Two-Way Automata Characterizations of L/poly versus NL

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Nondeterminism with Bounded Resources

- Time complexity

  $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$ *polynomial time*

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  $\mathsf{L} \stackrel{?}{=} \mathsf{NL}$ — *logarithmic space*

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  $PSPACE = NPSPACE$ — *polynomial space*

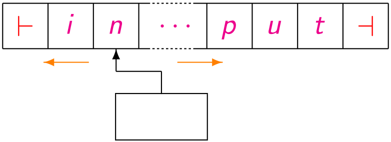
  $L \stackrel{?}{=} NL$ — *logarithmic space*

- State complexity

  $1D \subsetneq 1N$ — *one-way automata*

  $2D \stackrel{?}{=} 2N$ — *two-way automata*

Two-Way Automata



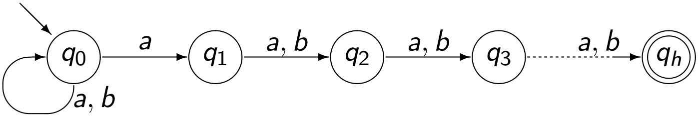
- The input head can be moved in both directions
- They recognize only regular language
- They can be *smaller* than one-way automata

Technical detail:

- Input surrounded by the *endmarkers* ⊢ and ⊣

An Example

$L_h = (a+b)^* a (a+b)^{h-1}$



- 1NFA: $h+1$ states
- 1DFA: 2^h states
- 2DFA: $h+2$ states

Classes

- Family of problems/languages $\mathcal{L} = (L_h)_{h \geq 1}$
- 2D class of families of problems solvable by poly-size 2DFAs:

  $\mathcal{L} \in$ 2D iff $\exists$polynomial p s.t.
  each L_h is solved by a 2DFA of size $p(h)$
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$L_h = (a+b)^* a (a+b)^{h-1}$ $\quad\quad$ $\mathcal{L} = (L_h)_{h \geq 1}$:

- 1NFA: $h+1$ states $\quad \Rightarrow \mathcal{L} \in$ 1N
- 1DFA: 2^h states $\quad \Rightarrow \mathcal{L} \notin$ 1D
- 2DFA: $h+2$ states $\quad \Rightarrow \mathcal{L} \in$ 2D $\subseteq$ 2N

The Question of Sakoda and Sipser

Problem ([Sakoda&Sipser '78])

Do there exist polynomial simulations of

- 1NFAs *by* 2DFAs
- 2NFAs *by* 2DFAs *?*

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Conjecture

Both simulations are not polynomial!
i.e., 1N $\neq$ 2D *and* 2N $\neq$ 2D

Two-Way Automata versus Logarithmic Space

Theorem ([Berman&Lingas '77])

If L = NL *then*
for every s-state σ-symbol 2NFA
there is a poly$(s\sigma)$-state 2DFA
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$\swarrow$

Theorem ([Geffert&P '11])

If L $=$ NL *then*
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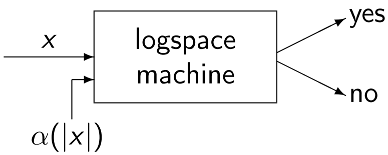
- L/poly
  class of languages accepted by deterministic logspace machines with a *polynomial advice*



Problem

L/poly $\supseteq$ NL ?

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Two-Way Automata versus Logarithmic Space

2N/unary := only *unary* inputs

Theorem ([Geffert&P '11])

L = NL ⇒ 2D ⊇ 2N/unary

2N/poly := only *short* inputs

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L/poly ⊇ NL ⇔ 2D ⊇ 2N/poly

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- What about the weaker hypothesis $L/\text{poly} \supseteq NL$?
- What about the converse of this statement?

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In this work:

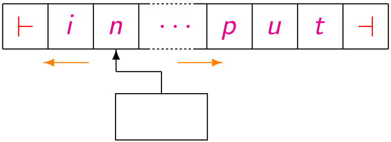
L/poly $\supseteq$ NL $\Leftrightarrow$ 2D $\supseteq$ 2N/unary

$\rightarrow$

Furthermore:

- Investigation of the common behavior unary/short
- Characterizations of L/poly vs NL

1st Tool: Outer Nondeterministic Automata (2OFA)



Nondeterministic choices are possible
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Lemma

For every s-state 2NFA *and integer l there is a $poly(sl)$-state* 2OFA *which agrees with it on all inputs of length $\leq l$*

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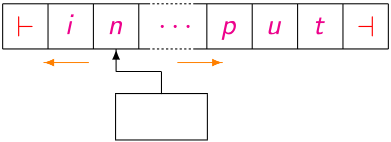

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Lemma ([Geffert et al. '03])

For every s-state unary 2NFA *there is an equivalent poly(s)-state* 2OFA

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2nd Tool: The Graph Accessibility Problem

GAP:

- Given $G = (V, E)$ an oriented graph, $s, t \in V$
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$\Rightarrow$
$(\text{GAP}_h) \in$ 2D iff
2D $\supseteq$ 2N/unary iff
2D $\supseteq$ 2N/poly iff
L/poly $\supseteq$ NL

Binary Encoding: The Family BGAP

- $G = (V_h, E)$, with $V_h = \{0, \ldots, h-1\}$
- *Binary encoding* of G:

  $\langle G \rangle_2 \in \{0,1\}^{h^2}$ standard encoding of the adjacency matrix
- $\text{BGAP}_h := \{\langle G \rangle_2 \mid G \text{ has a path from 0 to } h-1\}$
- 2NFA recognizing BGAP_h:
  - *input:* $x \in \{0,1\}^{h^2}$ *output:* $x \in \text{BGAP}_h$?
  - Nondeterministic choices only on the left endmarker
  - $O(h^3)$ states

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$\text{BGAP} \in 2O$

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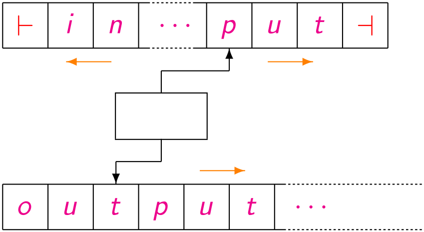
Reductions

Two-Way Deterministic Transducer (2DFT)



- $\mathcal{L} = (L_h)_{h \geq 1}$, $\mathcal{L}' = (L'_h)_{h \geq 1}$
- "Small" reduction:
  $\mathcal{L} \leq_{sm} \mathcal{L}'$ iff each L_h reduces to L'_h
  via "small" 2DFTs with "short" outputs

BGAP and Characterizations

Theorem

BGAP *is*

2N/poly-*complete*

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under $\leq_{sm}$

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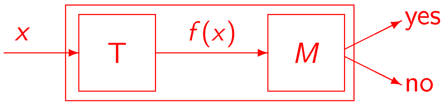
Standard machine composition



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Hence the following statements are equivalent:

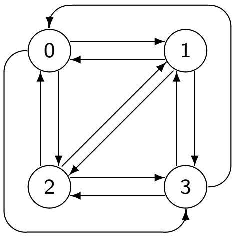
$2D \ni BGAP$

$2D \supseteq 2N/poly$ — $2D \supseteq 2O$

Unary Encoding: The Family UGAP

- $K_h :=$ complete directed graph with vertex set $V_h = \{0, \ldots, h-1\}$
- With each edge (i, j) we associate a different prime number $p_{(i,j)}$
- A subgraph $G = (V_h, E)$ of K_h is encoded by the string a^{m_G}, where

$$m_G = \prod_{(i,j) \in E} p_{(i,j)}$$



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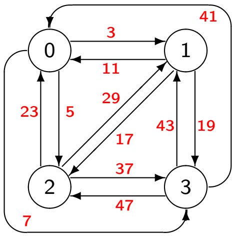

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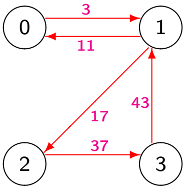

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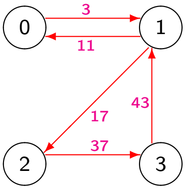

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- Graph $K_h(m)$: $\exists$ edge (i,j) iff $p_{(i,j)}$ divides m
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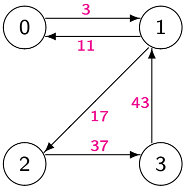

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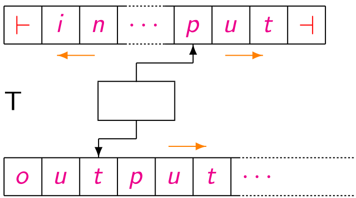
Prime Reductions



- Producing a unary output a^m could require too many states!
- Output: a list $z_1 \cdots z_k$ of prime powers factorizing m
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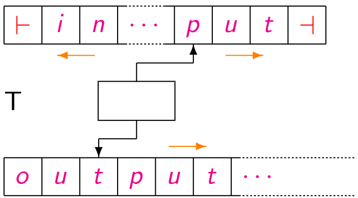

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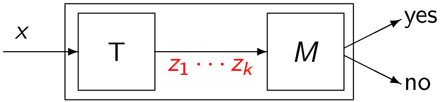
Machine composition



- Unary 2DFAs can be modified to read prime encodings
- This allows to prove that 2D is closed under $\preceq_{sm}$

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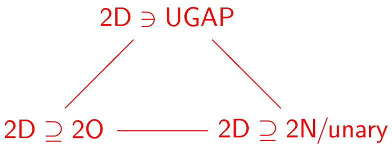

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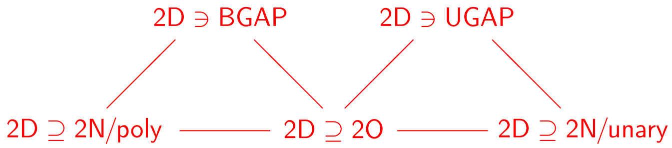

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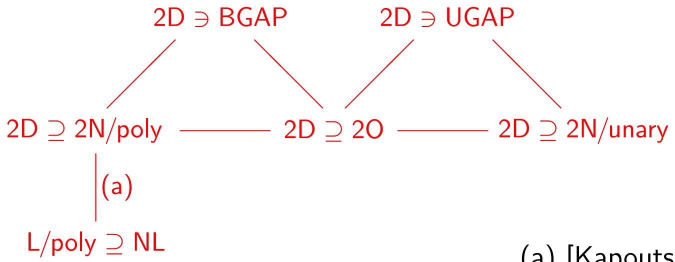

(a) [Kapoutsis '11]

Directions for Further Investigations

- Characterizations in terms of two-way automata of *uniform* L vs NL
- Comparison of two-way automata on unary vs short inputs
- Use of the reductions introduced in the paper for other purposes

Thank you for your attention!