrow}}{A \Rightarrow q_1{\downarrow}}\,\wedge E}\,\to E$$

The proof-tree has the open-leaf $A \Rightarrow p_1 \vee p_2{\uparrow}$, which must be $\uparrow$-expanded.

A proof-search example

- Gluing the above derivations by rule $\downarrow\uparrow$ we get:

$$A \;=\; p_1 \,\wedge\, (p_1 \vee p_2 \to q_1 \wedge q_2)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A\downarrow}\,\mathrm{Id}}{A \Rightarrow p_1 \vee p_2 \to q_1 \wedge q_2\downarrow}\,\wedge E \qquad A \Rightarrow p_1 \vee p_2\uparrow}{A \Rightarrow q_1 \wedge q_2\downarrow}\,\to E}{A \Rightarrow q_1\downarrow}\,\wedge E}{A \Rightarrow q_1\uparrow}\,\downarrow\uparrow}{\cdot \Rightarrow A \to q_1\uparrow}\,\to I_2$$

We have to $\uparrow$-expand $A \Rightarrow p_1 \vee p_2\uparrow$

A proof-search example

- $\uparrow$-expansion

$$\frac{A \Rightarrow p_1 \uparrow}{A \Rightarrow p_1 \vee p_2 \uparrow} \vee I$$

We get

$$A \;=\; p_1 \;\wedge\; (p_1 \vee p_2 \to q_1 \wedge q_2)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A\downarrow}\,\mathrm{Id}}{A \Rightarrow p_1 \vee p_2 \to q_1 \wedge q_2 \downarrow} \wedge E \qquad \color{red}{\dfrac{A \Rightarrow p_1 \uparrow}{A \Rightarrow p_1 \vee p_2 \uparrow} \vee I}}{A \Rightarrow q_1 \wedge q_2 \downarrow} \to E}{A \Rightarrow q_1 \downarrow} \wedge E}{A \Rightarrow q_1 \uparrow} \downarrow\uparrow}{\cdot \Rightarrow A \to q_1 \uparrow} \to I_2$$

We close the leaf $\color{red}{A \Rightarrow p_1 \uparrow}$ by extracting p_1 from A.

A proof-search example

- $\downarrow$-expansion

$$\dfrac{\dfrac{}{A \Rightarrow A\downarrow}\,\mathrm{Id}}{A \Rightarrow p_1\downarrow}\,\wedge E \qquad A \;=\; p_1 \wedge (p_1 \vee p_2 \to q_1 \wedge q_2)$$

By applying $\downharpoonleft\!\upharpoonright$, we get the closed derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A\downarrow}\,\mathrm{Id}}{A \Rightarrow p_1 \vee p_2 \to q_1 \wedge q_2\downarrow}\,\wedge E \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A\downarrow}\,\mathrm{Id}}{A \Rightarrow p_1\downarrow}\,\wedge E}{A \Rightarrow p_1\uparrow}\,\downharpoonleft\!\upharpoonright}{A \Rightarrow p_1 \vee p_2\uparrow}\,\vee I}{A \Rightarrow q_1 \wedge q_2\downarrow}\,\to E}{A \Rightarrow q_1\downarrow}\,\wedge E}{A \Rightarrow q_1\uparrow}\,\downharpoonleft\!\upharpoonright}{\cdot \Rightarrow A \to q_1\uparrow}\,\to I_2$$

Some problems

- The proof-search space is very huge.

  Recall that a $\downarrow$-search phase starts *selecting a formula from the context* (don't know non-determinism) and decomposing it by applying elimination rules.

  To keep the contexts small and minimize the possible choices, we have to avoid as much as possible the application of the context-extending rules:

$$\frac{A, \Gamma \Rightarrow B\uparrow}{\Gamma \Rightarrow A \to B\uparrow} \to I_2$$

$$\frac{\Gamma \Rightarrow A \vee B\downarrow \quad A, \Gamma \Rightarrow C\uparrow \quad B, \Gamma \Rightarrow C\uparrow}{\Gamma \Rightarrow C\uparrow} \vee E$$

Some problems

- Too many non-deterministic choices.

  For instance, in $\uparrow$-expansion of

$$\Gamma \Rightarrow C \wedge D\uparrow$$

  we have three choices:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow C\uparrow\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma \Rightarrow D\uparrow\end{matrix}}{\Gamma \Rightarrow C \wedge D\uparrow} \wedge I \quad (1) \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow C \wedge D\downarrow\end{matrix}}{\Gamma \Rightarrow C \wedge D\uparrow} \downarrow\uparrow \quad (2)$$

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow A \vee B\downarrow\end{matrix} \quad \begin{matrix}\vdots \\ A, \Gamma \Rightarrow C \wedge D\uparrow\end{matrix} \quad \begin{matrix}\vdots \\ B, \Gamma \Rightarrow C \wedge D\uparrow\end{matrix}}{\Gamma \Rightarrow C \wedge D\uparrow} \vee E \quad (3)$$

Our proposal

- We decorate the $\uparrow$-arrow with one of the *labels b, u*.

  We have now three kinds of sequents:

  $$\Gamma \Rightarrow A{\downarrow} \qquad \Gamma \Rightarrow A{\uparrow}^{b} \qquad \Gamma \Rightarrow A{\uparrow}^{u}$$

  Label *b blocks* some rule applications during $\uparrow$-expansion, and this decreases the degree of non-determinism.

- We impose additional constraints on rule applications.

$$\mathbf{NJ} + \text{labels } b, u + \text{constraints} = \mathbf{Nbu}$$

From **NJ** to **Nbu**

Main property of **Nbu**

Given a derivation

$$\begin{array}{c}\vdots\\ \Gamma \Rightarrow A{\uparrow}^{u}\end{array}$$

in **Nbu** (where the root sequent has label u) and erasing the labels, we get a derivation

$$\begin{array}{c}\vdots\\ \Gamma \Rightarrow A{\uparrow}\end{array}$$

of **NJ**.
This guarantees the soundness of **Nbu**

Labels b, u have been introduced to get a terminating proof-search procedure for the intuitionistic sequent calculus:

M. Ferrari, C. Fiorentini, and G. Fiorino. A terminating evaluation-driven variant of G3i. TABLEAUX 2013

From **NJ** to **Nbu**

- $\downarrow\!\uparrow$, $\bot E$
  - **NJ**

$$\frac{\Gamma \Rightarrow A\downarrow}{\Gamma \Rightarrow A\uparrow}\downarrow\!\uparrow \qquad \frac{\Gamma \Rightarrow \bot\downarrow}{\Gamma \Rightarrow A\uparrow}\bot E$$

  - **Nbu**

$$\frac{\Gamma \Rightarrow p\downarrow}{\Gamma \Rightarrow p\uparrow^{l}}\downarrow\!\uparrow \qquad \frac{\Gamma \Rightarrow \bot\downarrow}{\Gamma \Rightarrow F\uparrow^{l}}\bot E$$

$$p \in \mathcal{V} \qquad F \in \mathcal{V} \cup \{\bot\}$$

In the conclusion, we constraint the form of the right formula and we add a label $l \in \{b, u\}$ to the $\uparrow$-arrow.

From **NJ** to **Nbu**

Rules of **Nbu**

- $\wedge I$
  - **NJ**

$$\frac{\Gamma \Rightarrow A\uparrow \qquad \Gamma \Rightarrow B\uparrow}{\Gamma \Rightarrow A \wedge B\uparrow} \wedge I$$

  - **Nbu**

$$\frac{\Gamma \Rightarrow A\uparrow^{l} \qquad \Gamma \Rightarrow B\uparrow^{l}}{\Gamma \Rightarrow A \wedge B\uparrow^{l}} \wedge I \qquad l \in \{b, u\}$$

- $\vee I$
  - **NJ**

$$\frac{\Gamma \Rightarrow A_k\uparrow}{\Gamma \Rightarrow A_0 \vee A_1\uparrow} \vee I$$

  - **Nbu**

$$\frac{\Gamma \Rightarrow A_k\uparrow^{b}}{\Gamma \Rightarrow A_0 \vee A_1\uparrow^{l}} \vee I \qquad l \in \{b, u\}$$

Note that the label of the $\uparrow$-arrow in the premise is set to b.

From **NJ** to **Nbu**

- $\vee E$
  - **NJ**

$$\frac{\Gamma \Rightarrow A \vee B{\downarrow} \qquad A, \Gamma \Rightarrow C{\uparrow} \qquad B, \Gamma \Rightarrow C{\uparrow}}{\Gamma \Rightarrow C{\uparrow}} \vee E$$

This is the most problematic rule in proof-search.

  - **Nbu**

$$\frac{\Gamma \Rightarrow A \vee B{\downarrow} \qquad A, \Gamma \Rightarrow D{\uparrow^u} \qquad B, \Gamma \Rightarrow D{\uparrow^u}}{\Gamma \Rightarrow D{\uparrow^u}} \vee E$$

√ Condition on the right-hand side formula D:

$$D \in \mathcal{V} \cup \{\bot\} \quad \text{or} \quad D = D_0 \vee D_1$$

√ Conditions on $A \vee B$:

$$A \notin \Gamma \qquad B \notin \Gamma$$

√ The label of the $\uparrow$-arrow in the conclusion must be u

These constraints strongly reduce the non-determinism.

From NJ to Nbu

- $\rightarrow I_1$ and $\rightarrow I_2$
  - **NJ**

$$\frac{\Gamma \Rightarrow B\uparrow}{\Gamma \Rightarrow A \rightarrow B\uparrow} \rightarrow I_1 \qquad \frac{A, \Gamma \Rightarrow B\uparrow}{\Gamma \Rightarrow A \rightarrow B\uparrow} \rightarrow I_2$$

Non-deterministic choice between the two rules

  - **Nbu**

$$\frac{\Gamma \Rightarrow B\uparrow^{l}}{\Gamma \Rightarrow A \rightarrow B\uparrow^{l}} \rightarrow I_1 \qquad \frac{A, \Gamma \Rightarrow B\uparrow^{u}}{\Gamma \Rightarrow A \rightarrow B\uparrow^{l}} \rightarrow I_2$$

$$A \in \Gamma \qquad\qquad A \notin \Gamma$$

√ The choice between the two version is deterministic.

√ In bottom-up application, $\rightarrow I_1$ preserves the label, whereas $\rightarrow I_2$ always sets the label to u.

The calculus **Nbu**

$$\frac{}{A,\Gamma \Rightarrow A{\downarrow}}\ \mathrm{Id} \qquad \frac{\Gamma \Rightarrow p{\downarrow}}{\Gamma \Rightarrow p{\uparrow}^{l}}\ {\downarrow\uparrow} \qquad \frac{\Gamma \Rightarrow \bot{\downarrow}}{\Gamma \Rightarrow F{\uparrow}^{l}}\ \bot E$$

$$\frac{\Gamma \Rightarrow A{\uparrow}^{l} \qquad \Gamma \Rightarrow B{\uparrow}^{l}}{\Gamma \Rightarrow A \wedge B{\uparrow}^{l}}\ \wedge I \qquad \frac{\Gamma \Rightarrow A_0 \wedge A_1{\downarrow}}{\Gamma \Rightarrow A_k{\downarrow}}\ \wedge E \qquad \frac{\Gamma \Rightarrow A_k{\uparrow}^{b}}{\Gamma \Rightarrow A_0 \vee A_1{\uparrow}^{l}}\ \vee I$$

$$\frac{\Gamma \Rightarrow A \vee B{\downarrow} \qquad A,\Gamma \Rightarrow D{\uparrow}^{u} \qquad B,\Gamma \Rightarrow D{\uparrow}^{u}}{\Gamma \Rightarrow D{\uparrow}^{u}}\ \vee E$$

$$\frac{\Gamma \Rightarrow B{\uparrow}^{l}}{\Gamma \Rightarrow A \to B{\uparrow}^{l}}\ {\to} I_1 \qquad \frac{A,\Gamma \Rightarrow B{\uparrow}^{u}}{\Gamma \Rightarrow A \to B{\uparrow}^{l}}\ {\to} I_2 \qquad \frac{\Gamma \Rightarrow A \to B{\downarrow} \qquad \Gamma \Rightarrow A{\uparrow}^{b}}{\Gamma \Rightarrow B{\downarrow}}\ {\to} E$$

$p \in \mathcal{V}$, $F \in \mathcal{V} \cup \{\bot\}$, $D \in \mathcal{V} \cup \{\bot\}$ or $D = D_0 \vee D_1$

Theorem (Completeness of Nbu)

*$A \in$ **IPL** iff there exists a derivation of $\cdot \Rightarrow A{\uparrow}^{u}$ in **Nbu***

The calculus **Nbu**

- There is a trivial translation from **Nbu** into **NJ**:

Nbu	$\mapsto$	**NJ**
$\vdots$ $\Gamma \Rightarrow A{\uparrow}^u$	... erase the labels ...	$\vdots$ $\Gamma \Rightarrow A{\uparrow}$

- The definition of the converse map is not immediate.
  Indeed, to consistently add labels to an **NJ**-derivation, some preliminary non-trivial reduction steps could be required.
- Derivations in **Nbu** are in long normal form, since the switch from $\uparrow$-expansion and $\downarrow$-expansion (rule $\downarrow\!\uparrow$) is marked by an atomic formula.

The calculus **Nbu**

Example

The **Nbu**-derivation

$$\dfrac{\dfrac{\dfrac{}{p \lor q \Rightarrow p \lor q\downarrow}\ \mathrm{Id}}{p \lor q \Rightarrow p \lor q\uparrow}\ \downarrow\!\!\uparrow}{\cdot \Rightarrow p \lor q \to p \lor q\uparrow}\ \to I_2$$

cannot be labelled (rule $\downarrow\!\!\uparrow$ is applied to non atomic formula).

To get a **Nbu**-derivation, we need some reduction steps, so that $\downarrow\!\!\uparrow$ is applied to an atomic formula:

$$\dfrac{\dfrac{\dfrac{}{p \lor q \Rightarrow p \lor q\downarrow}\ \mathrm{Id} \qquad \dfrac{\dfrac{\dfrac{}{p, p \lor q \Rightarrow p\downarrow}\ \mathrm{Id}}{p, p \lor q \Rightarrow p\uparrow^{b}}\ \downarrow\!\!\uparrow}{p, p \lor q \Rightarrow p \lor q\uparrow^{u}}\ \lor I \qquad \dfrac{\dfrac{\dfrac{}{q, p \lor q \Rightarrow q\downarrow}\ \mathrm{Id}}{q, p \lor q \Rightarrow q\uparrow^{b}}\ \downarrow\!\!\uparrow}{q, p \lor q \Rightarrow p \lor q\uparrow^{u}}\ \lor I}{p \lor q \Rightarrow p \lor q\uparrow^{u}}\ \lor E}{\cdot \Rightarrow p \lor q \to p \lor q\uparrow^{u}}\ \to I_2$$

The calculus **Nbu**

- Note that label *b* introduce a sort of *focus* on the right.
  Indeed, in $\uparrow$-expansion of $\Gamma \Rightarrow A{\uparrow}^{b}$, with A non-atomic, we are forced to apply an I-rule to decompose A.

$$\dfrac{\dfrac{\dfrac{}{A \to p, \Gamma \Rightarrow A \to p{\downarrow}}\,\mathrm{Id} \qquad \overset{(\star)}{A \to p, \Gamma \Rightarrow A{\uparrow}^{b}}}{A \to p, \Gamma \Rightarrow p{\downarrow}}\,{\to}E}{A \to p, \Gamma \Rightarrow p{\uparrow}^{u}}\,{\downarrow\uparrow}$$

In $(\star)$, we have to decompose A as much as possible; application of $\vee E$ is not allowed.

Proof-search

- We can define a goal-oriented proof-search strategy for **Nbu**, where all the rules are applied bottom-up.
- **Nbu**-trees can contain loops:

$$\Gamma \;=\; p_1,\;\; p_1 \to p_2,\, p_2 \to p_1$$

$$\dfrac{\dfrac{}{\Gamma \Rightarrow p_2 \to p_1 \downarrow} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \Rightarrow p_1 \to p_2 \downarrow} \qquad \dfrac{\dfrac{\vdots}{\color{red}{\Gamma \Rightarrow p_1 \downarrow}}}{\Gamma \Rightarrow p_1 \uparrow^b}\,{\downharpoonleft\upharpoonright}}{\Gamma \Rightarrow p_2 \downarrow}\,{\to E}}{\Gamma \Rightarrow p_2 \uparrow^b}\,{\downharpoonleft\upharpoonright}}{\dfrac{\color{red}{\Gamma \Rightarrow p_1 \downarrow}}{\Gamma \Rightarrow p_1 \uparrow^u}\,{\downharpoonleft\upharpoonright}}\,{\to E}$$

To get termination, we we make use of *history sets* (sets of atoms).

Evaluations

We have observed that in proof-search it is desirable to extend a context as little as possible, so to narrow the search space.

In **Nbu**, the context-extending rules $\to I_2$ and $\vee E$ can only be applied if the formulas to be added are not in the context.

$$\frac{A, \Gamma \Rightarrow B{\uparrow}^u}{\Gamma \Rightarrow A \to B{\uparrow}^l} \to I_2 \qquad\qquad \frac{\Gamma \Rightarrow A \vee B{\downarrow} \quad A, \Gamma \Rightarrow D{\uparrow}^u \quad B, \Gamma \Rightarrow D{\uparrow}^u}{\Gamma \Rightarrow D{\uparrow}^u} \vee E$$

$$A \notin \Gamma \qquad\qquad A,\ B \notin \Gamma$$

We introduce a more general condition which prevents the application of $\to I_2$ and $\vee E$ whenever the formulas A, B do not add "significant information" to the context Γ.

We formalize this by the notion of evaluation relation.

Evaluations

Definition (Evaluation)

An evaluation is any decidable relation $\models_\theta$ between finite multisets of formulas Γ and formulas A such that:

(1) $A \in \Gamma$ implies $\Gamma \models_\theta A$

(2) $\Gamma \models_\theta A$ and $\mathbf{Nbu} \vdash_d A, \Gamma \Rightarrow B{\uparrow}^l$ imply $\mathbf{Nbu} \vdash_d \Gamma \Rightarrow B{\uparrow}^l$
(a sort of cut rule).

$\mathbf{Nbu} \vdash_d \Gamma \Rightarrow A{\uparrow}^l \quad \rightsquigarrow$ there is an **Nbu**-derivation of $\Gamma \Rightarrow A{\uparrow}^l$ with depth at most d

Intuitively, $\Gamma \models_\theta A$ means:

the information conveyed by A is already available in Γ

We avoid to extend a context Γ with A whenever $\Gamma \models_\theta A$.

The calculus $\mathbf{Nbu}_\theta$

Let θ be an evaluation

$$\mathbf{Nbu}_\theta \quad = \quad \mathbf{Nbu} \text{ with } \to I_1,\ \to I_2,\ \vee E \text{ modified as follows:}$$

$$\frac{\Gamma \Rightarrow B\uparrow^{l}}{\Gamma \Rightarrow A \to B\uparrow^{l}} \to I_1 \qquad \Gamma \models_\theta A \qquad \textit{no need to add } A$$

$$\frac{A, \Gamma \Rightarrow B\uparrow^{u}}{\Gamma \Rightarrow A \to B\uparrow^{l}} \to I_2 \qquad \Gamma \not\models_\theta A$$

$$\frac{\Gamma \Rightarrow A \vee B\downarrow \quad A, \Gamma \Rightarrow D\uparrow^{u} \quad B, \Gamma \Rightarrow D\uparrow^{u}}{\Gamma \Rightarrow D\uparrow^{u}} \vee E \qquad \begin{array}{l} \Gamma \not\models_\theta A \\ \Gamma \not\models_\theta B \end{array}$$

$D \in \mathcal{V} \cup \{\bot\}$ or $D = D_0 \vee D_1$

The calculus $\mathbf{Nbu}_\theta$

Examples of evaluation relations:

- Minimum evaluation relation $\models_{\theta_{\min}}$

$$\Gamma \models_{\theta_{\min}} A \quad \text{iff} \quad A \in \Gamma \qquad \textit{membership}$$

  Note that $\mathbf{Nbu}_{\theta_{\min}} \equiv \mathbf{Nbu}$

- Cover relation $\models_{cov}$

  $\Gamma \models_{cov} E$ iff E has the form

$$E ::= G \mid E \wedge E \mid E \vee A \mid A \vee E \mid A \to E \qquad \begin{array}{l} G \in \Gamma \\ A \text{ any formula} \end{array}$$

The cover relation has been introduced in

S. Buss and R. Iemhoff. The depth of intuitionistic cut free proofs. Manuscript, 2003.

T. Franzen. Algorithmic aspects of intuitionistic propositional logic. T. R. 1988.

to study the depth of derivations in sequent calculi.

- The stronger the evaluation relation is, the better is the gain in proof-search.
  For instance, let $\Gamma = p, q$

$$\Gamma \models_{cov} p \land q$$

$$\Gamma \not\models_{\theta_{\min}} p \land q \qquad (p \land q \notin \Gamma)$$

  Thus, $\models_{cov}$ is better than $\models_{\theta_{\min}}$.
- Can we define evaluation relations stronger than cover?

  Can *semantics* help?

Concluding remarks

- We have presented **Nbu**, a variant of the Natural Deduction calculus for **IPL** which allows goal-oriented proof-search.
- **Nbu** internalizes some aspects of such proof-search procedure using three mechanisms:
  (1) orientation of sequents by labelled arrows $\uparrow^u$, $\uparrow^b$ and $\downarrow$;
  (2) side conditions involving rules $\vee E$, $\rightarrow I_1$ and $\rightarrow I_2$;
  (3) restrictions on the conclusion of the rules $\downarrow\!\uparrow$, $\bot E$ and $\vee E$
- The sequent image of **Nbu** is a labelled variant of Herbelin's calculus [CSL,94] (the sequent calculus isomorphic to Natural Deduction).
- We have implemented the proof-search procedure in JTabWb (a Java framework for developing provers):

  `http://www.dista.uninsubria.it/~ferram/`

  We have performed some experiments and the results are competitive with those of the state-of-the-art provers for **IPL**.