Limited Automata and Unary Languages

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Limited Automata [Hibbard '67]

One-tape Turing machines with restricted rewritings

Definition

Fixed an integer $d \geq 1$, a *d-limited automaton* is

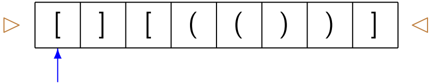
- a one-tape Turing machine
- which is allowed to rewrite the content of each tape cell *only in the first d visits*

Example: 2-LA for the Dyck Language over $\{[], ()\}$

Idea:

- Move to the right to search a closed bracket and rewrite it
- Then move to the left, to search an open bracket.

  If it is of the same type, then rewrite it and repeat



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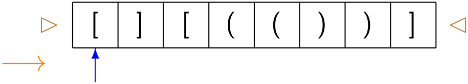

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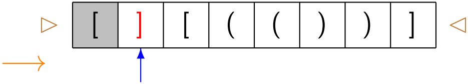

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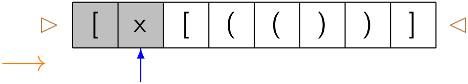

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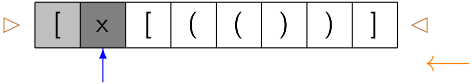

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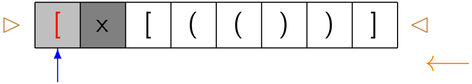

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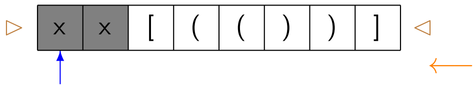

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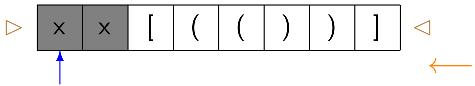

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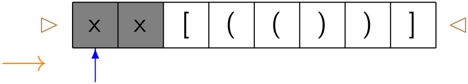

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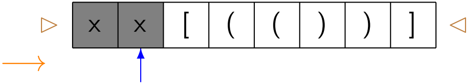

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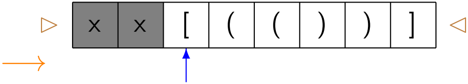

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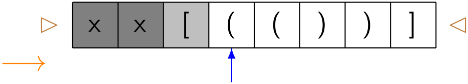

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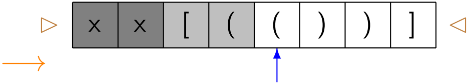

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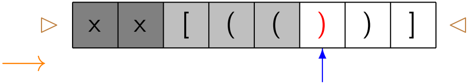

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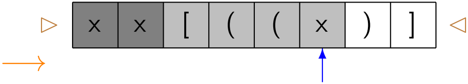

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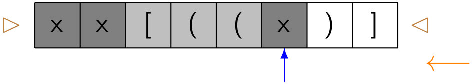

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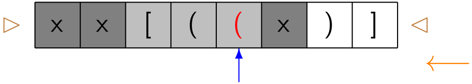

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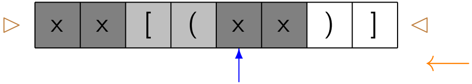

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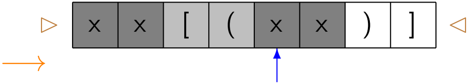

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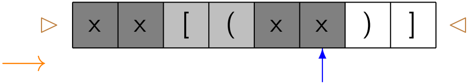

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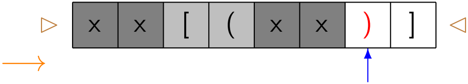

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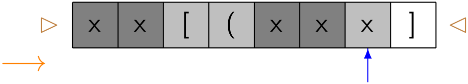

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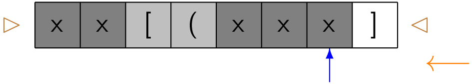

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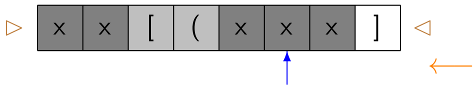

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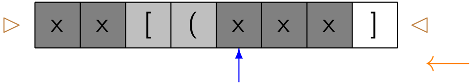

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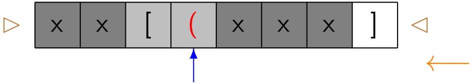

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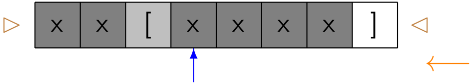

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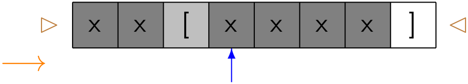

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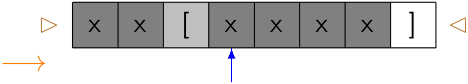

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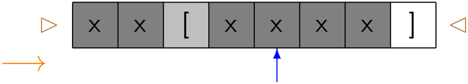

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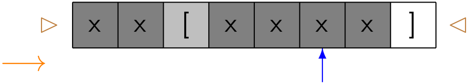

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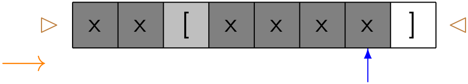

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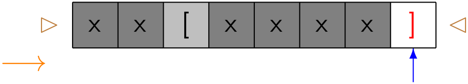

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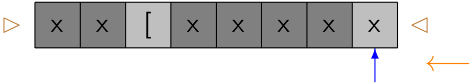

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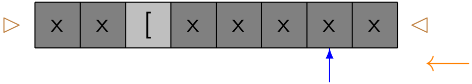

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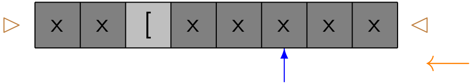

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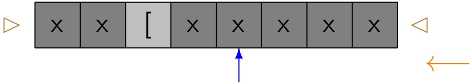

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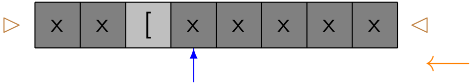

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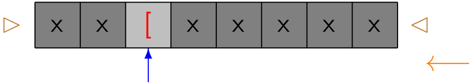

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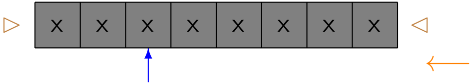

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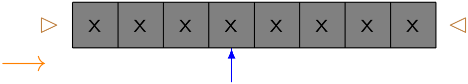

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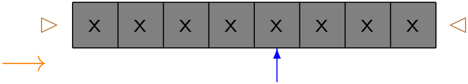

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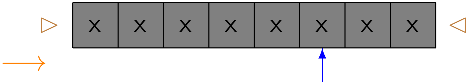

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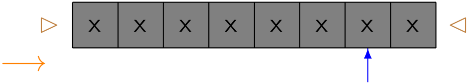

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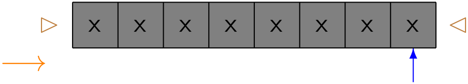

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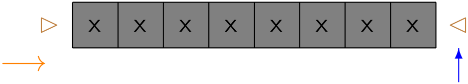

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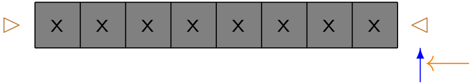

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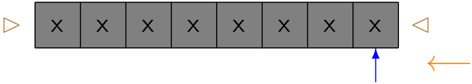

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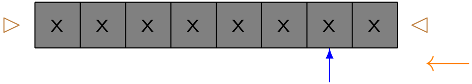

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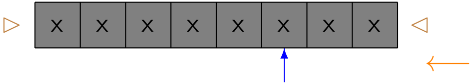

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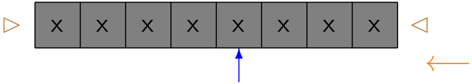

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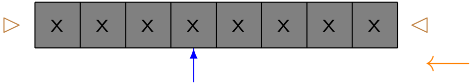

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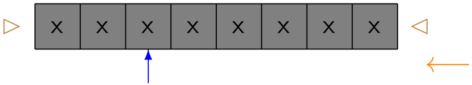

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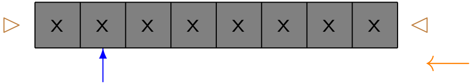

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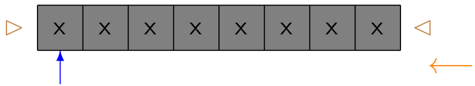

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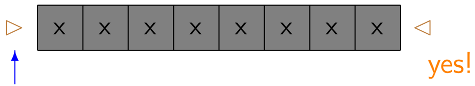

Each cell is rewritten only in the first 2 visits!

Limited Automata [Hibbard '67]

One-tape Turing machines with restricted rewritings

Definition

Fixed an integer $d \geq 1$, a *d-limited automaton* is

- a one-tape Turing machine
- which is allowed to rewrite the content of each tape cell *only in the first d visits*

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Computational power

- For each $d \geq 2$, d-limited automata characterize context-free languages [Hibbard '67]
- 1-limited automata characterize regular languages [Wagner&Wechsung '86]

Descriptional Complexity: Limited Automata vs PDAs

- $d = 2$ [P.&Pisoni '15]

Descriptional Complexity: Limited Automata vs PDAs

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2-LAs $\rightarrow$ PDAs

Exponential gap

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- $d > 2$ [Kutrib&P.&Wendlandt *to app.*]

d-LAs $\rightarrow$ PDAs

Still exponential!

Descriptional Complexity: Limited vs Finite Automata

- $d = 1$ [P.&Pisoni '14]

n-state 1-LAs $\rightarrow$ finite automata

	DFA	NFA
nondet. 1-LA		
det. 1-LA		

- The gaps are optimal! (binary witness)

What about the unary case?

Descriptional Complexity: Limited vs Finite Automata

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What about the unary case?

The Unary Case, $d = 1$

Theorem ([P.&Pisoni '14])

For n prime, the language $\{a^{n^2}\}^*$*:*

- *is accepted by a* 1-LA *with* $n + 1$ *states and a constant size tape alphabet*
- *requires* n^2 *many states to be accepted be a 2NFA*

$\Rightarrow$ Quadratic lower bound for the simulation of unary 1-LAs by finite automata

The Unary Case, $d = 1$

Theorem ([Kutrib&Wendlandt '15])

For n prime, the language $\{a^{n \cdot F(n)}\}$:

- *is accepted by a 1-LA with $4n$ states and a tape alphabet with $n + 1$ symbols*
- *requires $n \cdot F(n)$ many states to be accepted be a 2NFA*

where $F(n) = e^{\sqrt{n \cdot \ln(n)}(1+o(1))}$ (Landau function)

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This paper: Exponential lower bound

The Unary Case, $d > 1$

- d-LA $\equiv$ CFLs ($d > 1$)
- Each unary CFL is regular [Ginsburg&Rice '62]

$\Rightarrow$ unary d-LA $\equiv$ unary REG

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For $n > 0$, the language $\{a^{2^n}\}^*$:

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This paper: Same lower bound for the simulation of unary 1-LAs

Unary 1-LA vs Finite Automata

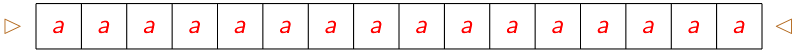
The Exponential Separation

The Witness Language

- Fixed $n > 0$: $L_n = \{a^{2^n}\}$
- The smallest NFA accepting L_n has $2^n + 1$ many states
- We show the existence of a *deterministic* 1-LA of $O(n)$ size accepting L_n

A Linear Bounded Automaton for $L_n = \{a^{2^n}\}$

Idea: “divide” the input n times by 2



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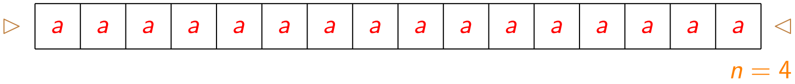

- Make n sweeps of the tape
- At each sweep overwrite each "odd" a

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▷	X	a	X	a	X	a	X	a	X	a	X	a	X	a	X	a	◁

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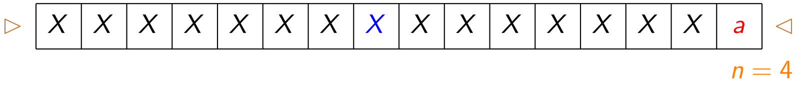

- Make n sweeps of the tape
- At each sweep overwrite each "odd" a
- Accept if only one a is left on the tape

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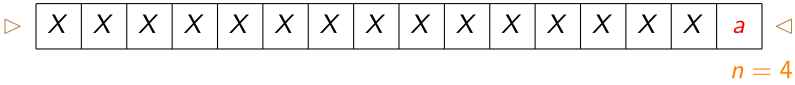

- Make n sweeps of the tape
- At each sweep overwrite each “odd” a
- Accept if only one a is left on the tape
- $O(n)$ states

A Linear Bounded Automaton for $L_n = \{a^{2^n}\}$

Idea: “divide” the input n times by 2

$\triangleright$	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	$\triangleleft$

$n = 4$

Possible variation:

- Rewrite input symbols with the number of current sweep

A Linear Bounded Automaton for $L_n = \{a^{2^n}\}$

Idea: "divide" the input n times by 2

▷	0	a	0	a	0	a	0	a	0	a	0	a	0	a	0	a	◁

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Idea: "divide" the input n times by 2

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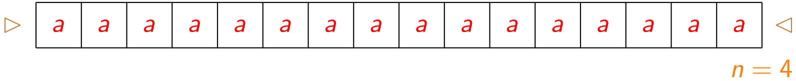
$n = 4$

Possible variation:

- Rewrite input symbols with the number of current sweep

We can build a 1-LA that, for each tape cell,
guesses the number of the sweep
in which this linear bounded automaton rewrites the cell

A 1-Limited Automaton for $L_n = \{a^{2^n}\}$



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- *1st sweep:*
  For each cell, guess and write a symbol in $\{0, 1, \ldots, n\}$
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  Verify that the symbol i occurs in all odd positions,
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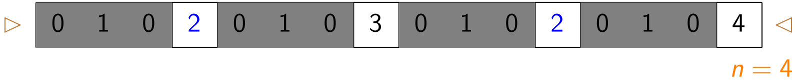

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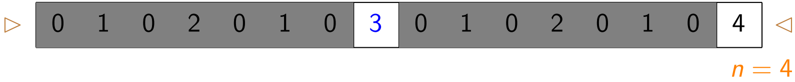

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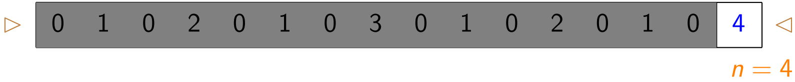

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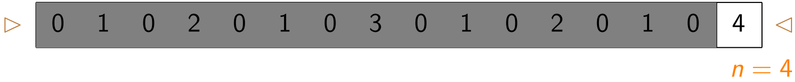

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We can do better!

Size $O(n)$, only *deterministic* transitions

The Binary Carry Sequence

The string written by the above linear bounded automaton is a prefix of the *binary carry sequence*:

- First two elements: 0 1
- Next elements: $w \rightarrow ww'$
  - w part already constructed,
  - w' copy of w, with the last symbol replaced by its successor

0 1

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The Binary Carry Sequence: Properties

- $w_j :=$ prefix of length j of the binary carry sequence
- $BIS(w_j) :=$ *Backward Increasing Sequence* of w_j

  longest increasing sequence obtained with the greedy method by inspecting w_j from the end

$w_{11} = \quad 0 \quad 1 \quad 0 \quad 2 \quad 0 \quad 1 \quad 0 \quad 3 \quad 0 \quad 1 \quad 0$

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The Binary Carry Sequence: Properties

- w_j := prefix of length j of the binary carry sequence
- $BIS(w_j)$:= *Backward Increasing Sequence* of w_j

  longest increasing sequence obtained with the greedy method by inspecting w_j from the end

$w_{11} =$ 0 1 0 2 0 1 0 3 0 1 0

$BIS(w_{11}) =$ 0 1 3

$11 = 2^0 + 2^1 + 2^3$

Property 1

$BIS(w_j) =$ positions of 1s in the binary representation of j

The Binary Carry Sequence: Properties

$w_{11} = \quad 0 \quad 1 \quad 0 \quad 2 \quad 0 \quad 1 \quad 0 \quad 3 \quad 0 \quad 1 \quad 0$

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Property 2

The symbol of the binary carry sequence in position $j+1$ is the smallest nonnegative integer that does not occur in $BIS(w_j)$

A Deterministic 1-LA for $L_n = \{a^{2^n}\}$

Idea: Write on the tape prefixes of the binary carry sequence

$n = 4$

$\triangleright$	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	$\triangleleft$

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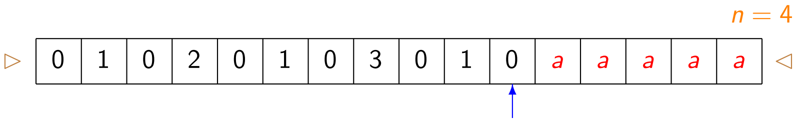

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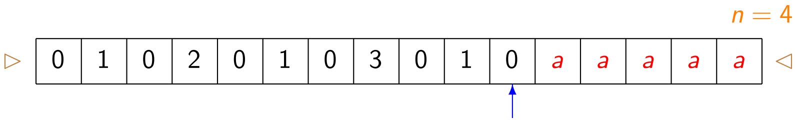

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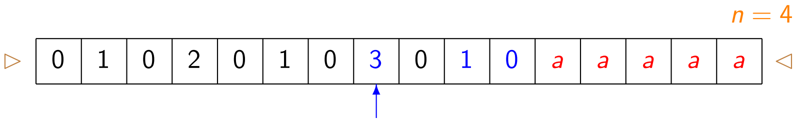

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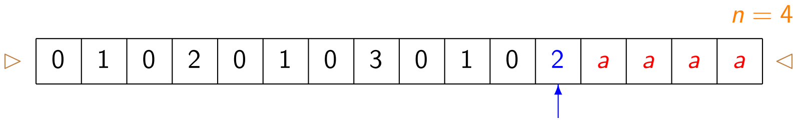

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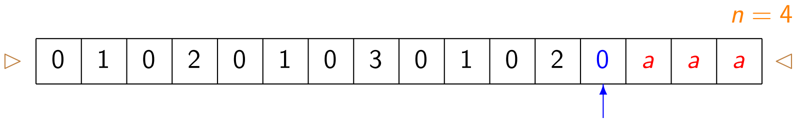

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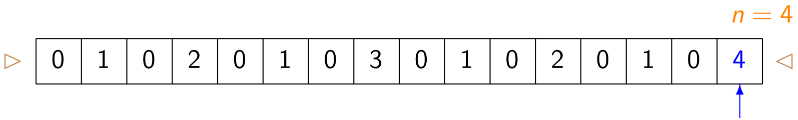

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- Total size of the description $O(n)$

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Unary 1-LA vs Finite Automata: Upper and Lower Bounds

det-1-LAs $\rightarrow$ NFAs/DFAs
ndet-1-LAs $\rightarrow$ NFAs

Exponential gap

- l.b. our result
- u.b. general case

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Problem
Can we reduce the distance between l.b. and u.b.?

From Unary Finite Automata to 1-LAs

An exponential reduction is not always achievable:

Theorem

There is a constant c s.t. for each sufficiently large n there is a unary n-state DFA s.t. all equivalent d-LAs have descriptions of size $> c \cdot n^{1/2}$, for each $d > 0$

Unary CFGs vs Limited Automata

Unary Context-Free Languages

Theorem ([Ginsburg&Rice '62])

Each unary context-free language is regular

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Study the size relationships between unary CFGs and limited automata

[This work]

The conversion
unary CFGs $\rightarrow$ 1-LAs
is polynomial in size

A Variant of the Chomsky-Schützenberger Theorem

Extended Dyck Language $\widehat{D}_\Omega$

- Balanced brackets padded with neutral symbols
- Ex. $\Omega = \{(,), [,], |\}$, strings $||(|)$, $(([|]|)[]||)|()[]|$, ...

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Theorem ([Okhotin '12])

$L \subseteq \Sigma^*$ is context-free iff $L = h(\widehat{D}_\Omega \cap R)$, where

- *Ω is an extended bracket alphabet*
- *$R \subseteq \Omega^*$ is regular*
- *$h : \Omega \to \Sigma$ is a letter-to-letter homomorphism*

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Remarks

- The size of Ω is *polynomial* wrt the size of a given CFG G specifying L
- The language R is *local*
- Strings in $\widehat{D}_\Omega \cap R$ encode derivation trees of G

Chomsky-Schützenberger Theorem in the Unary Case

- $G = (V, \{a\}, P, S)$ unary CFG generating $L(G)$
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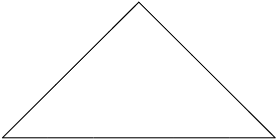

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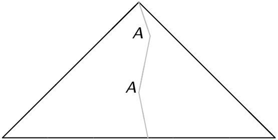

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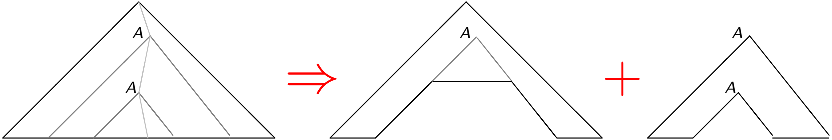

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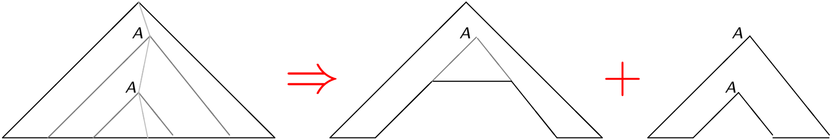

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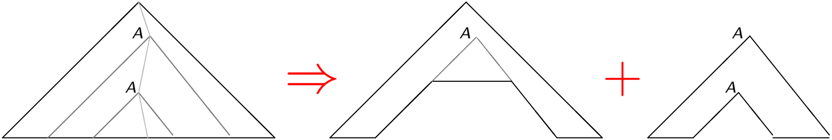

Then $$L(G) = h(\widehat{D}^{(\#V)}_{\Omega_G} \cap R)$$

The "restricted extended" Dyck Language $\widehat{D}^{(\#V)}_{\Omega_G} \subset \widehat{D}_{\Omega_G}$

- contains only the strings with bracket nesting depth $\leq \#V$
- is recognized by a 2DFA of size polynomial wrt the size of G

A 1-LA Accepting $L(G) = h(\widehat{D}_{\Omega_G}^{(\#V)} \cap R)$

1. Input a^m
2. Guess $w \in h^{-1}(a^m)$
   - Scan the tape from left to right
   - Rewrite each input cell with a symbol from Ω_G
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Summing up:

- Each cell is rewritten only in the first visit
- The total size of the resulting 1-LA is polynomial

We proved that

Theorem

The conversion of unary CFGs into 1-LAs is polynomial in size

Unary CFGs vs Limited Automata

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Problems

- Cost of the converse conversion, i.e., (unary) 1-LAs $\rightarrow$ CFGs
  General alphabets: 2-LAs $\rightarrow$ CFGs is *exponential* in size
- Conversion of unary CFGs into *deterministic* limited automata

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Thank you for your attention!