Algoritmo del sottogradiente

Input : $L(\lambda)$: funzione da massimizzare; λ^0: punto iniziale; T: massimo numero di iterazioni senza miglioramento, α_0: parametro di arresto.

1. Inizializzazione $t := 1$, $\alpha_t := 2$;
2. Per un dato valore $\lambda^{(t-1)}$, calcola la soluzione $x^{(t)}$ del problema Lagrangiano

$$\min c^T x + \lambda^T (b - Ax)$$
$$Cx \geq d$$
$$x \in Z_+^n$$

3. Dato il punto $x^{(t)}$, calcola un sottogradiente $s^{(t)}$ di $L(\lambda)$, ad esempio

$$s^{(t)} := b - Ax.$$

Se $s^{(t)} = 0$ allora stop, altrimenti vai al passo 4.

4. $\lambda^{(t)} := \lambda^{(t-1)} + \theta^{(t)} s^{(t)}$, con $\theta^{(t)}$ scelto in modo tale che le condizioni

$$\lim_{t \to \infty} \theta^{(t)} = 0$$
$$\sum_{t=0}^{k} \theta^{(t)} \to \infty \text{ per } k \to \infty$$

siano soddisfatte, ad esempio:

$$\theta^{(t)} := \alpha_t \frac{UB - L(\lambda^{(t)})}{\left\| s^{(t)} \right\|^2}$$

.

5. Poni $t := t + 1$, se il lower bound non è migliorato nelle ultime T iterazioni, poni

$$\alpha_t := \alpha_t / 2.$$

6. Se $\alpha_t < \alpha_0$ stop, altrimenti vai al passo passo 2.

Trasporto e localizzazione di al più L impianti di produzione

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$
$$\sum_{j=1}^{n} x_{ij} \le s_i y_i \qquad i=1,\dots,n$$
$$\sum_{i=1}^{n} x_{ij} \ge d_j \qquad j=1,\dots,n$$
$$\sum_{i=1}^{n} y_i \le L$$
$$x_{ij} \ge 0\ \forall i,j; \quad y_i \in \{0,1\}\ \forall i$$

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n}\lambda_j\left(d_j - \sum_{i=1}^{n} x_{ij}\right)$$
$$\sum_{j=1}^{n} x_{ij} \le s_i y_i \qquad i=1,\dots,n$$
$$\sum_{i=1}^{n} y_i \le L$$
$$x_{ij} \ge 0\ \forall i,j; \quad y_i \in \{0,1\}\ \forall i$$

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$
$$\sum_{j=1}^{n} x_{ij} \le s_i y_i \qquad i=1,\dots,n$$
$$\sum_{i=1}^{n} x_{ij} \ge d_j \qquad j=1,\dots,n$$
$$\sum_{i=1}^{n} y_i \le L$$
$$0 \le x_{ij} \le d_j \forall i,j; \quad y_i \in \{0,1\}\ \forall i$$

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n}\lambda_j\left(d_j - \sum_{i=1}^{n} x_{ij}\right)$$
$$\sum_{j=1}^{n} x_{ij} \le s_i y_i \qquad i=1,\dots,n$$
$$\sum_{i=1}^{n} y_i \le L$$
$$0 \le x_{ij} \le d_j \forall i,j; \quad y_i \in \{0,1\}\ \forall i$$

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$
$$\sum_{j=1}^{n} x_{ij} \le s_i \qquad i=1,\dots,n$$
$$\sum_{i=1}^{n} x_{ij} \ge d_j \qquad j=1,\dots,n$$
$$x_{ij} \le d_j y_i \qquad \forall i,j$$
$$\sum_{i=1}^{n} y_i \le L$$
$$x_{ij} \ge 0\ \forall i,j; \quad y_i \in \{0,1\}\ \forall i$$

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n}\lambda_j\left(d_j - \sum_{i=1}^{n} x_{ij}\right)$$
$$\sum_{j=1}^{n} x_{ij} \le s_i \qquad i=1,\dots,n$$
$$x_{ij} \le d_j y_i \qquad \forall i,j$$
$$\sum_{i=1}^{n} y_i \le L$$
$$x_{ij} \ge 0\ \forall i,j; \quad y_i \in \{0,1\}\ \forall i$$

Multi Zaino

$$\max \sum_{i=1}^{m}\sum_{j=1}^{n} p_j x_{ij}$$

$$\sum_{j=1}^{n} w_{ij} x_{ij} \le b_i \qquad i=1,\dots,m$$

$$\sum_{i=1}^{m} x_{ij} \le 1 \qquad j=1,\dots,n$$

$$x_{ij} \in \{0,1\} \quad \forall i,j$$

$$\max \sum_{i=1}^{m}\sum_{j=1}^{n} p_j x_{ij} + \sum_{j=1}^{n} \lambda_j \left(1-\sum_{i=1}^{m} x_{ij}\right) \qquad \sum_{j=1}^{n}\lambda_j + \sum_{i=1}^{m} \max \sum_{j=1}^{n} (p_j - \lambda_j) x_{ij}$$

$$\sum_{j=1}^{n} w_{ij} x_{ij} \le b_i \qquad i=1,\dots,m \qquad\qquad \sum_{j=1}^{n} w_{ij} x_{ij} \le b_i$$

$$x_{ij} \in \{0,1\} \quad \forall i,j \qquad\qquad x_{ij} \in \{0,1\} \quad \forall i,j$$

$$\max \sum_{i=1}^{m}\sum_{j=1}^{n} p_j x_{ij} + \sum_{i=1}^{m} \lambda_i \left(b_i-\sum_{j=1}^{n} w_{ij} x_{ij}\right) \qquad \sum_{i=1}^{m}\lambda_i b_i + \max \sum_{i=1}^{m}\sum_{j=1}^{n} (p_j - \lambda_i w_{ij}) x_{ij}$$

$$\sum_{i=1}^{m} x_{ij} \le 1 \qquad j=1,\dots,n \qquad\qquad \sum_{i=1}^{m} x_{ij} \le 1 \qquad j=1,\dots,n$$

$$x_{ij} \in \{0,1\} \quad \forall i,j \qquad\qquad x_{ij} \in \{0,1\} \quad \forall i,j$$

Bin Packing

$$\min \sum_{i=1}^{m} c_i y_i$$

$$\sum_{j=1}^{n} w_{ij} x_{ij} \le b_i y_i \quad i = 1,\dots,m$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad j = 1,\dots,n$$

$$x_{ij} \in \{0,1\} \ \ \forall i, j; \quad y_i \in \{0,1\} \ \ \forall i$$

$$\min \sum_{i=1}^{m} c_i y_i + \sum_{j=1}^{n} \lambda_j \left(1 - \sum_{i=1}^{m} x_{ij} \right)$$

$$\sum_{j=1}^{n} w_{ij} x_{ij} \le b_i y_i \quad i = 1,\dots,m$$

$$x_{ij} \in \{0,1\} \ \ \forall i, j; \quad y_i \in \{0,1\} \ \ \forall i$$

$$\sum_{j=1}^{n} \lambda_j + \min \sum_{i=1}^{m} c_i y_i - \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_j x_{ij}$$

$$\sum_{j=1}^{n} w_{ij} x_{ij} \le b_i y_i \quad i = 1,\dots,m$$

$$x_{ij} \in \{0,1\} \ \ \forall i, j; \quad y_i \in \{0,1\} \ \ \forall i$$

$$\sum_{j=1}^{n} \lambda_j - \sum_{i=1}^{m} \max \left\{ \sum_{j=1}^{n} \lambda_j x_{ij} - c_i y_i \right\}$$

$$\sum_{j=1}^{n} w_{ij} x_{ij} \le b_i y_i$$

$$x_{ij} \in \{0,1\} \ \ \forall i, j; \quad y_i \in \{0,1\} \ \ \forall i$$

$$z_i = \max \sum_{j=1}^{n} \lambda_j x_j$$

$$\sum_{j=1}^{n} w_{ij} x_j \le b_i$$

$$x_j \in \{0,1\} \ \ \forall j$$

$$L(\lambda) = \sum_{j=1}^{n} \lambda_j - \sum_{i=1}^{m} \begin{cases} 0 & \text{se } z_i \le c_i \\ z_i - c_i & \text{altrimenti} \end{cases}$$

$$\min \sum_{i=1}^{m} c_i y_i + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} w_{ij} x_{ij} - b_i y_i \right)$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad j = 1,\dots,n$$

$$x_{ij} \in \{0,1\} \ \ \forall i, j; \quad y_i \in \{0,1\} \ \ \forall i$$

$$\min \sum_{i=1}^{m} (c_i - \lambda_i b_i) y_i + \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i w_{ij} x_{ij}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad j = 1,\dots,n$$

$$x_{ij} \in \{0,1\} \ \ \forall i, j; \quad y_i \in \{0,1\} \ \ \forall i$$

Assegnamento e sequenziamento

$$\min z$$
$$\sum_{j=1}^{n} p_{ij}x_{ij} \le z \quad i=1,\dots,m$$
$$\sum_{i=1}^{m} x_{ij} = 1 \quad j=1,\dots,n$$
$$x_{ij} \in \{0,1\} \;\; \forall i, j; \quad z \ge 0$$

$$\min\; z + \sum_{j=1}^{n} \lambda_j \left(1 - \sum_{i=1}^{m} x_{ij}\right)$$
$$\sum_{j=1}^{n} p_{ij}x_{ij} \le z \quad i=1,\dots,m$$
$$x_{ij} \in \{0,1\} \;\; \forall i, j; \quad z \ge 0$$

$$\sum_{j=1}^{n} \lambda_j + \min\left\{ z - \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_j x_{ij} \right\}$$
$$\sum_{j=1}^{n} p_{ij}x_{ij} \le z \quad i=1,\dots,m$$
$$x_{ij} \in \{0,1\} \;\; \forall i, j; \quad z \ge 0$$

$$\sum_{j=1}^{n} \lambda_j - \max\left\{ \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_j x_{ij} - z \right\}$$
$$\sum_{j=1}^{n} p_{ij}x_{ij} \le z \quad i=1,\dots,m$$
$$x_{ij} \in \{0,1\} \;\; \forall i, j; \quad z \ge 0$$

$$\min\; z + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} p_{ij}x_{ij} - z \right)$$
$$\sum_{i=1}^{m} x_{ij} = 1 \quad j=1,\dots,n$$
$$x_{ij} \in \{0,1\} \;\; \forall i, j; \quad z \ge 0$$

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_i p_{ij} x_{ij} + \left(1 - \sum_{i=1}^{m} \lambda_i\right) z$$
$$\sum_{i=1}^{m} x_{ij} = 1 \quad j=1,\dots,n$$
$$x_{ij} \in \{0,1\} \;\; \forall i, j; \quad z \ge 0$$

Se $\sum_{i=1}^{m} \lambda_i > 1$ allora $L(\lambda) = -\infty$ ponendo $z = +\infty$. Quindi

per $\sum_{i=1}^{m} \lambda_i \le 1$

$$L(\lambda) = \min \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_i p_{ij} x_{ij}$$
$$\sum_{i=1}^{m} x_{ij} = 1 \quad j=1,\dots,n$$
$$x_{ij} \in \{0,1\} \;\; \forall i, j$$

Assegnamento lineare con vincoli di budget

$$\max \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad i = 1, \dots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, \dots, n$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \le b$$

$$x_{ij} \in \{0,1\} \quad \forall i, j$$

Assegnamento lineare

$$\lambda b + \max \sum_{i=1}^{m} \sum_{j=1}^{n} (p_{ij} - \lambda w_{ij}) x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad i = 1, \dots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, \dots, n$$

$$x_{ij} \in \{0,1\} \quad \forall i, j$$

Knapsack Multiple Choice

$$\sum_{i=1}^{n} \lambda_i + \max \sum_{i=1}^{n} \sum_{j=1}^{n} (p_{ij} - \lambda_i) x_{ij}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, \dots, n$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \le b$$

$$x_{ij} \in \{0,1\} \quad \forall i, j$$

Knapsack

$$\sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{n} \mu_j + \max \sum_{i=1}^{n} \sum_{j=1}^{n} (p_{ij} - \lambda_i - \mu_j) x_{ij}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \le b$$

$$x_{ij} \in \{0,1\} \quad \forall i, j$$

Cutting Stock

$$\min \sum_{i=1}^{m} y_i$$

$$\sum_{j=1}^{n} w_j x_{ij} \le W y_i \qquad i = 1,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} \ge d_j \qquad j = 1,\ldots,n$$

$$x_{ij} \in Z^+ \quad \forall i,j; \qquad y_i \in \{0,1\} \quad \forall i$$

$$\sum_{j=1}^{n} \lambda_j d_j + \min \sum_{i=1}^{m} y_i - \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_j x_{ij}$$

$$\sum_{j=1}^{n} w_j x_{ij} \le W y_i \qquad i = 1,\ldots,m$$

$$x_{ij} \in Z^+ \quad \forall i,j; \qquad y_i \in \{0,1\} \quad \forall i$$

$$L(\lambda) = \sum_{j=1}^{n} \lambda_j d_j + \sum_{i=1}^{m} \min \left\{ y_i - \sum_{j=1}^{n} \lambda_j x_{ij} \right\}$$

$$\sum_{j=1}^{n} w_j x_{ij} \le W y_i$$

$$x_{ij} \in Z^+ \quad \forall i,j; \qquad y_i \in \{0,1\} \quad \forall i$$

$$z(\lambda) = \max \sum_{j=1}^{n} \lambda_j x_j$$

$$\sum_{j=1}^{n} w_j x_j \le W$$

$$x_j \in Z^+ \quad \forall j$$

$$L(\lambda) = \sum_{j=1}^{n} \lambda_j d_j + \begin{cases} 0 & \text{se } z(\lambda) \le 1 \\ m(1 - z(\lambda)) & \text{altrimenti} \end{cases}$$

SSTDMA

$$\min \sum_{k=1}^{n} z_k$$

$$\sum_{j=1}^{n} x_{ijk} \le 1 \quad i = 1,\dots,n$$

$$\sum_{i=1}^{n} x_{ijk} \le 1 \quad j = 1,\dots,n$$

$$\sum_{k=1}^{n} x_{ijk} = 1 \quad \forall i, j$$

$$w_{ij} x_{ijk} \le z_k \quad \forall i, j, k$$

$$x_{ijk} \in \{0,1\} \ \ \forall i, j, k;\ z_k \ge 0 \ \ \forall k$$

$$\min \sum_{k=1}^{n} z_k + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} (1 - \sum_{k=1}^{n} x_{ijk})$$

$$\sum_{j=1}^{n} x_{ijk} \le 1 \quad i = 1,\dots,n$$

$$\sum_{i=1}^{n} x_{ijk} \le 1 \quad j = 1,\dots,n$$

$$w_{ij} x_{ijk} \le z_k \quad \forall i, j, k$$

$$x_{ijk} \in \{0,1\} \ \ \forall i, j, k;\ z_k \ge 0 \ \ \forall k$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} - \max \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_{ij} x_{ijk} - \sum_{k=1}^{n} z_k \right\}$$

$$\sum_{j=1}^{n} x_{ijk} \le 1 \quad i = 1,\dots,n$$

$$\sum_{i=1}^{n} x_{ijk} \le 1 \quad j = 1,\dots,n$$

$$w_{ij} x_{ijk} \le z_k \quad \forall i, j, k$$

$$x_{ijk} \in \{0,1\} \ \ \forall i, j, k;\ z_k \ge 0 \ \ \forall k$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} - \sum_{k=1}^{n} \max \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} x_{ijk} - z_k \right\}$$

$$\sum_{j=1}^{n} x_{ijk} \le 1 \quad i = 1,\dots,n$$

$$\sum_{i=1}^{n} x_{ijk} \le 1 \quad j = 1,\dots,n$$

$$w_{ij} x_{ijk} \le z_k \quad \forall i, j, k$$

$$x_{ijk} \in \{0,1\} \ \ \forall i, j, k;\ z_k \ge 0 \ \ \forall k$$

$$w = \max \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} x_{ij} - z \right\}$$

$$\sum_{j=1}^{n} x_{ij} \le 1 \quad i = 1,\dots,n$$

$$\sum_{i=1}^{n} x_{ij} \le 1 \quad j = 1,\dots,n$$

$$w_{ij} x_{ij} \le z \quad \forall i, j$$

$$x_{ij} \in \{0,1\} \ \ \forall i, j;\ z \ge 0$$

$$L(\lambda) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} - nw$$

Cutting Stock

$$\min \sum_{i=1}^{m} y_i$$
$$\sum_{j=1}^{n} w_j x_{ij} \le W y_i \quad i=1,\dots,m$$
$$\sum_{i=1}^{m} x_{ij} \ge d_j \quad j=1,\dots,n$$
$$x_{ij} \in Z^+ \ \ \forall i,j; \quad y_i \in \{0,1\} \ \ \forall i$$

$$\min \sum_{i=1}^{K} z_i$$
$$\sum_{i=1}^{K} a_{ij} z_i \ge d_j \quad j=1,\dots,n$$
$$z_i \in \{0,1\} \ \ \forall i$$

$$P) \quad \min 1^T z$$
$$Az \ge d$$
$$z \in \{0,1\}^K$$

$$A_h = \begin{pmatrix} a_{1h} \\ a_{2h} \\ \vdots \\ a_{nh} \end{pmatrix} \text{ dove } a_{jh} = \begin{cases} 1 & \text{item } j \text{ coperto da taglio } h \\ 0 & \text{altrimenti} \end{cases}$$

$$PR) \quad \min 1^T \bar{z} \qquad \bar{A}\bar{z} \ge d \qquad \bar{z} \in \{0,1\}^I \qquad \text{con } I < K$$

$$\min 1^T \bar{z} \qquad \bar{A}\bar{z} \ge d \qquad 0 \le \bar{z} \le 1$$

z^* soluzione ottima primale rilassamento continuo Primale Ridotto (PR)

λ^* soluzione ottima duale rilassamento continuo (PR)

$\exists\, A_h \in A / \bar{A}$ t.c. $r_h = 1 - A_h\lambda^* < 0$?

r_h =costo ridotto variabile h

Da a_{jh} a x_j

$$\min 1 - \sum_{j=1}^{n} \lambda_j x_j$$
$$\sum_{j=1}^{n} w_j x_j \le W$$
$$x_j \in Z^+ \ \ \forall j$$

$$z_i = \max \sum_{j=1}^{n} \lambda_j x_j$$
$$\sum_{j=1}^{n} w_j x_j \le W$$
$$x_j \in Z^+ \ \ \forall j$$

if $z_i > 1$ then
  $r_h = 1 - A_h\lambda^* < 0$
  con $a_{jh} := x_j$. Aggiungiamo colonna A_h alla matrice $\bar{A}$ e risolviamo nuovamente il problema PR)
else
  z^* soluzione ottima primale del rilassamento continuo del Primale P)

5.1.1 Modified subgradient ascent Classical subgradient ascent suffers from strong oscillations in the direction of the subgradient vector s, which updates λ. This suggests to update λ taking into account not only the new subgradient vector, but also the previous one, since this is still likely to include some useful information.

The updating formula (see Camerini et al. (1975)) is

$$\lambda_{k+1} = \max\left(\lambda_k + t_k \sigma_{\lambda_k} \frac{|UB - z_{\lambda_k}|}{\|\sigma_{\lambda_k}\|^2}, 0\right) \tag{8}$$

where

$$\sigma_{\lambda_k} = \begin{cases} s_{\lambda_k} & \text{when } s'_{\lambda_k}\sigma_{\lambda_{k-1}} \geq 0 \\ s_{\lambda_k} + \beta s_{\lambda_{k-1}} & \text{when } s'_{\lambda_k}\sigma_{\lambda_{k-1}} < 0 \end{cases} \tag{9}$$

t_k is a decreasing *step coefficient*, $s'_{\lambda_k}\sigma_{\lambda_{k-1}}$ is the dot product of the two vectors and

$$\beta = \frac{\|s_{\lambda_k}\|}{\|s_{\lambda_{k-1}}\|} \tag{10}$$

which corresponds to scaling $s_{\lambda_{k-1}}$ with respect to s_{λ_k}, in order to superpose them effectively.