Parikh Equivalence and Descriptional Complexity

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NFAs vs DFAs

Subset construction: [Rabin&Scott '59]

NFA n states	$\Longrightarrow$	DFA 2^n states

The state bound cannot be reduced

[Lupanov '63, Meyer&Fischer '71, Moore '71]

What happens if we do not care of the order of symbols in the strings?

This problem is related to the concept of *Parikh Equivalence*

Parikh Equivalence

- $\Sigma = \{a_1, \ldots, a_m\}$ alphabet of m symbols
- *Parikh's map* $\psi : \Sigma^* \to \mathbb{N}^m$:

$$\psi(w) = (|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_m})$$

  for each string $w \in \Sigma^*$
- *Parikh's image* of a language $L \subseteq \Sigma^*$:

$$\psi(L) = \{\psi(w) \mid w \in L\}$$

- $w' =_\pi w''$ iff $\psi(w') = \psi(w'')$
- $L' =_\pi L''$ iff $\psi(L') = \psi(L'')$

Parikh's Theorem

Theorem ([Parikh '66])

The Parikh image of a context-free language is a semilinear set, i.e, each context-free language is Parikh equivalent to a regular language

Example:

- $L = \{a^n b^n \mid n \geq 0\}$
- $R = (ab)^*$

$$\psi(L) = \psi(R) = \{(n, n) \mid n \geq 0\}$$

Different proofs after the original one of Parikh, e.g.

- [Goldstine '77]: a simplified proof
- [Aceto&Ésik&Ingólfsdóttir '02]: an equational proof
- ...
- [Esparza&Ganty&Kiefer&Luttenberger '11]: complexity aspects

Our Goal

We want to convert nondeterministic automata and context-free grammars into *small Parikh equivalent* deterministic automata

Problem (NFAs to DFAs)

NFA *n states*	$\Longrightarrow_\pi$	*DFA* *how many states?*

Problem (CFGs to DFAs)

CFG *size n*	$\Longrightarrow_\pi$	*DFA* *how many states?*

Why?

- Interesting theoretical properties:
  wrt Parikh equivalence regular and context-free languages are indistinguishable [Parikh '66]
- Connections of with:
  - Semilinear sets
  - Presburger Arithmetics [Ginsburg&Spanier '66]
  - Petri Nets [Esparza '97]
  - Logical formulas [Verma&Seidl&Schwentick '05]
  - Formal verification
    [Dang&Ibarra&Bultan&Kemmerer&Su'00, Göller&Mayr&To'09]
  - ...
- Unary case:
  size costs of the simulations of CFGs and PDAs by DFAs
  [Pighizzini&Shallit&Wang '02]

Converting NFAs

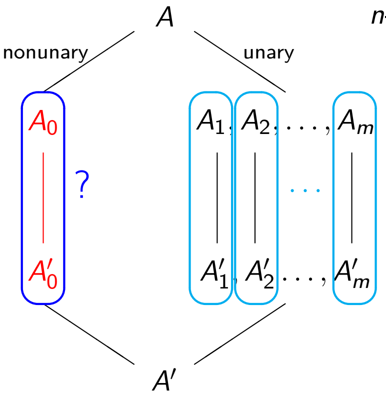
Problem (NFAs to DFAs)

NFA		*DFA*
n states	$\Longrightarrow_\pi$	*how many states?*

- *Upper bound*
  Subset construction: 2^n
- *Lower bound*
  Conversion *NFAs* $\rightarrow$ *DFAs* in the unary case: $e^{\Theta(\sqrt{n \ln n})}$

[Chrobak '86]

Converting NFAs: General Idea



n-state NFA over $\Sigma = \{a_1, \ldots, a_m\}$

$L(A_i) = L(A) \cap a_i^*$, $i \geq 1$

$L(A_0) = L - \bigcup_{i=0}^{m} L(A_i)$

Chrobak conversion: $e^{O(\sqrt{n \ln n})}$ states

Parikh equivalent DFAs

DFA Parikh equivalent to A

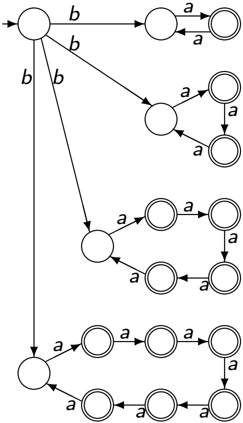
How much is the state cost of the conversion of NFAs accepting *only nonunary strings* into Parikh equivalent DFAs?

Only polynomial!

(less than in unary case)

An Example

$L = \{ba^n \mid n \bmod 210 \neq 0\}$



$L_1 = \{ba^n \mid n \bmod 2 \neq 0\}$

$L_1' = \{ba^n \mid n \bmod 2 \neq 0\}$

$L_2 = \{ba^n \mid n \bmod 3 \neq 0\}$

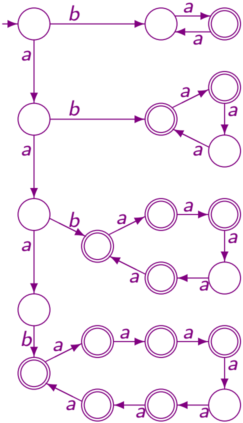
$L_2' = \{aba^{n-1} \mid n \bmod 3 \neq 0\}$

$L_3 = \{ba^n \mid n \bmod 5 \neq 0\}$

$L_3' = \{a^2ba^{n-2} \mid n \bmod 5 \neq 0\}$

$L_4 = \{ba^n \mid n \bmod 7 \neq 0\}$

$L_4' = \{a^3ba^{n-3} \mid n \bmod 7 \neq 0\}$



$DFA \geq 211$ states

$L' = L_1' \cup L_2' \cup L_3' \cup L_4'$

DFA with only 21 states!

Converting NFAs Accepting Only Nonunary Strings

The conversion uses a modification of the following result:

Theorem ([Kopczyński&To '10])

Given $\Sigma = \{a_1, \ldots, a_m\}$, there is a polynomial p s.t. for each n-state NFA A over Σ,

$$\psi(L(A)) = \bigcup_{i \in I} Z_i$$

where:

- *I is a set of at most $p(n)$ indices*
- *for $i \in I$, $Z_i \subseteq \mathbb{N}^m$ is a linear set of the form:*

  $$Z_i = \{\alpha_0 + n_1\alpha_1 + \cdots + n_k\alpha_k \mid n_1, \ldots, n_k \in \mathbb{N}\}$$

  with

  - $0 \leq k \leq m$
  - *the components of α_0 are bounded by $p(n)$*
  - *$\alpha_1, \ldots, \alpha_k$ are linearly independent vectors from $\{0, 1, \ldots, n\}^m$*

Converting NFAs Accepting Only Nonunary Strings

Outline: linear sets

Each above linear set

$$Z_i = \{\alpha_0 + n_1\alpha_1 + \cdots + n_k\alpha_k \mid n_1, \ldots, n_k \in \mathbb{N}\}$$

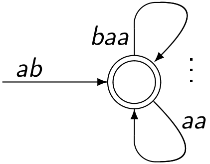
can be converted into a poly size DFA accepting a language

$$R_i = w_0(w_1 + \cdots + w_k)^*$$

s.t. $\psi(w_j) = \alpha_j$, $j = 0, \ldots, k$, and
$w_1, \ldots, w_k$ *begin with different letters*

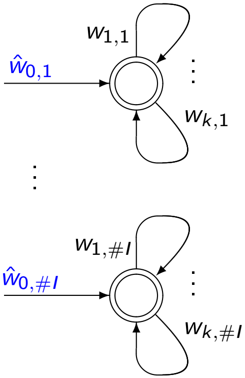
Example:

- $\{(1,1) + n_1(2,1) + n_2(2,0) \mid n_1, n_2 \geq 0\}$
- $ab(baa + aa)^*$



Converting NFAs Accepting Only Nonunary Strings

Outline: from linear to semilinear

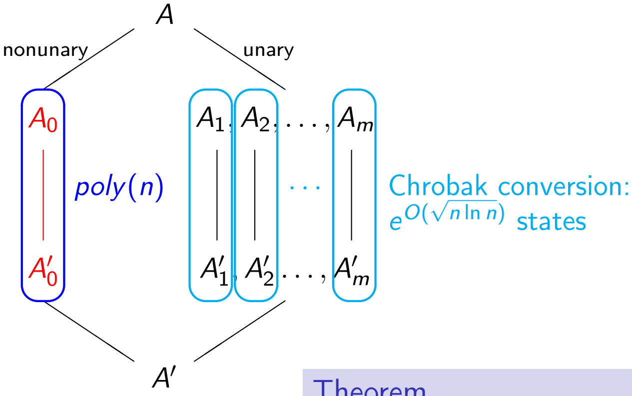


- Standard construction for union of DFAs: number of states = *product*
  $\#I \leq p(n)$ $\Rightarrow$ Too large!!!
- Strings $w_{0,i}$ can be replaced by Parikh equivalent strings $\hat{w}_{0,i}$ in such a way that $W_0 = \{\hat{w}_{0,i} \mid i \in I\}$ is a *prefix code*
- After this change: number of states $\leq$ *sum* Polynomial!!!

Theorem

For each n-state NFA accepting a language none of whose words are unary, there exists a Parikh equivalent DFA with a number of states polynomial in n

Converting NFAs: Back to the General Case



Theorem

For each n-state NFA there exists a Parikh equivalent DFA with $e^{O(\sqrt{n \ln n})}$ states. Furthermore this cost is tight

Converting CFGs

Problem (CFGs to NFAs and DFAs)

CFG *size h*	$\Longrightarrow_\pi$	*NFA/DFA* *how many states?*

- We consider CFGs in Chomsky Normal Form
- As a measure of size we consider the *number of variables*

[Gruska '73]

Problem (CFGs to NFAs)

CFG *Chomsky normal form* *h variables*	$\Longrightarrow_\pi$	*NFA* *how many states?*

Upper bound:

- $2^{2^{O(h^2)}}$ implicit construction from classical proof of Parikh's Th.
- $O(4^h)$ [Esparza&Ganty&Kiefer&Luttenberger '11]

Lower bound: $\Omega(2^h)$ Folklore

Problem (CFGs to DFAs)

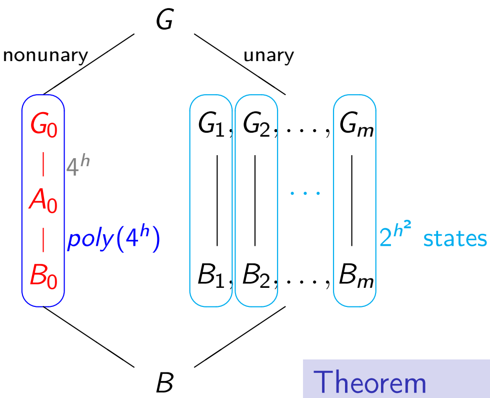
CFG
Chomsky normal form
h variables

$\Longrightarrow_\pi$

DFA
how many states?

- Upper bound: $2^{O(4^h)}$ — subset construction
- Lower bound: 2^{ch^2} — tight bound for the unary case $2^{\Theta(h^2)}$ [Pighizzini&Shallit&Wang '02]

Converting CFGs into Parikh Equivalent DFAs



CFG with h variables

Parikh equivalent NFA

Parikh equivalent DFAs

Theorem

For any CFG in Chomsky normal form with h variables, there exists a Parikh equivalent DFA with at most $2^{O(h^2)}$ states. Futhermore this bound is tight

Final Considerations

We obtained the following tight conversions:

	DFA	
NFA n states	$e^{O(\sqrt{n \ln n})}$ states	
CFG Cnf h variables	$2^{O(h^2)}$ states	

- In both cases the most expensive part is the unary one
- It could be interesting to investigate other conversions, e.g., automata minimization under Parkih equivalence, and computational complexity aspects

Final Considerations

Conversions into *two-way deterministic automata* (2DFAs)

	DFA	2DFA
NFA n states	$e^{O(\sqrt{n \ln n})}$ states	$poly(n)$ states
CFG Cnf h variables	$2^{O(h^2)}$ states	$2^{O(h)}$ states

Thank you for your attention!