Restricted Turing Machines and Language Recognition

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Hennie Machines & C

Part II: One-Tape Turing Machines
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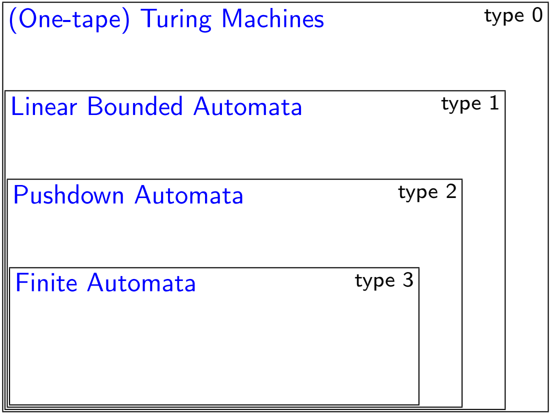
The Chomsky Hierarchy



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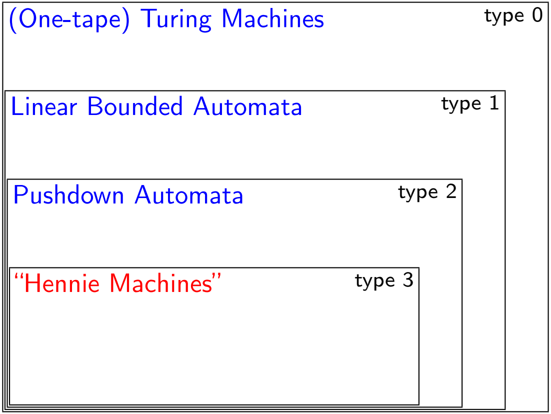

Outline

- Limited automata
- Equivalence with CFLs
- Determinism vs nondeterminism
- Descriptional complexity aspects
- 1-limited automata and regular languages
- Related models

Part II: One-Tape TMs with Rewriting Restrictions

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Limited Automata [Hibbard '67]

One-tape Turing machines with restricted rewritings

Definition

Fixed an integer $d \geq 1$, a *d-limited automaton* is

- a one-tape Turing machine
- which is allowed to rewrite the content of each tape cell *only in the first d visits*

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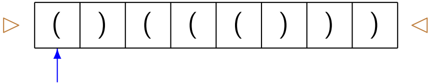

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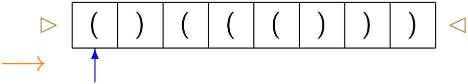

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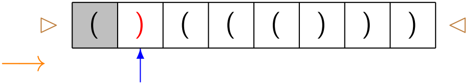

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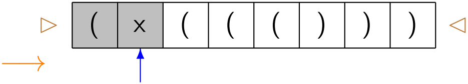

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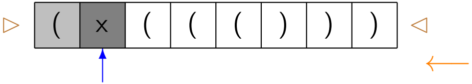

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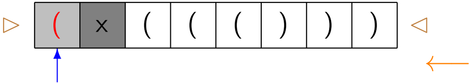

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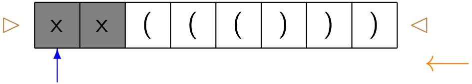

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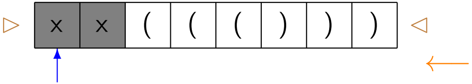

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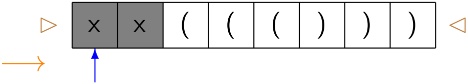

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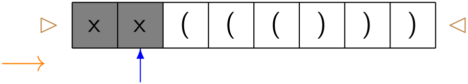

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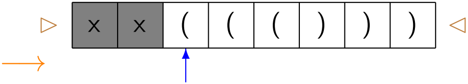

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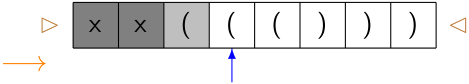

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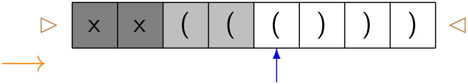

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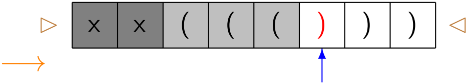

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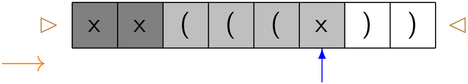

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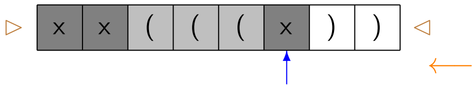

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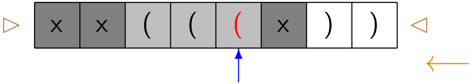

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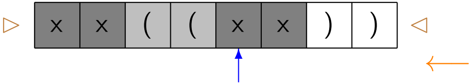

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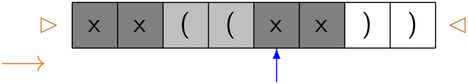

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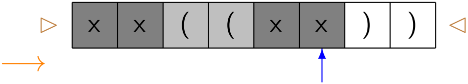

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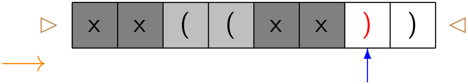

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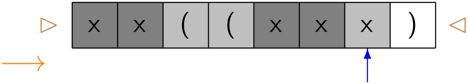

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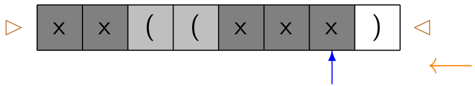

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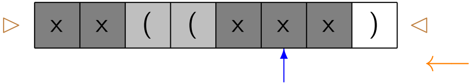

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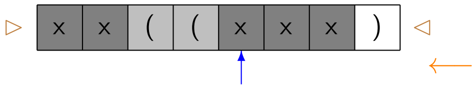

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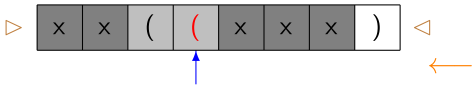

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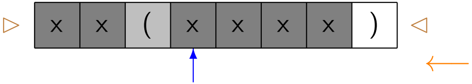

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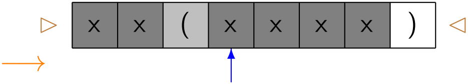

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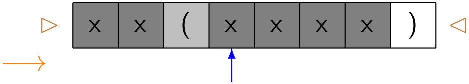

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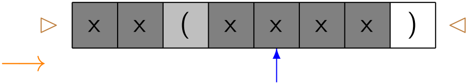

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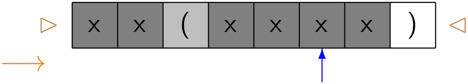

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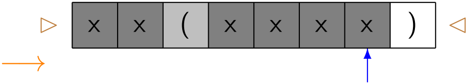

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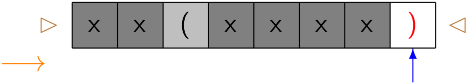

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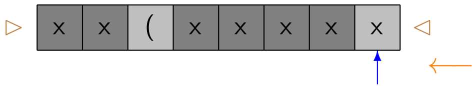

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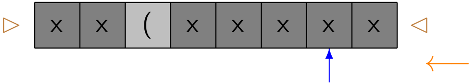

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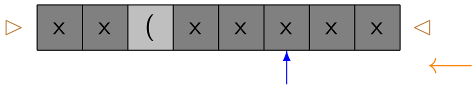

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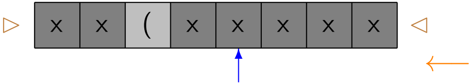

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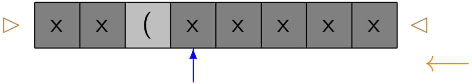

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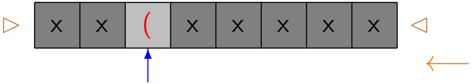

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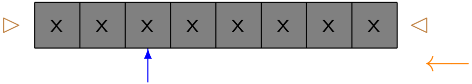

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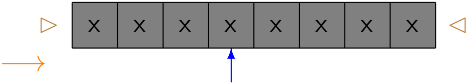

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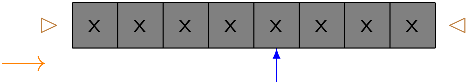

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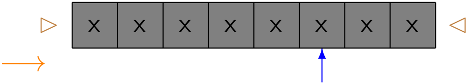

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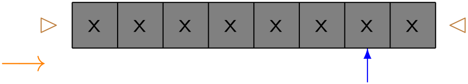

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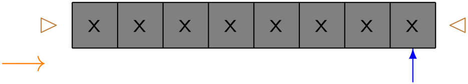

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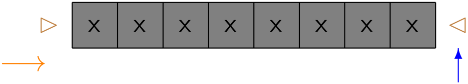

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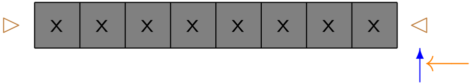

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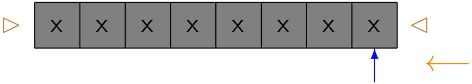

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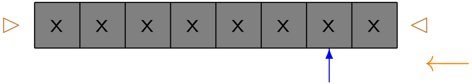

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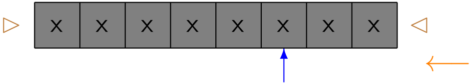

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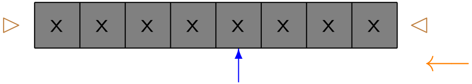

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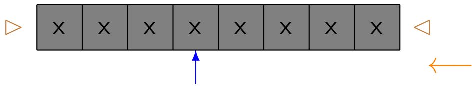

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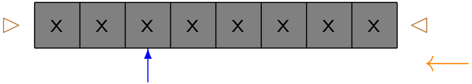

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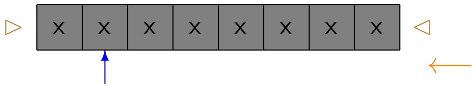

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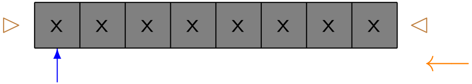

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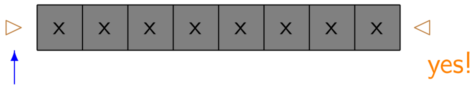

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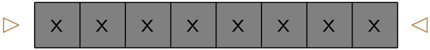

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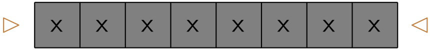

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Each cell is rewritten only in the first 2 visits!

Limited Automata [Hibbard '67]

One-tape Turing machines with restricted rewritings

Definition

Fixed an integer $d \geq 1$, a *d-limited automaton* is

- a one-tape Turing machine
- which is allowed to rewrite the content of each tape cell *only in the first d visits*

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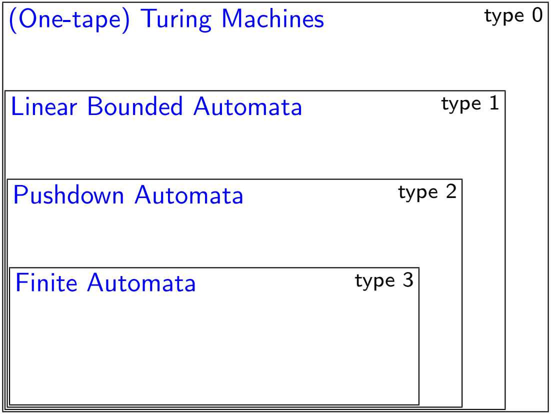
Computational power

- For each $d \geq 2$, d-limited automata characterize context-free languages [Hibbard '67]
- 1-limited automata characterize regular languages [Wagner&Wechsung '86]

The Chomsky Hierarchy



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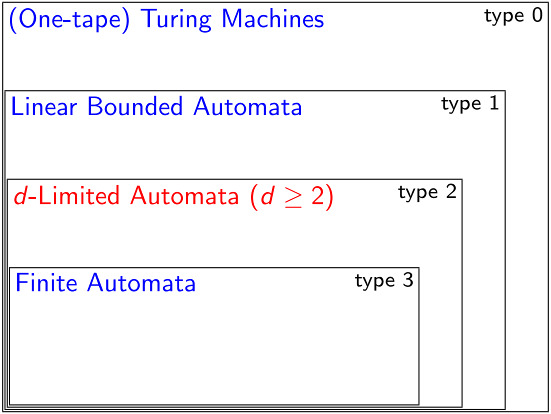

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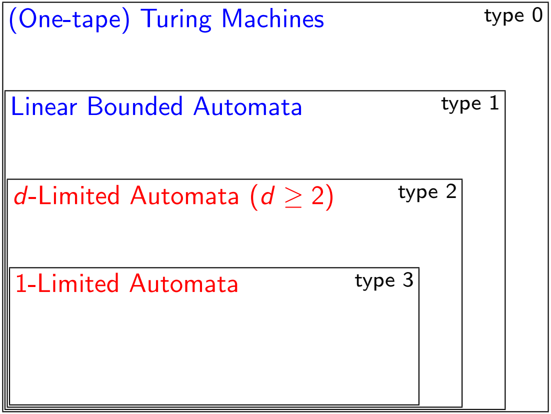

Why Each CFL is Accepted by a 2-LA [P.&Pisoni '14]

Main tool:

Theorem ([Chomsky&Schützenberger '63])

Every context-free language $L \subseteq \Sigma^*$ can be expressed as

$$L = h(D_k \cap R)$$

where, for $\Omega_k = \{(_1,)_1, (_2,)_2, \ldots, (_k,)_k\}$:

- $D_k \subseteq \Omega_k^*$ *is a Dyck language*
- $R \subseteq \Omega_k^*$ *is a regular language*
- $h : \Omega_k \to \Sigma^*$ *is an homomorphism*

Furthermore, it is possible to restrict to *non-erasing* homomorphisms [Okhotin '12]

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L context-free language, with $L = h(D_k \cap R)$

- T nondeterministic transducer computing h^{-1}
- A_D 2-LA accepting the Dyck language D_k
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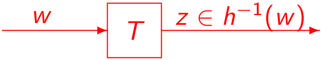

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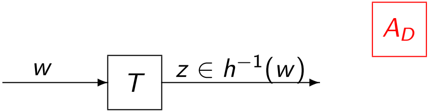

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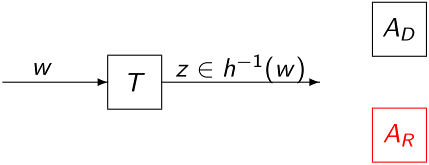

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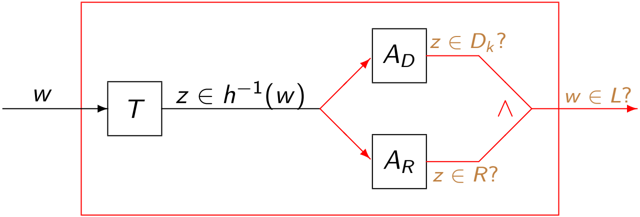

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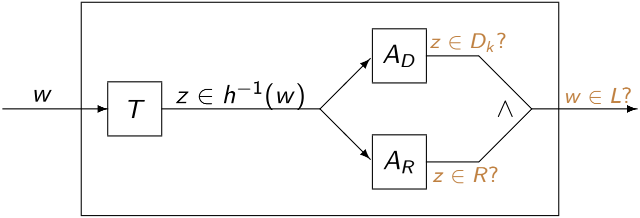

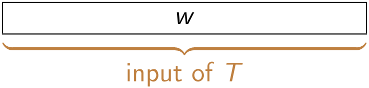

$$z = \sigma_1 \sigma_2 \cdots \sigma_k \in h^{-1}(w)$$

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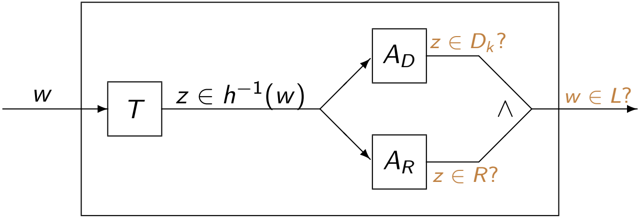

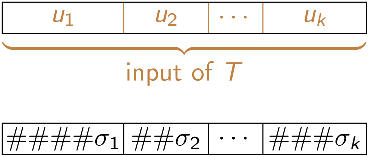

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Non erasing homomorphism!

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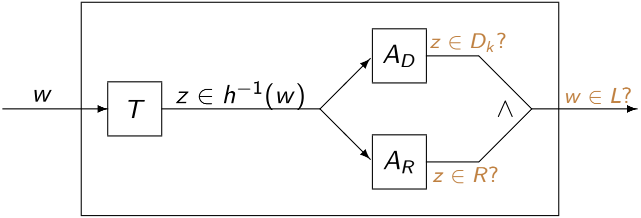

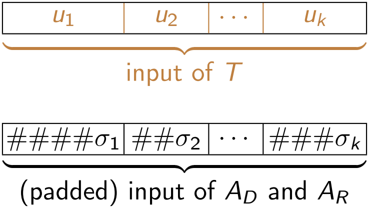

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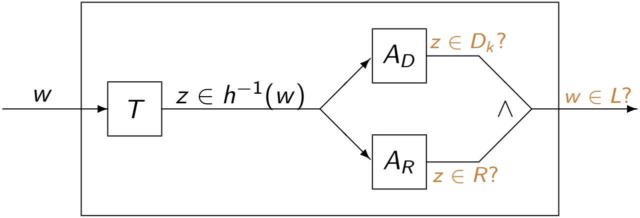

u_1	u_2	$\cdots$	u_k

input of T

$$z = \sigma_1\sigma_2\cdots\sigma_k \in h^{-1}(w)$$

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$####\sigma_1$	$##\sigma_2$	$\cdots$	$###\sigma_k$

(padded) input of A_D and A_R

Non erasing homomorphism!

Not stored into the tape!

Each σ_i is produced "on the fly"

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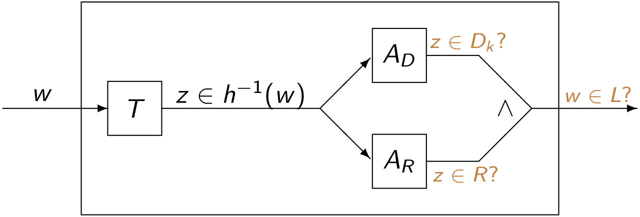

w

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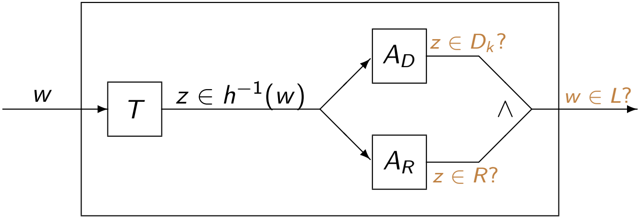

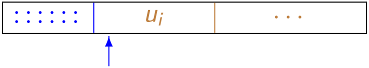

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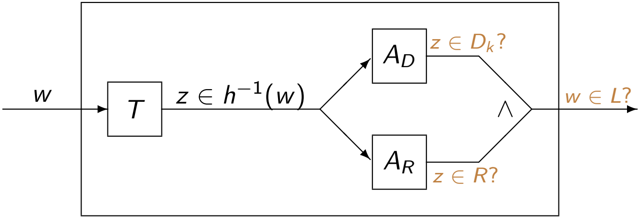

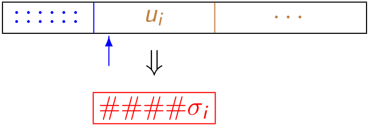

$$w = \cdots u_i \cdots$$

$$\Downarrow$$

$$h(\sigma_i) = u_i$$

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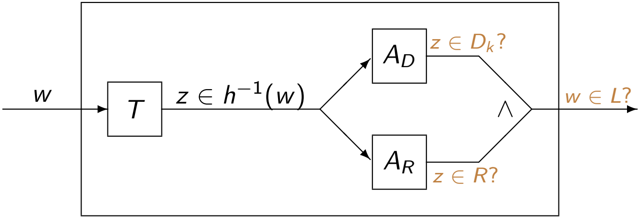

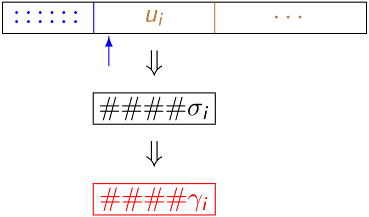

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γ_i: first rewriting by A_D

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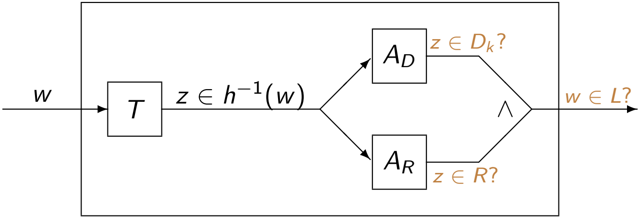

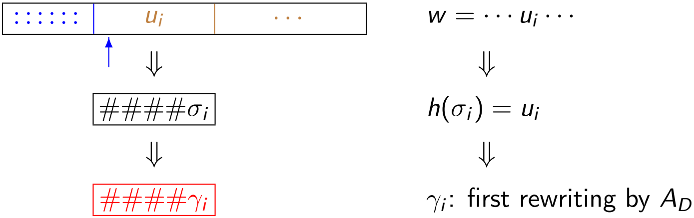

- On the tape, u_i is replaced directly by $\#\#\#\#\gamma_i$
- One move of A_R on input σ_i is also simulated

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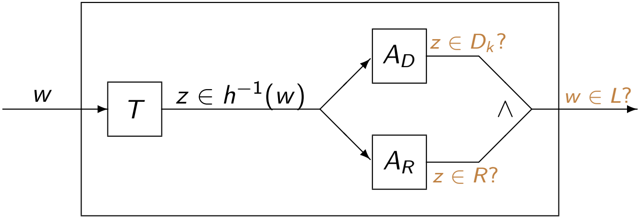

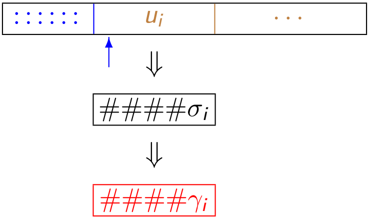

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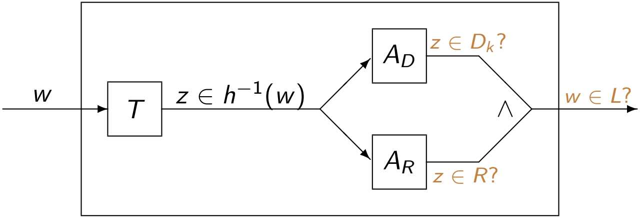

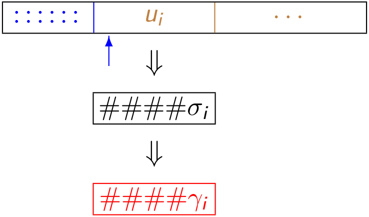

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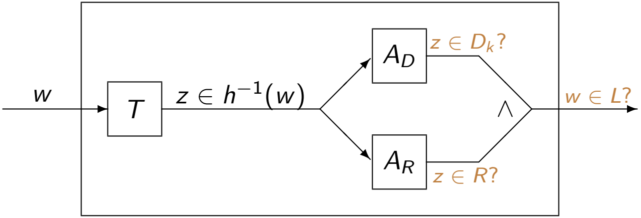
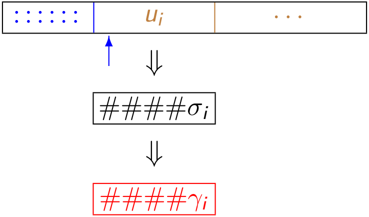
The resulting machine is a 2-LA recognizing the given CFL

Why Each CFL is Accepted by a 2-LA





$w = \cdots u_i \cdots$

$\Downarrow$

$h(\sigma_i) = u_i$

$\Downarrow$

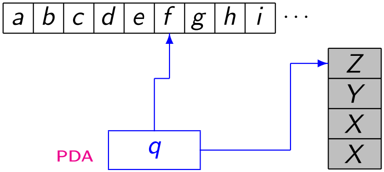
γ_i: first rewriting by A_D

- On the tape, u_i is replaced directly by $\#\#\#\#\gamma_i$
- One move of A_R on input σ_i is also simulated

The resulting machine is a 2-LA recognizing the given CFL

PDAs vs Limited Automata

Simulation of Pushdown Automata by 2-Limited Automata



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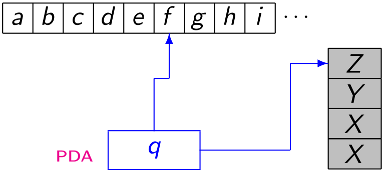

Normal form for (D)PDAs:

- at each step, the stack height increases at most by 1
- ϵ-moves cannot push on the stack

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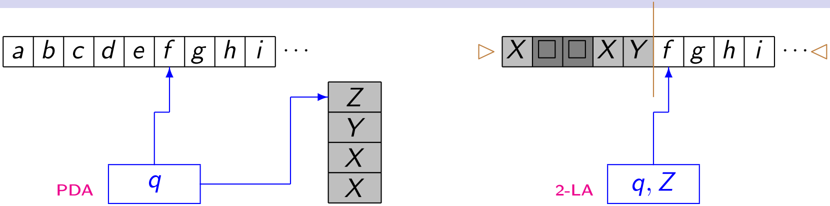

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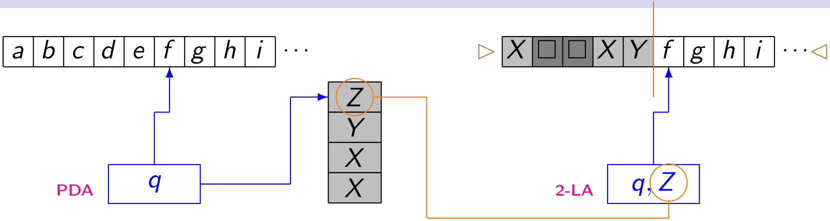

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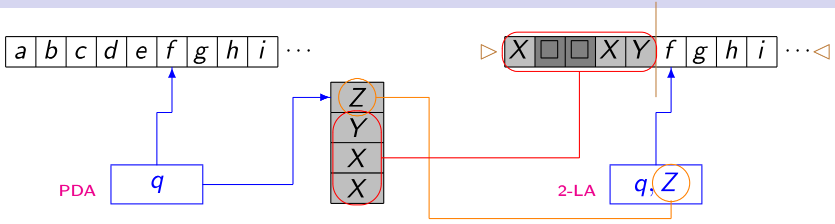

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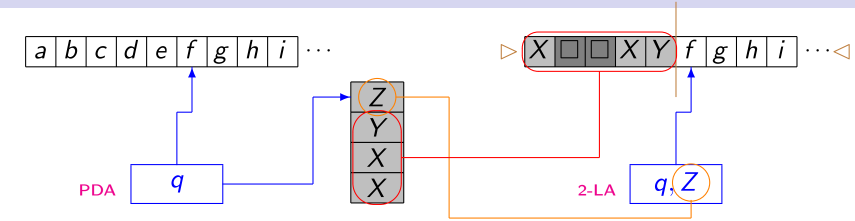

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- Polynomial size
- Determinism is preserved

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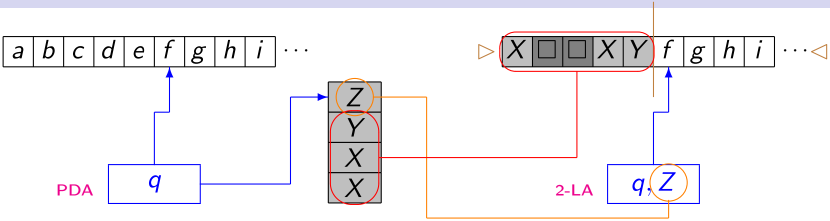

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Simulation of 2-Limited Automata by Pushdown Automata

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What about the converse simulation, namely that of 2-LAs *by PDAs?*

[Hibbard '67]

Original simulation

[P.&Pisoni '15]

Reformulation

- Exponential cost
- Determinism is preserved (extra costs)

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Transition Tables of 2-LAs

- Fixed a 2-limited automaton
- *Transition table* τ_w $\quad$ w is a "frozen" string

$$\tau_w \subseteq Q \times \{-1, +1\} \times Q \times \{-1, +1\}$$

$(q, d', p, d'') \in \tau_w$ iff M on a tape segment containing w has a computation path:

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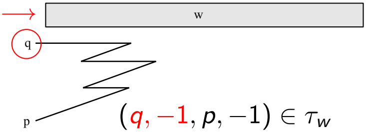

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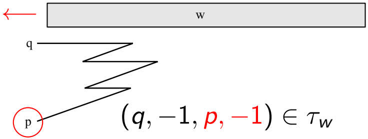

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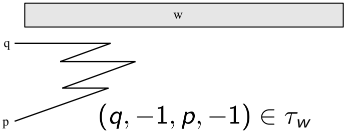

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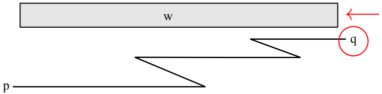

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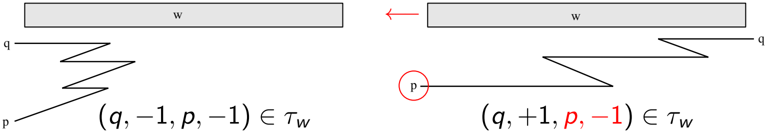

$(q,-1,p,-1) \in \tau_w$ $\qquad$ $(q,+1,p,-1) \in \tau_w$

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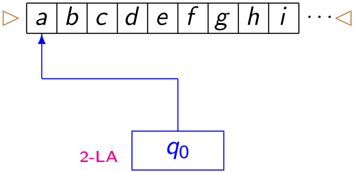
Simulation of 2-LAs by PDAs

Initial configuration



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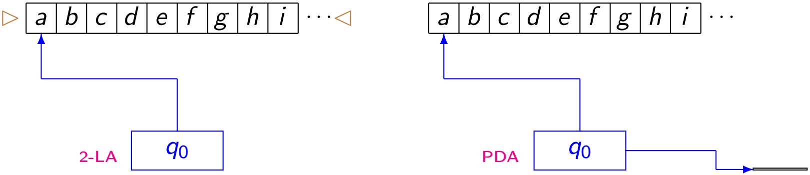

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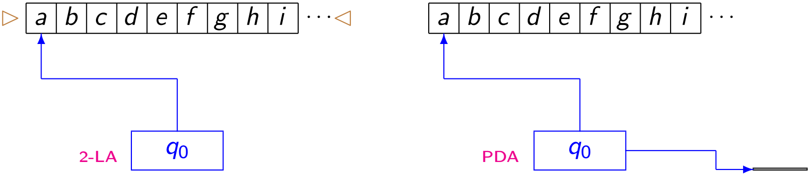

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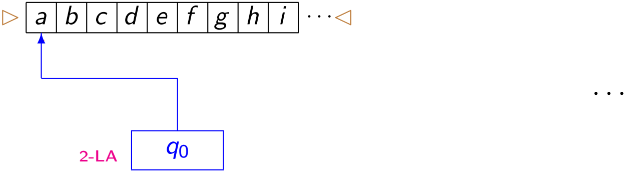

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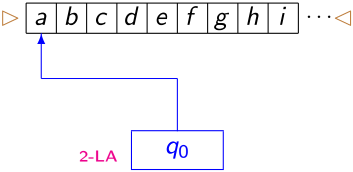

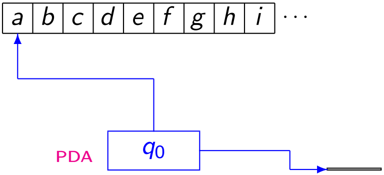

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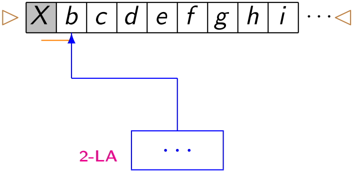

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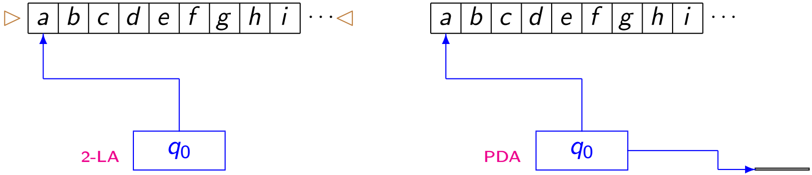

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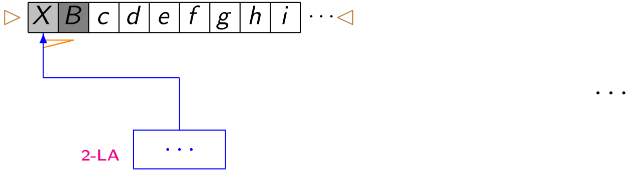

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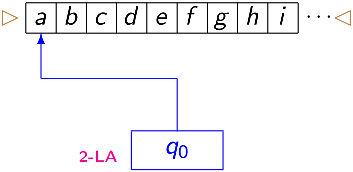

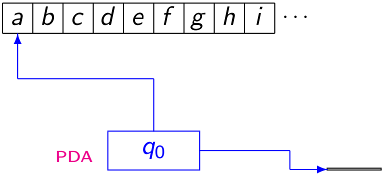

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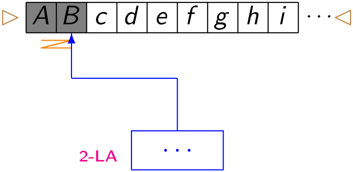

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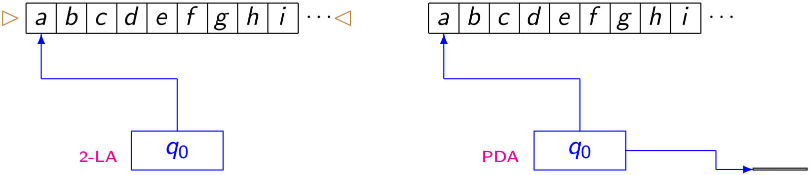

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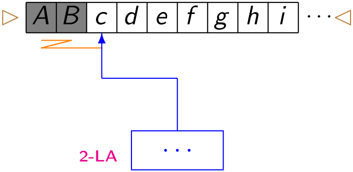

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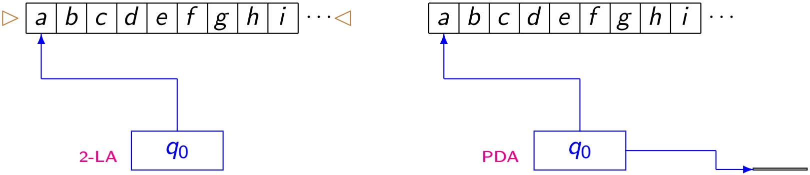

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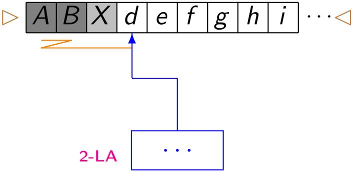

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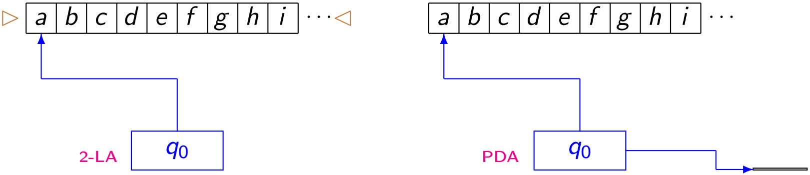

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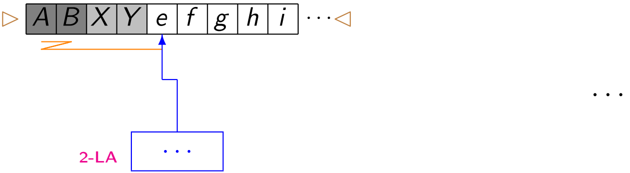

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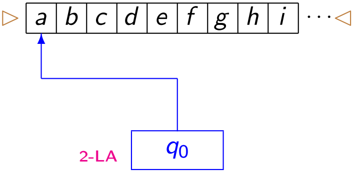

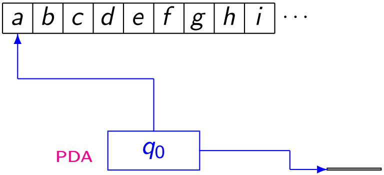

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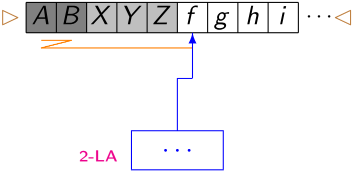

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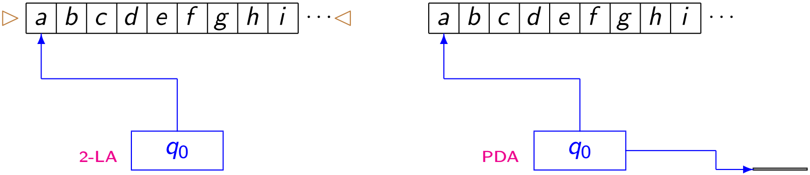

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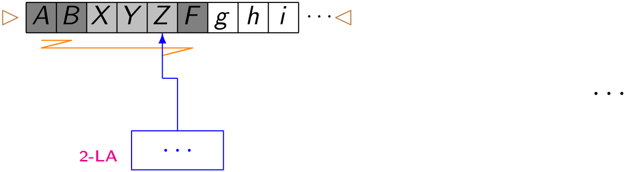

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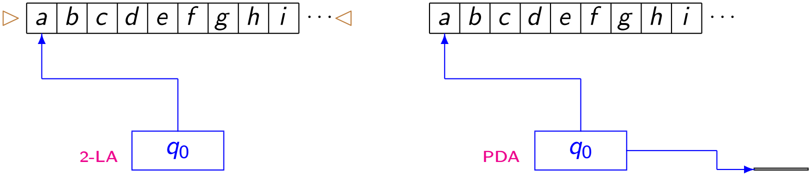

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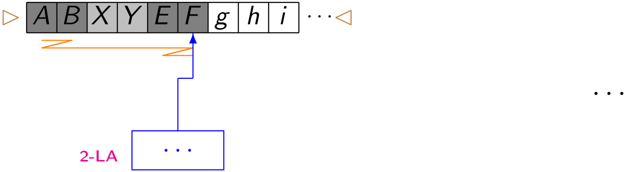

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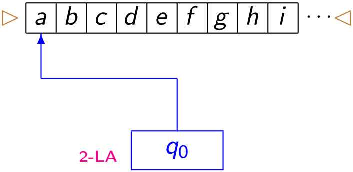

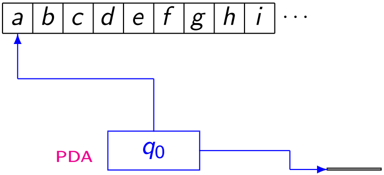

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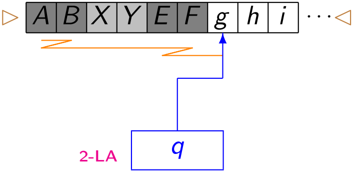

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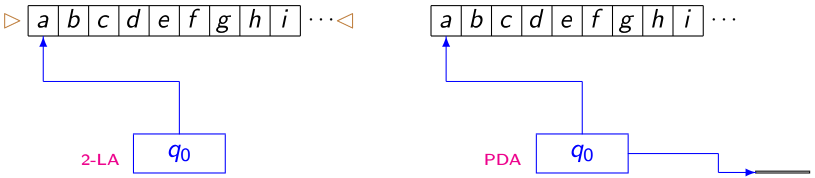

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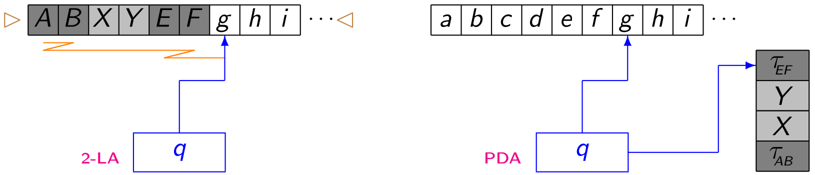

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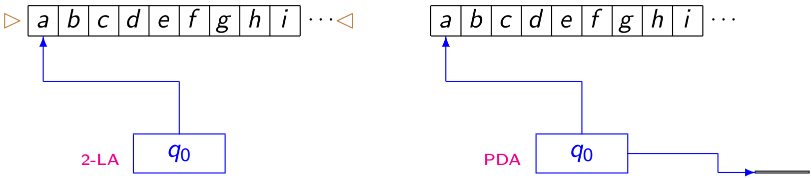

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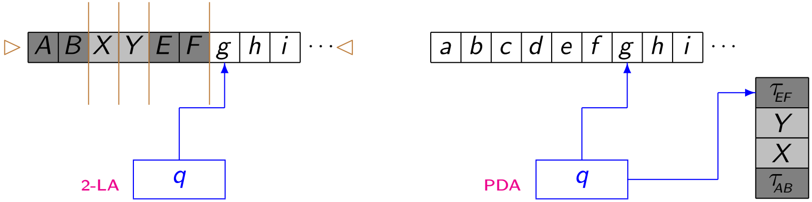

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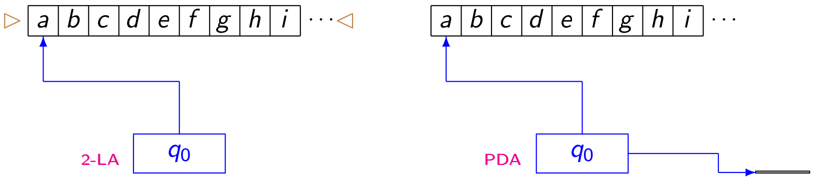

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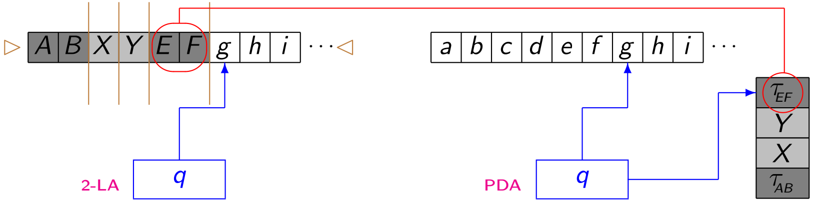

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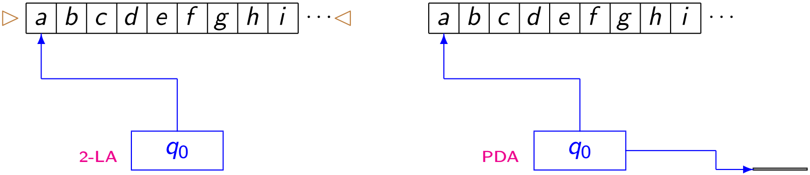

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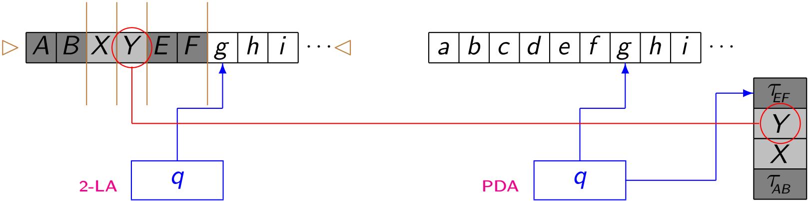

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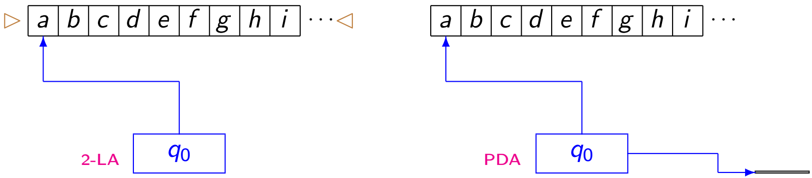

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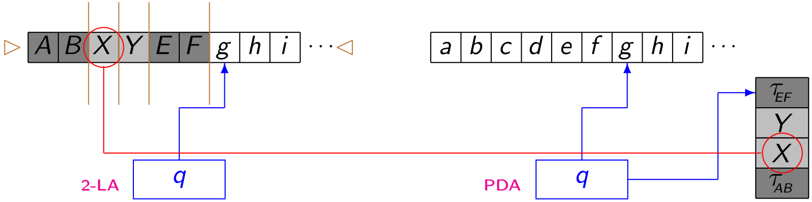

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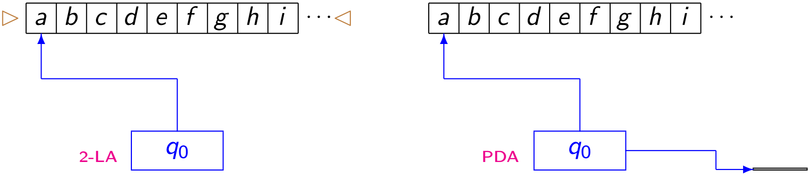

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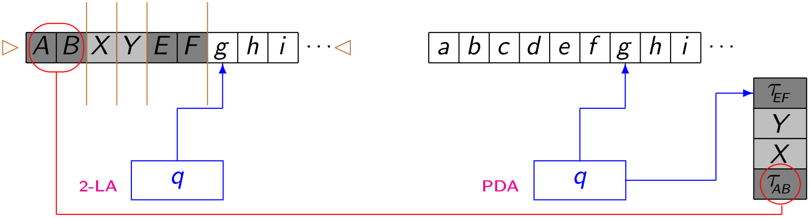

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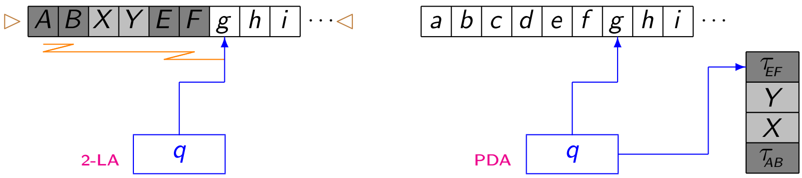

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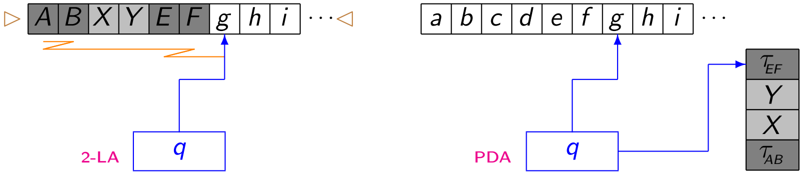

$\delta(q, g) \ni (p, Z, +1)$

move to the right

$\Downarrow$

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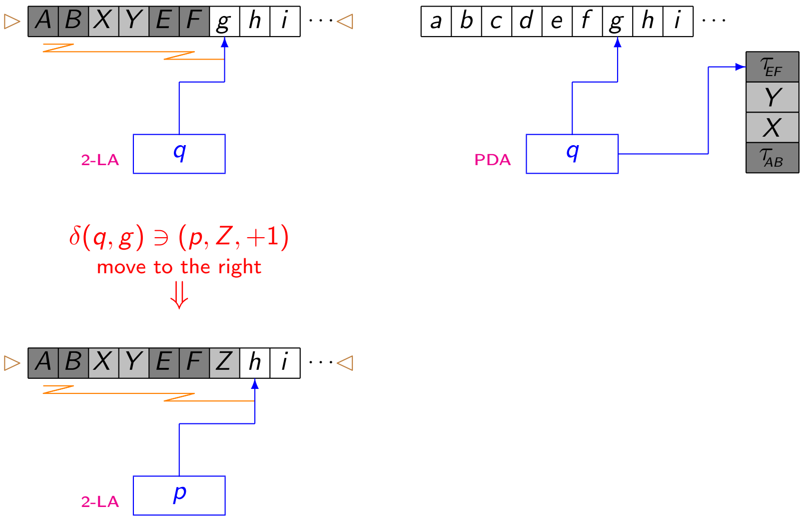

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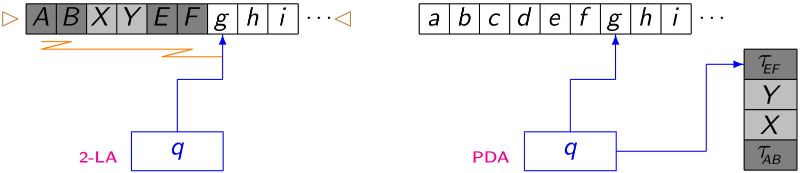

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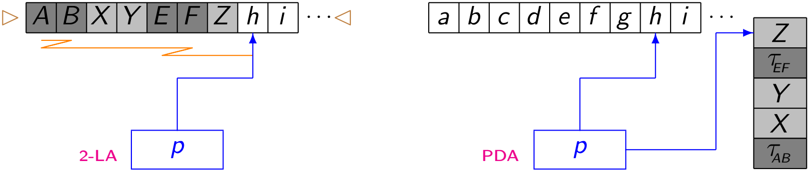
move to the right

$\Downarrow$

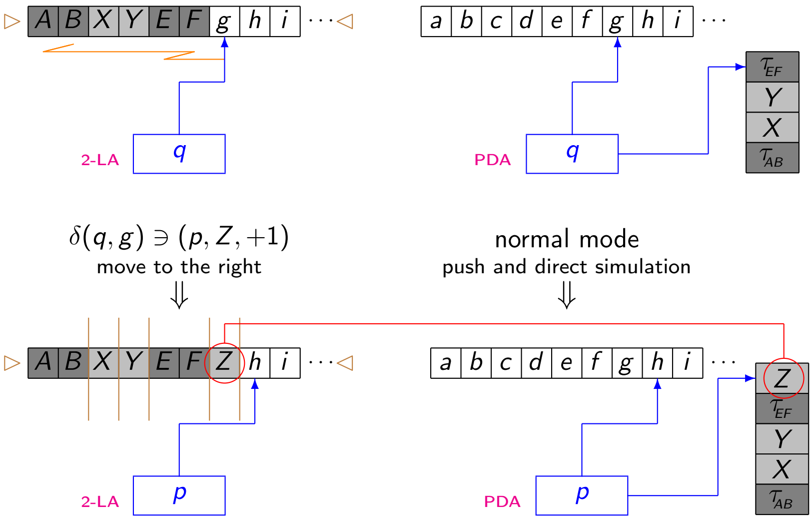
normal mode

push and direct simulation

$\Downarrow$



Simulation of 2-LAs by PDAs



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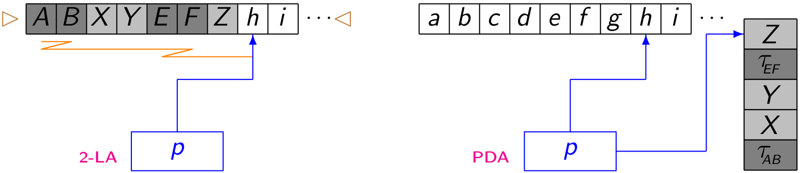

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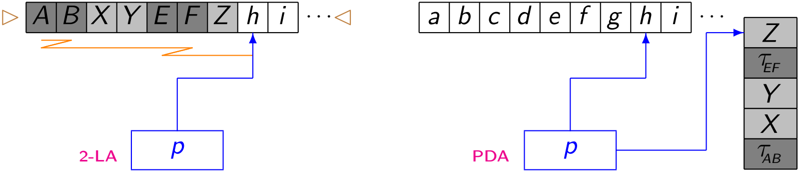

$\delta(p, h) \ni (r, H, -1)$

move to the left

$\Downarrow$

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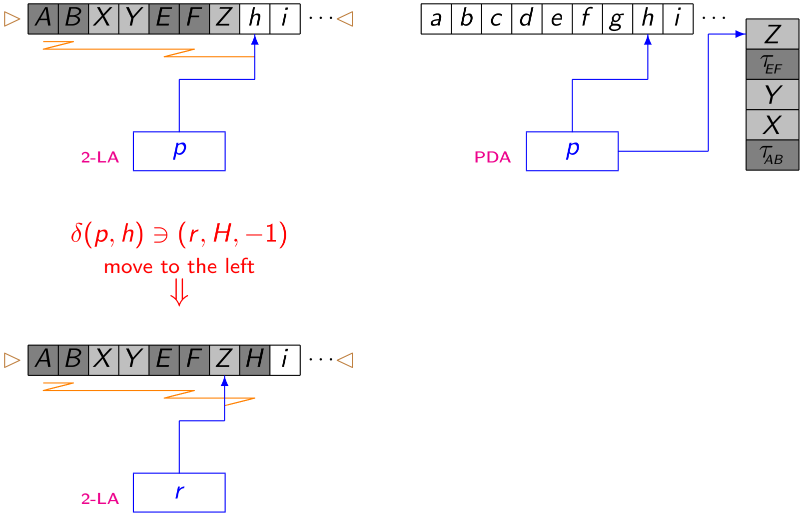

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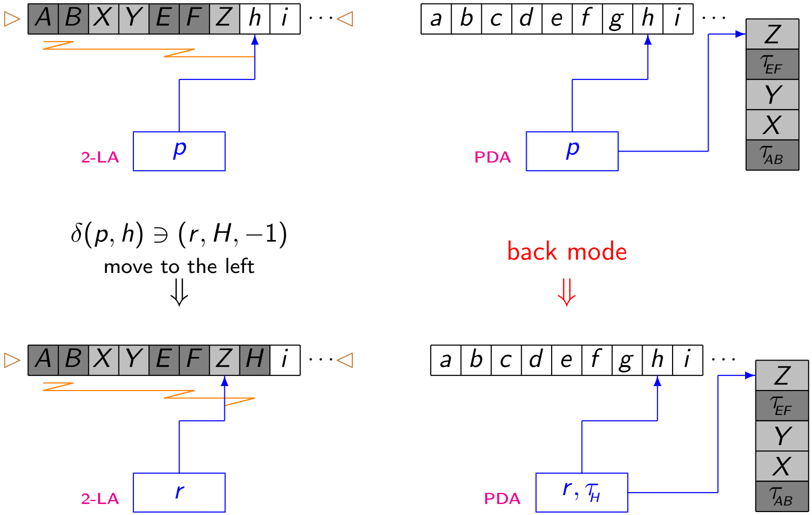

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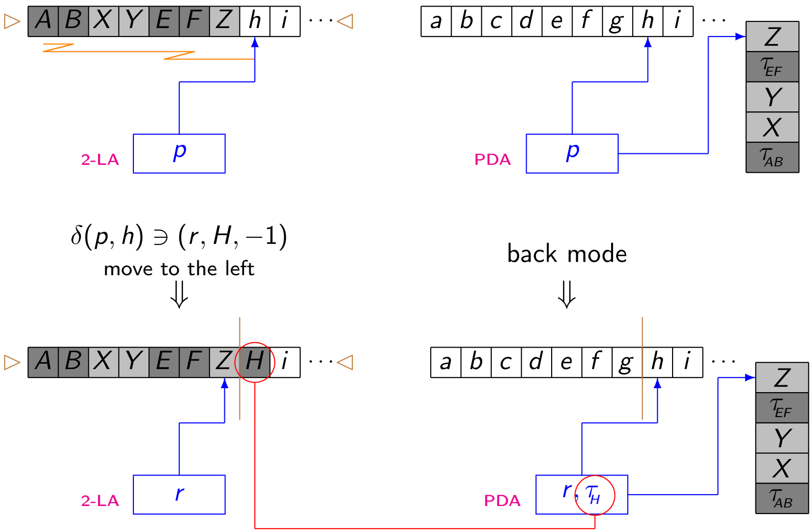

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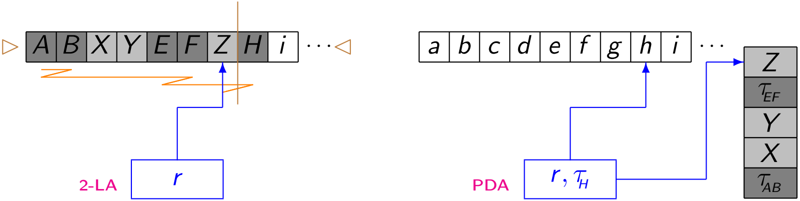

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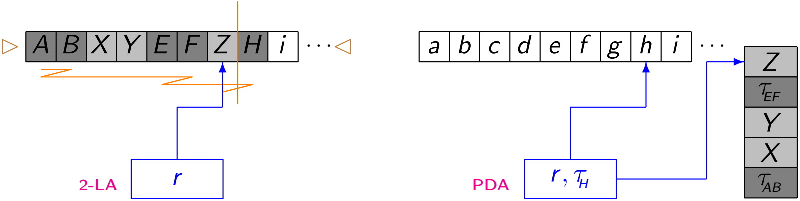

$\delta(r, Z) \ni (q, G, -1)$

move to the left

$\Downarrow$

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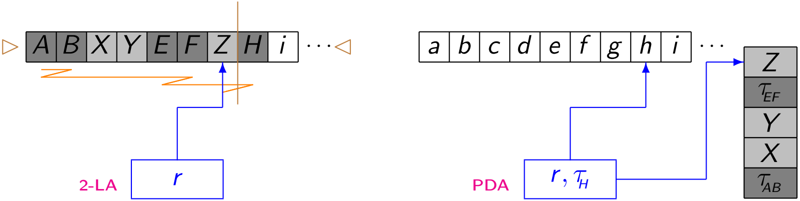

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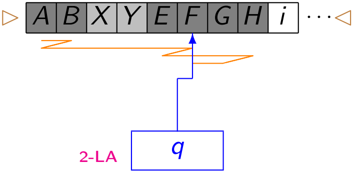

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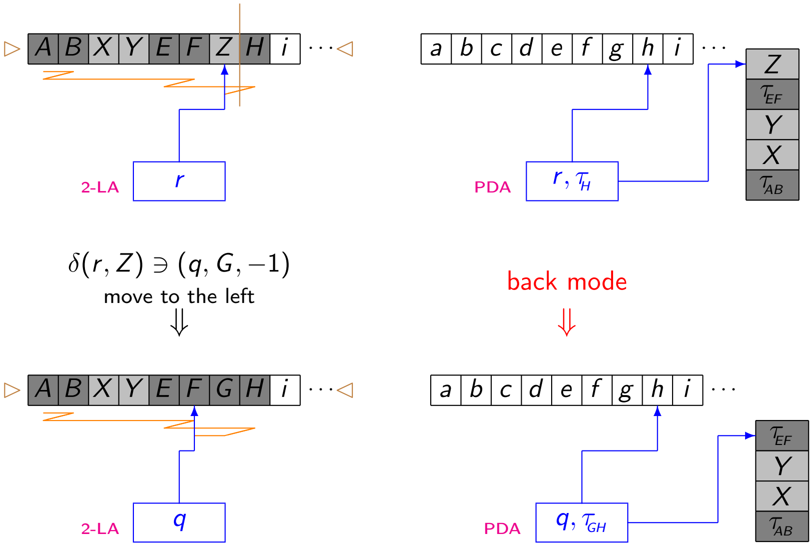

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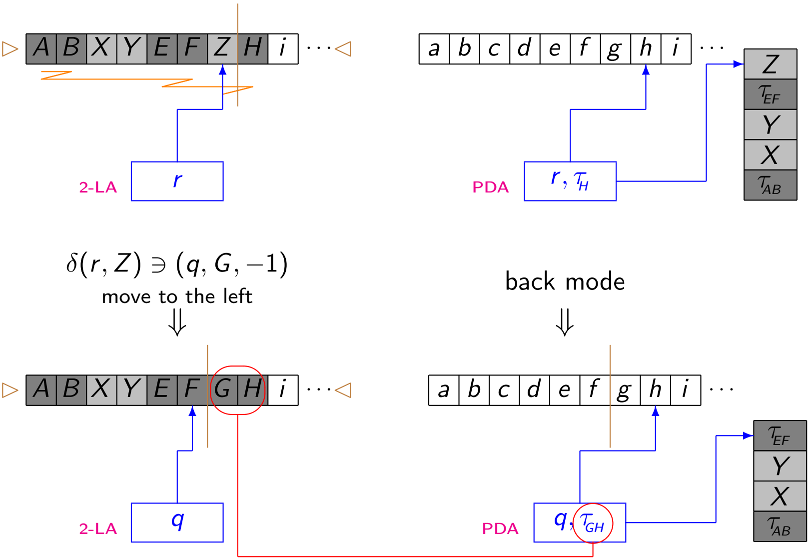

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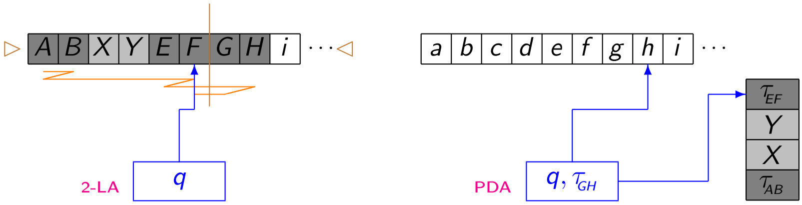

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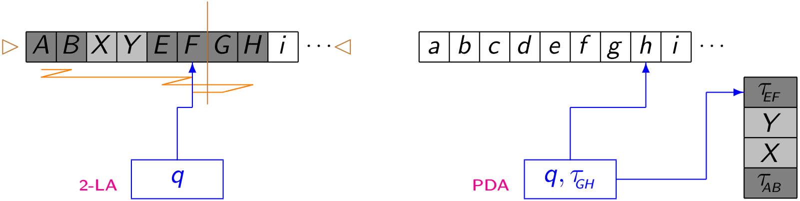

$(q, +1, s, -1) \in \mathcal{T}_{EF}$

exit to the left

$\Downarrow$

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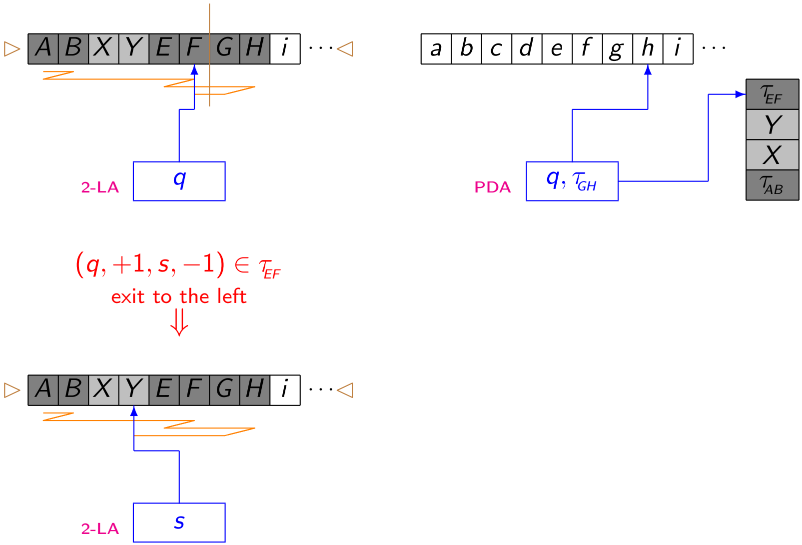

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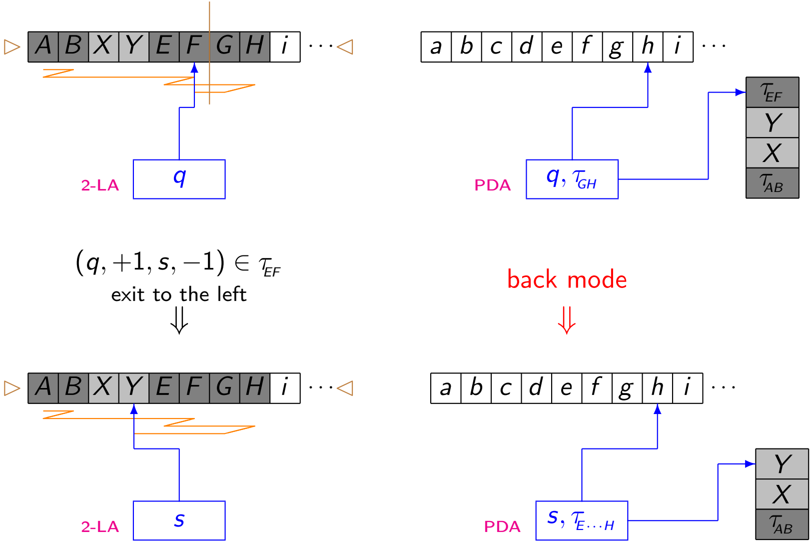

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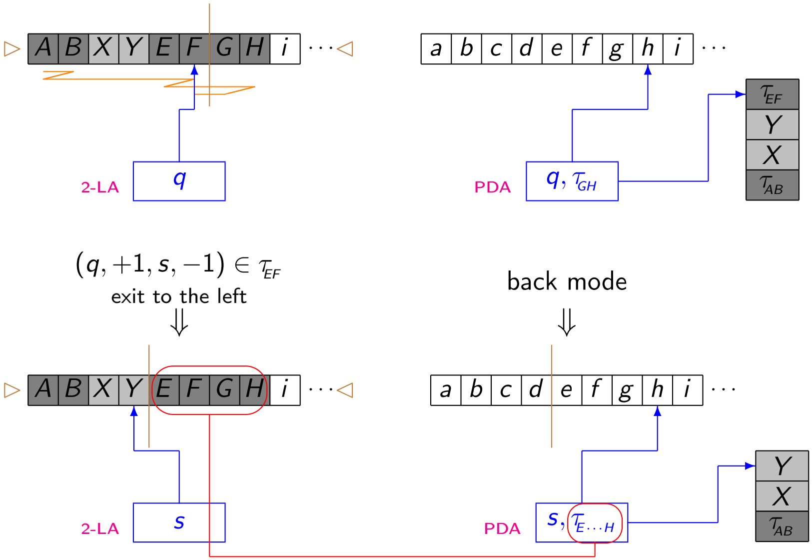

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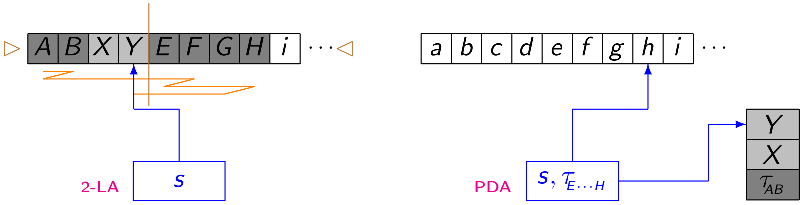

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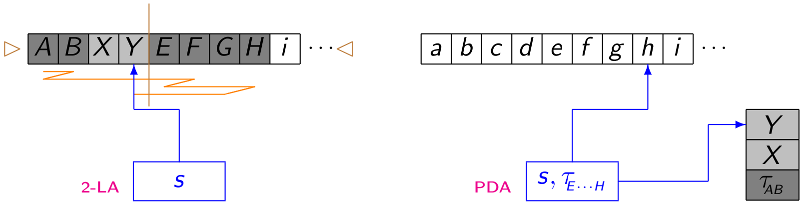

$\delta(s, Y) \ni (p, D, +1)$

move to the right

$\Downarrow$

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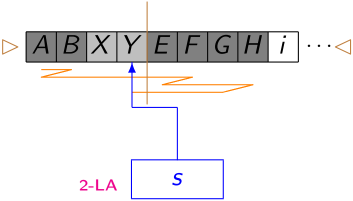

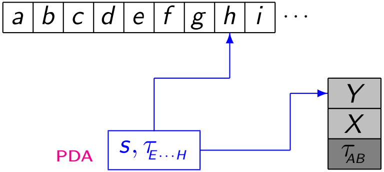

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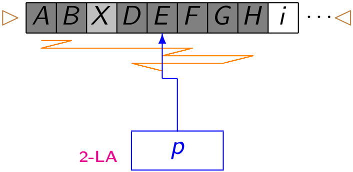

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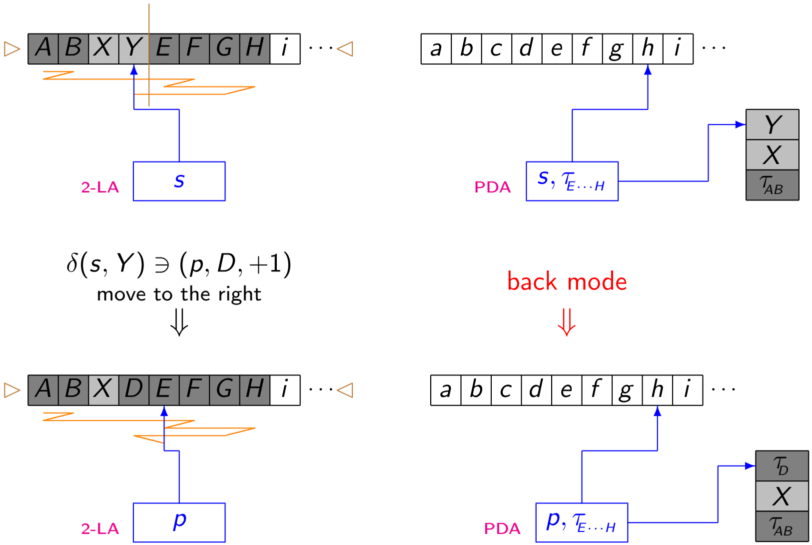

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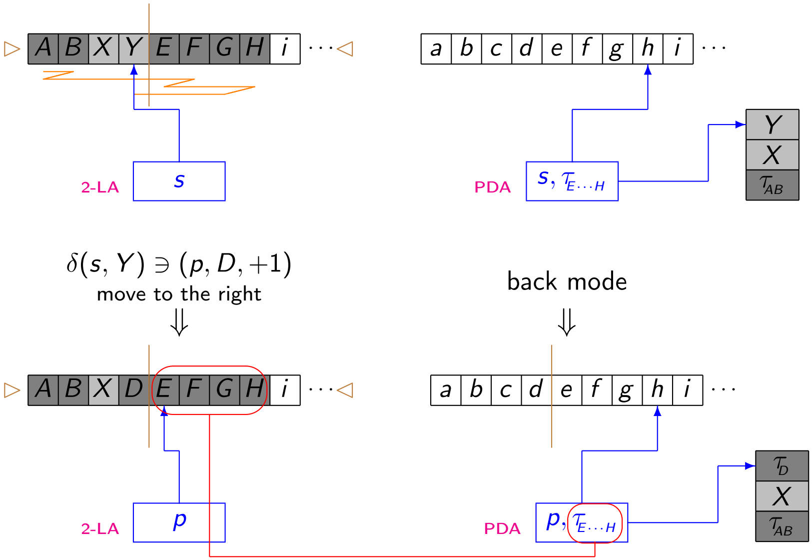

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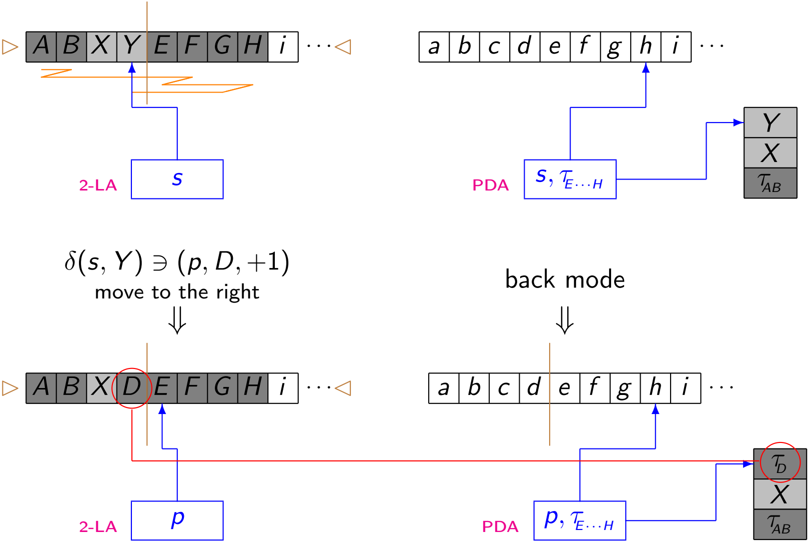

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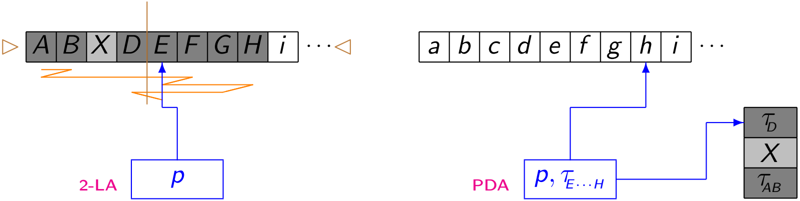

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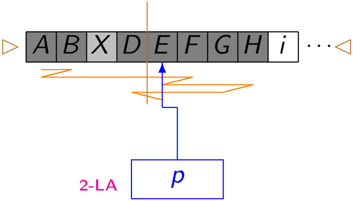

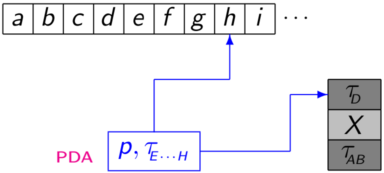

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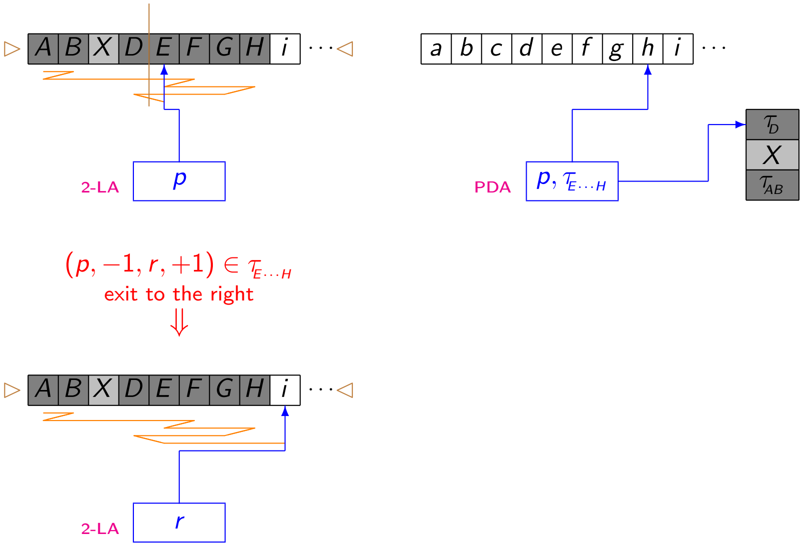

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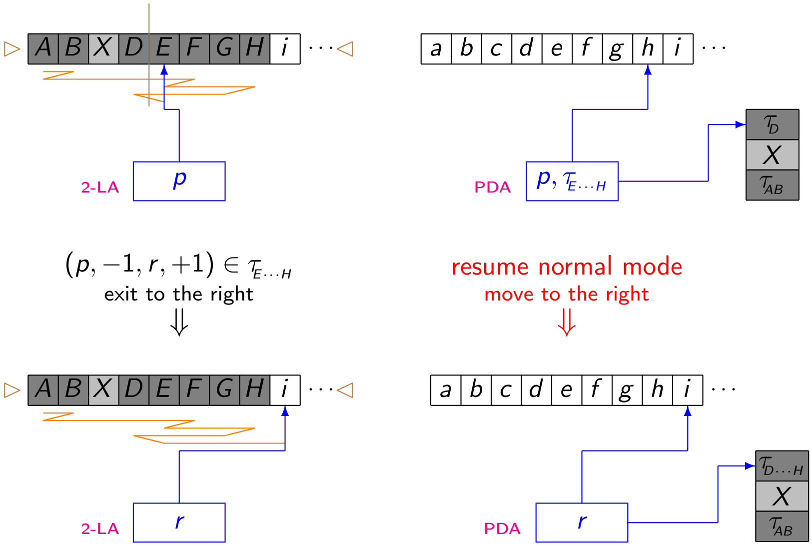

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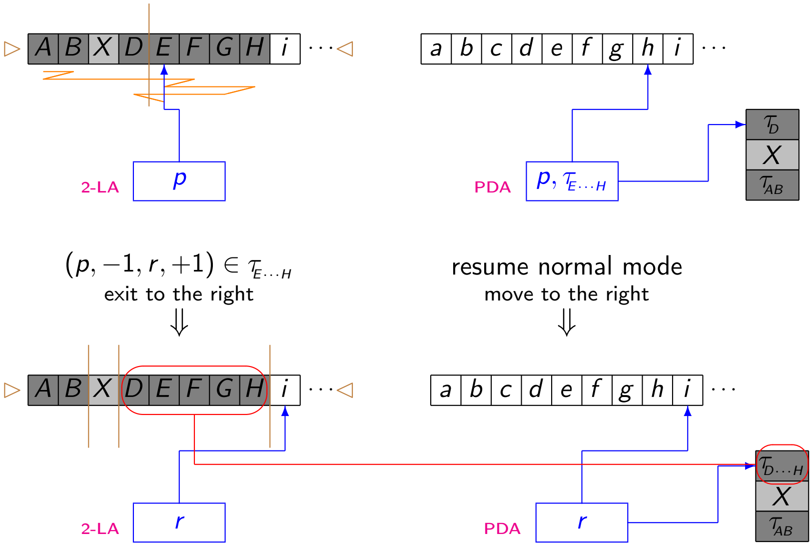

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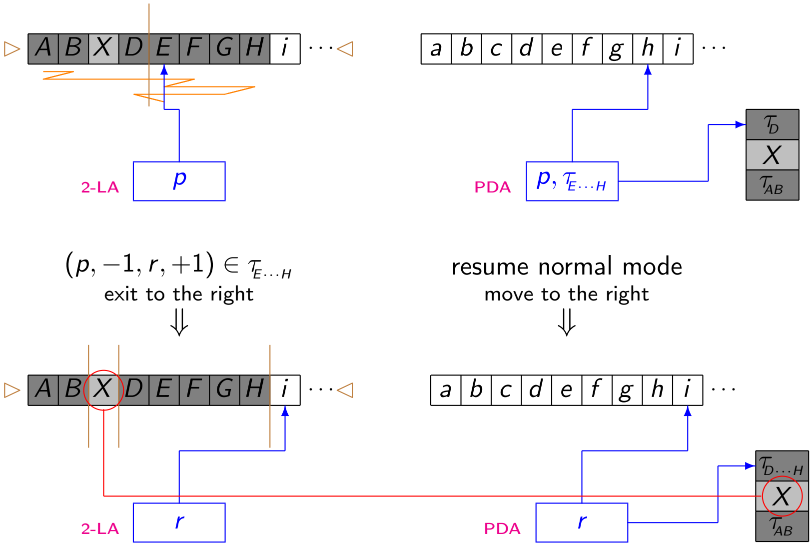

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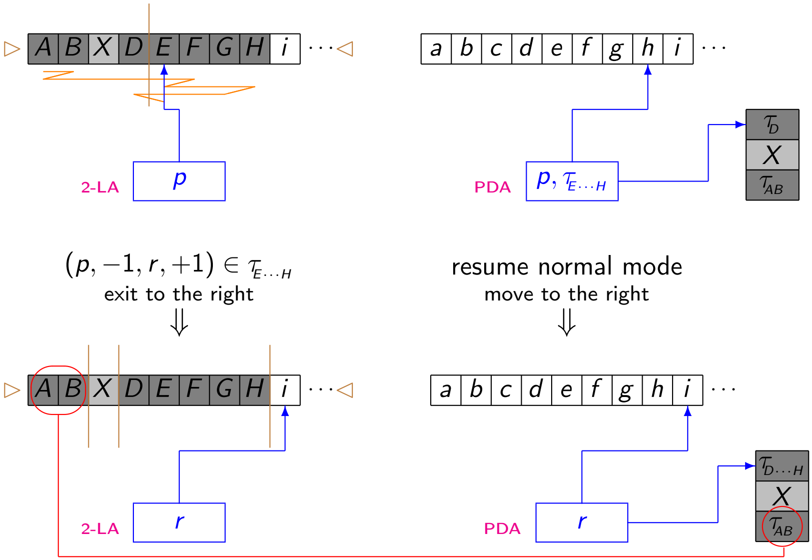

Simulation of 2-LAs by PDAs

Summing up...

Given a 2-LA M with:

- n states
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Resulting PDA:

- States
  Normal mode: states of M
  Back mode: (q, τ)
  q state of M, τ transition table

States
$2n(2^{4n^2}+1)+1$

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2-LAs → PDAs

Exponential cost

Optimality: the Witness Languages K_n

Given $n \geq 1$:

$$K_n \quad =$$

Optimality: the Witness Languages K_n

Given $n \geq 1$:

$$\underbrace{a_1 \quad a_2 \quad \cdots \quad a_n}_{x_1} \quad \cdots \quad \underbrace{a_{n+1}\, a_{n+2} \quad \cdots \quad a_{2n}}_{x_k} \quad \underbrace{b_1 \quad b_2 \quad \cdots \quad b_n}_{x}$$

$$K_n \quad = \{x_1 x_2 \cdots x_k x \mid \quad k \geq 0,\ x_1, x_2, \ldots, x_k, x \in \{0,1\}^n,$$

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$\cdots$

At least n of these blocks are equal
to the last block x

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Example ($n = 3$): 0 0 1 1 1 0 0 1 1 1 1 0 1 1 0 1 1 1 1 1 0

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How to Recognize K_n

0 0 1 1 1 0 0 1 1 1 1 0 1 1 0 1 1 1 1 1 0 $(n = 3)$

1. Scan all the tape from left to right
2. Start to move to the left and mark the rightmost n symbols
3. Compare each block of length n (from the right), symbol by symbol, with the last block
4. When the left end of the tape is reached accept if and only if the number of block equal to the last one is $\geq n$

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Complexity:

- K_n is accepted by a deterministic 2-LA with $O(n^2)$ states and a fixed working alphabet
- Each PDA accepting K_n has size at least exponential in n (Proof based on the *interchange lemma* for CFLs)

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Simulation of 2-LAs by PDAs

Cost of the simulation

- Exponential size for the simulation of 2-LAs by PDAs
- Optimal

Computational Power of Limited Automata

From the simulations:

- 2-Limited Automata $\equiv$ CFLs

What about d-Limited Automata, with $d > 2$?

- They are still characterize CFLs [Hibbard '67]
- They can be simulated by exponentially larger PDAs [Kutrib&P.&Wendlandt subm.]

What about 1-Limited Automata?

- Regular languages [Wagner&Wechsung '86]

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Deterministic 2-Limited Automata ≡ DCFLs

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- Determinism is preserved by the exponential simulation of 2-limited automata by PDAs
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Determinism vs Nondeterminism

What about *deterministic* d-Limited Automata, $d > 2$?

- $L = \{a^n b^n c \mid n \geq 0\} \cup \{a^n b^{2n} d \mid n \geq 0\}$
  is accepted by a *deterministic* 3-LA, but is not a DCFL
- Infinite hierarchy [Hibbard '67]

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1-Limited Automata

Simulation of 1-Limited Automata by Finite Automata

Main idea: transformation of *two-way* NFAs into *one-way* DFAs
[Shepherdson '59]

- First visit to a cell: direct simulation
- Further visits: *transition tables*
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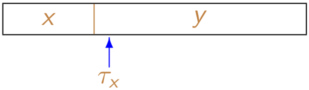

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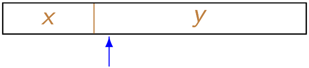
Simulation of 1-LAs: [Wagner&Wechsung '86]



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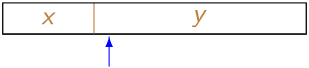

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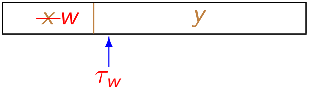

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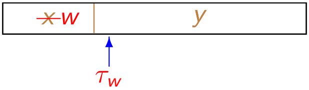

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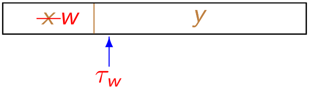

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The simulating DFA keeps in its finite control a *sets of transition tables*

1-Limited Automata $\rightarrow$ Finite Automata: Upper Bounds

Theorem

Let M be a 1-LA *with n states.*

- *There exists an equivalent* DFA *with $2^{n \cdot 2^{n^2}}$ states.*
- *There exists an equivalent* NFA *with $n \cdot 2^{n^2}$ states.*

If M is deterministic then there exists an equivalent DFA *with no more than $n \cdot (n+1)^n$ states.*

	DFA	NFA
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These upper bounds do not depend on the alphabet size of M!

The gaps are optimal!

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Optimality: the Witness Languages [P.&Pisoni '14]

Fixed $n \geq 1$:

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How to Recognize L_n: 1-Limited Automata

- Nondeterministic strategy:
  Guess the leftmost positions of n input blocks containing the same factor and *Verify*

- Implementation (3 tape scans):
  1. Mark n tape cells
  2. Count the tape modulo n to check whether or not:
     - the input length is a multiple of n, and
     - the marked cells correspond to the leftmost symbols of some blocks of length n
  3. Compare, symbol by symbol, each two consecutive blocks of length n that start from the marked positions

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- Idea:
  - For each $x \in \{0,1\}^n$ count how many blocks coincide with x
  - Accept if and only if one of the counters reaches the value n
- State upper bound:
  - Finite control:
    a counter (up to n) for each possible block of length n
  - There are 2^n possible different blocks of length n
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    more precisely $(2^n - 1) \cdot n^{2^n} + n$
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The state gap between 1-LAs and DFAs is double exponential!

How to Recognize L_n: Nondeterministic Finite Automata

- Idea:
  - *Guess* $x \in \{0,1\}^n$
  - *Verify* whether or not n blocks in the input contains x
- State upper bound:
  - Finite control: a counter $\leq n$ for the occurrences of x, and a counter modulo n for input positions
  - Number of states: $O(n^2 \cdot 2^n)$
- State lower bound:
  - $n^2 \cdot 2^n$ (fooling set technique)

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  - Finite control: a counter $\leq n$ for the occurrences of x, and a counter modulo n for input positions
  - Number of states: $O(n^2 \cdot 2^n)$
- State lower bound:
  - $n^2 \cdot 2^n$ (fooling set technique)

How to Recognize L_n: Nondeterministic Finite Automata

- Idea:
  - *Guess* $x \in \{0, 1\}^n$
  - *Verify* whether or not n blocks in the input contains x
- State upper bound:
  - Finite control: a counter $\leq n$ for the occurrences of x, and a counter modulo n for input positions
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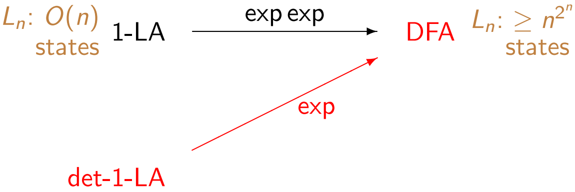
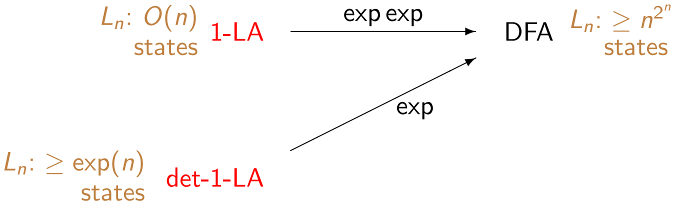
Nondetermism vs. Determinism in 1-LAs

L_n: $O(n)$ states **1-LA** $\xrightarrow{\text{exp exp}}$ **DFA** L_n: $\geq n^{2^n}$ states

Nondetermism vs. Determinism in 1-LAs



Nondetermism vs. Determinism in 1-LAs



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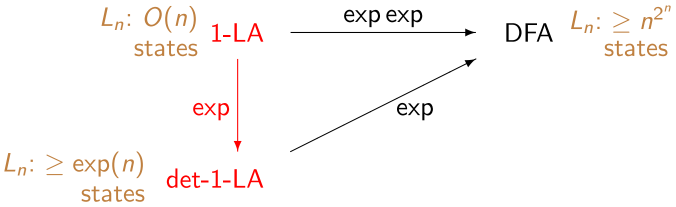
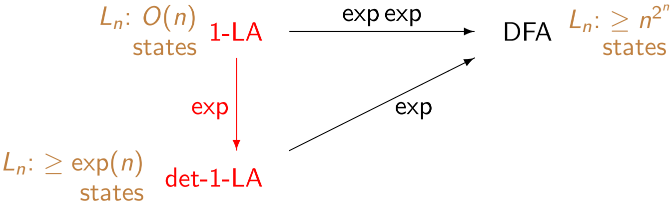


Corollary

Removing nondeterminism from 1-LAs requires exponentially many states

Nondetermism vs. Determinism in 1-LAs



Corollary

Removing nondeterminism from 1-LAs *requires exponentially many states*

Cfr. Sakoda and Sipser question [Sakoda&Sipser '78]:

How much it costs in states to remove nondeterminism from two-way finite automata?

Strongly Limited Automata

Different Restrictions

- Dyck languages are accepted without fully using capabilities of 2-limited automata
- Chomsky-Schützenberger Theorem: Recognition of CFLs can be reduced to recognition of Dyck languages

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Is it possible to restrict 2-limited automata without affecting their computational power?

Different Restrictions

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Question

Is it possible to restrict 2-limited automata without affecting their computational power?

YES!

Forgetting Automata
[Jancar&Mráz&Plátek '96]

- The content of any cell can be erased in the 1st or 2nd visit (using a fixed symbol)
- No other changes of the tape are allowed

Strongly Limited Automata [P.'15]

- Model inspired by the algorithm used by 2-limited automata to recognize Dyck languages
- Restrictions on
  - state changes
  - head reversals
  - rewriting operations

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Dyck Language Recognition

▷ | (|) | (| [| [|] |] |) | ◁

- Moves to the right:
  - to search a closed bracket
- Moves to the left:
  - to search an open bracket
  - to check the tape content in the final scan from right to left
- Rewritings:
  - each closed bracket is rewritten in the first visit
  - each open bracket is rewritten in the second visit
  - no rewritings in the final scan

Dyck Language Recognition

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Only one state q_0!

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Strongly Limited Automata

- Alphabet
  - Σ input
  - Γ working

- States and moves
  - q_0 initial state, moving from left to right
  - Q_L moving from right to left
  - Q_F final scan
    when $\lhd$ is reached move from right to left and
    test the membership of the tape content to a "local" language

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- States and moves

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  when $\lhd$ is reached move from right to left and

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Strongly Limited Automata: Palindromes

$\Sigma = \{a, b\}$, $\Gamma = \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$
q_0
$Q_L = \{q_a, q_b\}$

$\triangleright$	a	b	b	b	a	$\triangleleft$

Strongly Limited Automata: Palindromes

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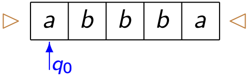

Transitions:

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other possibility in cell not yet rewritten:

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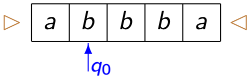

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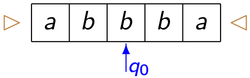

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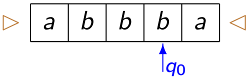

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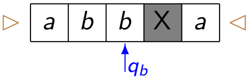

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q_σ moving from right to left

cells already rewritten: $\dashleftarrow$ *move to the left*

cells containing $\gamma \in \{a, b\}$, nondeterministically select between:

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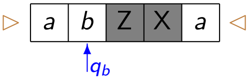

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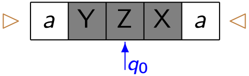

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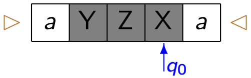

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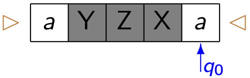

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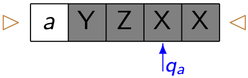

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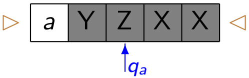

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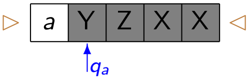

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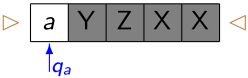

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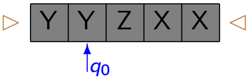

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other possibility in cell not yet rewritten:

$q_\sigma \xleftarrow{X}$ write $X \in \Gamma$, enter state $q_\sigma \in Q_L$, *turn to the left*

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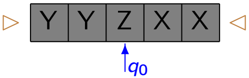

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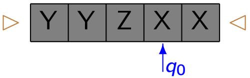

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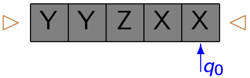

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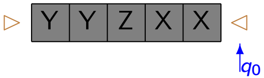

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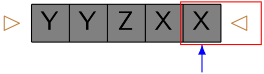

Final phase:

- The string between the end-markers should belong to

$$Y^*ZX^* + Y^*X^*$$

  with the exceptions of inputs of length ≤ 1

- The following two-letter factors are allowed:

  $\triangleright Y \quad YY \quad YZ \quad ZX \quad YX \quad XX \quad X\triangleleft$

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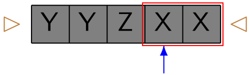

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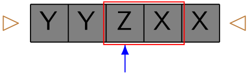

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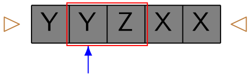

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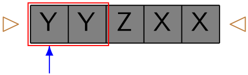

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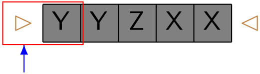

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  - PDAs
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What is the power of *deterministic* strongly limited automata?

- Each deterministic strongly limited automaton can be simulated by a deterministic 2-LA
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Proper subclass of deterministic context-free languages

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Final Remarks

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[Wechsung&Brandstädt '79]

Thank you for your attention!