Two-Way Finite Automata

Old and Recent Results

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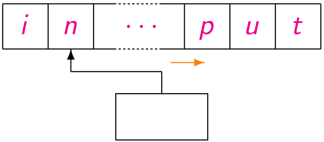
Finite State Automata



One-way version

At each step the input head is moved
one position to the right

- 1DFA: *deterministic* transitions
- 1NFA: *nondeterministic* transitions

A Very Preliminary Example

$\Sigma = \{a, b\}$, fixed $n > 0$:

$$H_n = (a+b)^{n-1}a(a+b)^*$$

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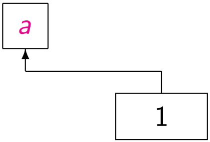

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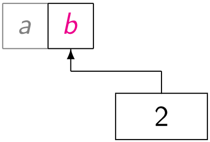

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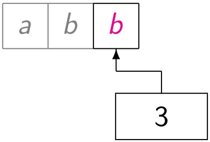

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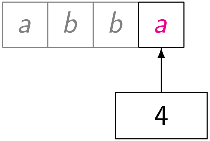

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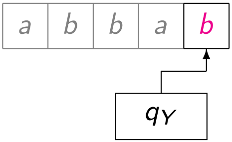

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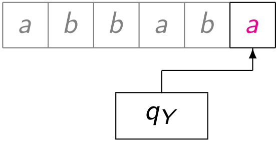

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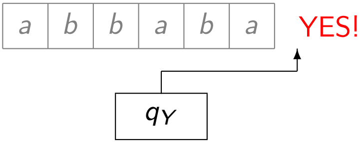

1DFA: $n + 2$ states

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How to locate it?

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Guess Reading the symbol *a* the automaton can guess that it is the *n*th symbol from the right

Verify In the next steps the automaton verifies such a guess

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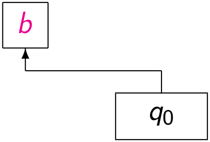

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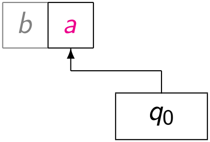

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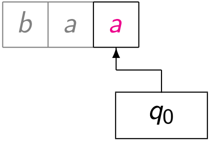

guess
4th symbol from the right

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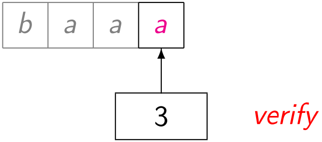

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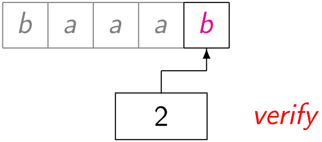

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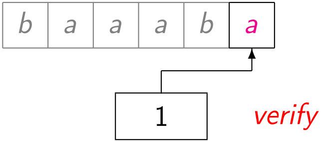

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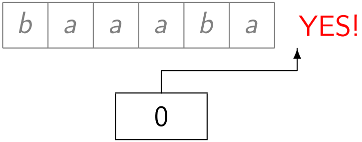

1NFA: $n + 1$ states

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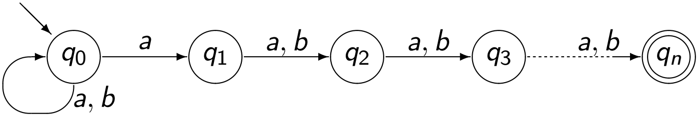

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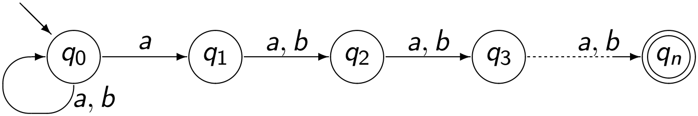

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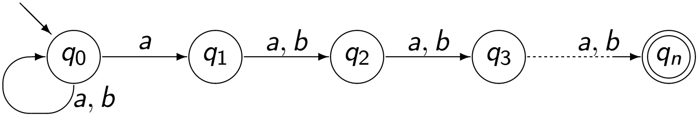

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Remember the previous n input symbols!

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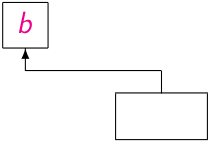

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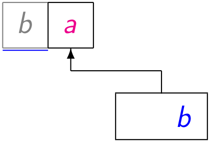

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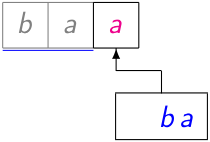

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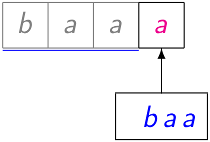

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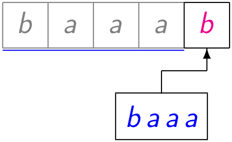

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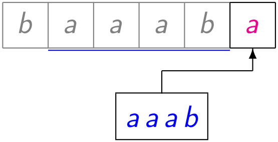

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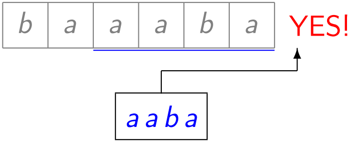

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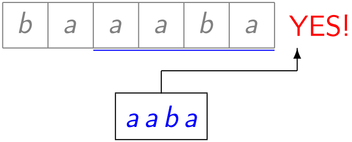

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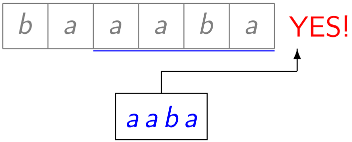

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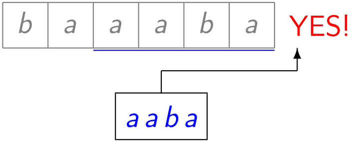

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This is the smallest one!

However...

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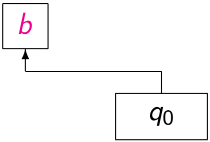

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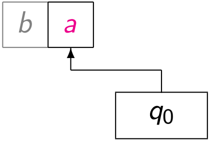

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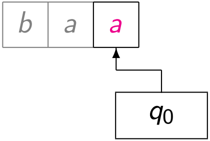

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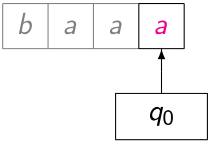

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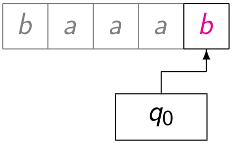

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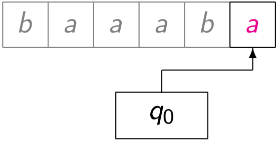

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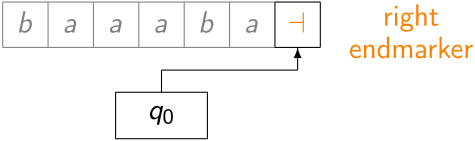

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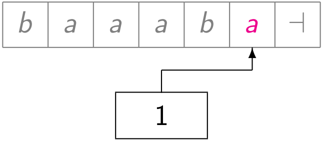

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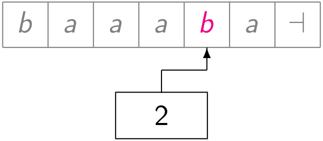

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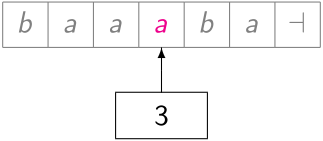

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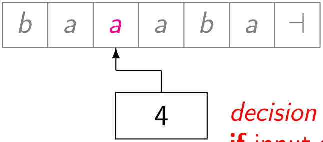

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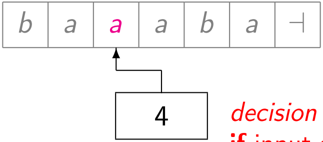

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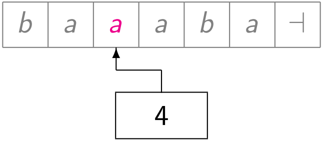

YES!

Two-way deterministic automaton (2DFA): $n + \dots$ states

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Summing up, l_n is accepted by

- a 1NFA and a 2DFA with approximatively the same number of states $n + \ldots$
- each 1DFA is exponentially larger ($\geq 2^n$ states)

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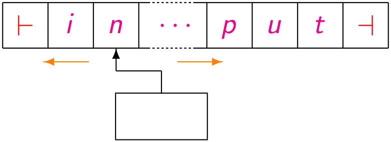
In this example,
nondeterminism can be removed using two-way motion keeping approximatively the same number of states

Two-Way Automata: Technical Details



- Input surrounded by the *endmarkers* ⊢ and ⊣
- Moves
  - to the *left*
  - to the *right*
  - *stationary*
- Initial configuration
- Accepting configuration
- Infinite computations are possible
- *Deterministic* (2DFA) and *nondeterministic* (2NFA) versions

1DFA, 1NFA, 2DFA, 2NFA

What about the power of these models?

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They share the same computational power, namely they characterize the class of *regular languages*,

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They share the same computational power, namely they characterize the class of *regular languages*, however...

...some of them are more succinct

Main Example: $L_n = (a+b)^* a (a+b)^{n-1} a (a+b)^*$

1NFA: $n + 2$ states

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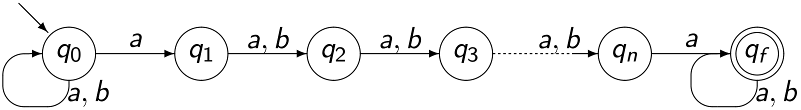

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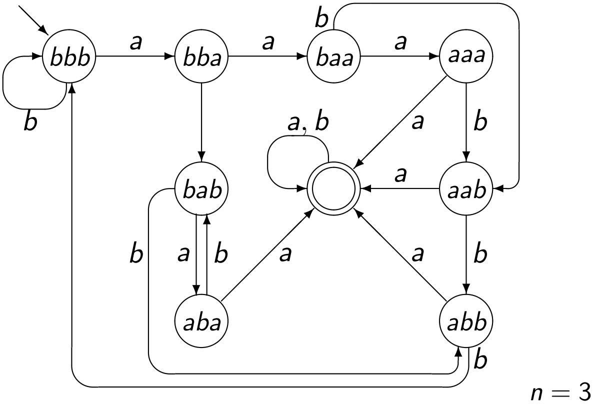

Minimum 1DFA: $2^n + 1$ states

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2DFA ?

Even scanning from the right it seems that
we need to remember a "window" of n symbols

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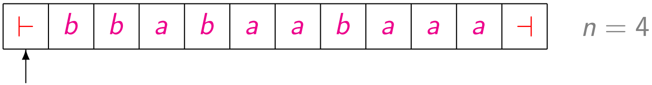

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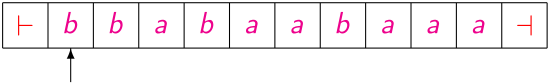

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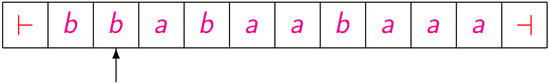

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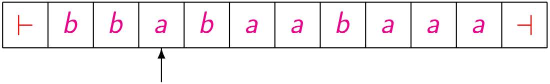

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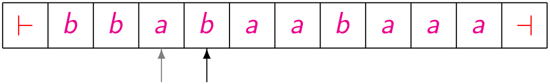

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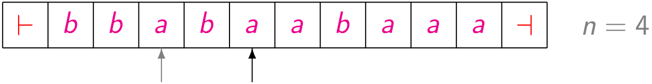

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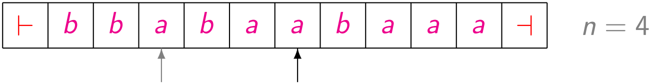

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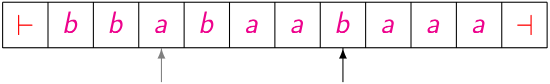

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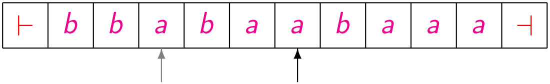

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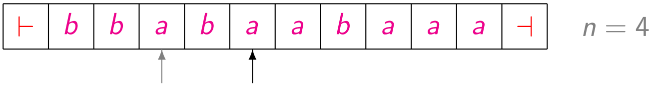

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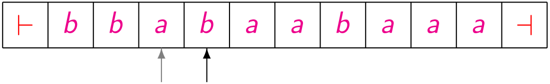

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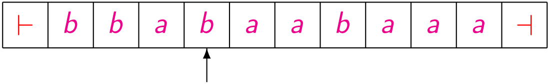

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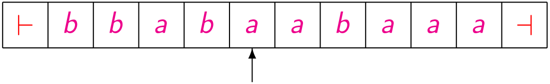

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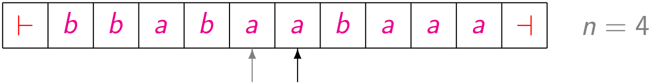

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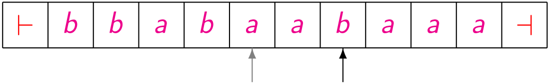

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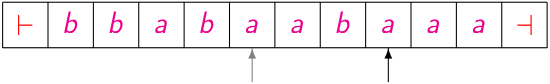

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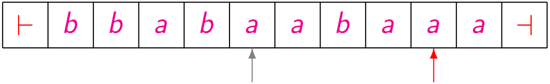

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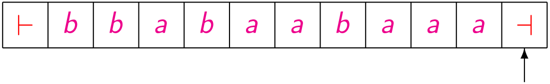

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2DFA: $2n + \ldots$ states

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A different algorithm

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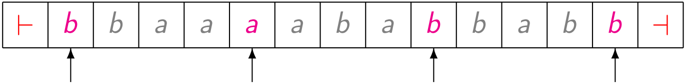

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Check positions k s.t. $k \equiv 1 \pmod{n}$

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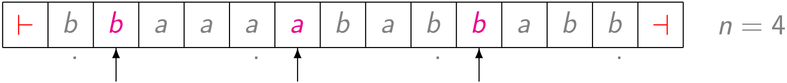

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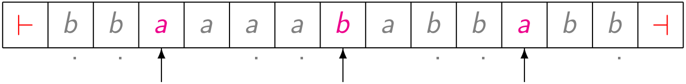

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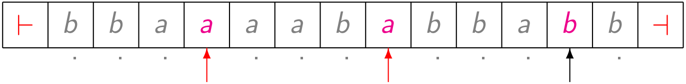

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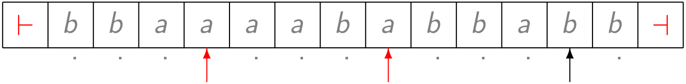

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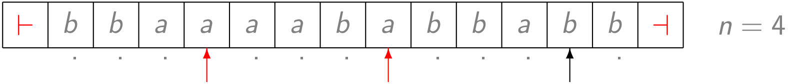

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Sweeping automata:

- Deterministic transitions
- Head reversals *only at the endmarkers*

Main Example: $L_n = (a+b)^* a (a+b)^{n-1} a (a+b)^*$

Summing up,

- L_n is accepted by
  - a 1NFA
  - a 2DFA
  - a sweeping automaton

  with $O(n)$ states
- Each 1DFA is exponentially larger

Also for this example,
nondeterminism can be removed using two-way motion
keeping a linear number of states

Is it always possible
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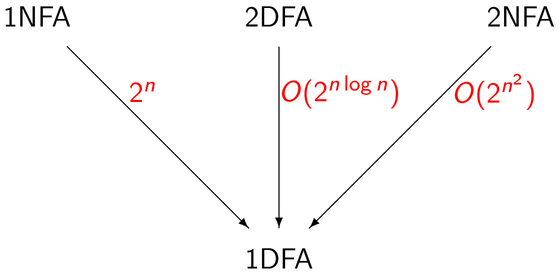
Costs of the Optimal Simulations Between Automata



[Rabin&Scott '59, Shepardson '59, Meyer&Fischer '71, . . .]

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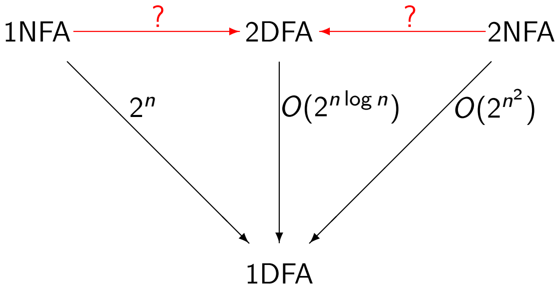

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Question

How much the possibility of moving the input head forth and back is useful to eliminate the nondeterminism?

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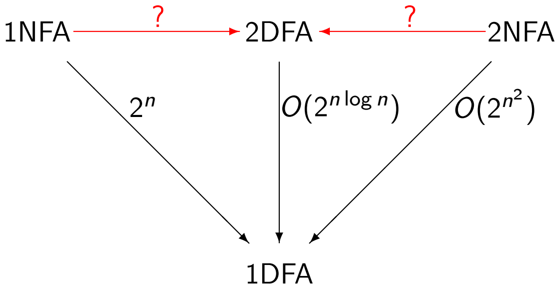

Problem ([Sakoda&Sipser '78])

Do there exist polynomial simulations of

- *1NFAs by 2DFAs*
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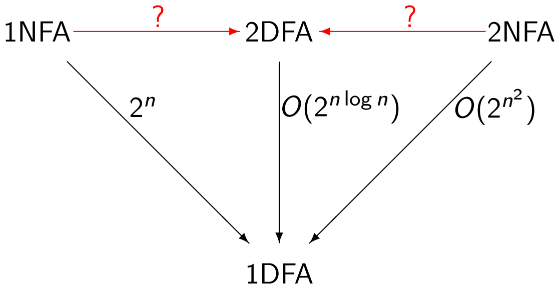

Problem ([Sakoda&Sipser '78])

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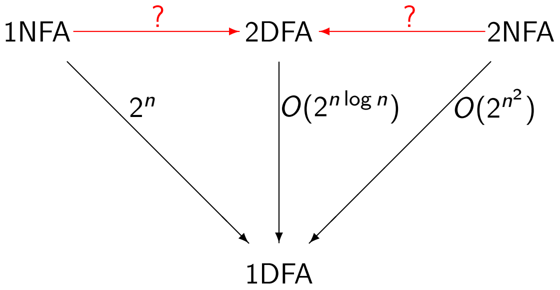
Conjecture

These simulations are not polynomial

Costs of the Optimal Simulations Between Automata



- Exponential upper bounds
  deriving from the simulations of 1NFAs and 2NFAs by 1DFAs
- Polynomial lower bound
  $\Omega(n^2)$ for the cost of the simulation of 1NFAs by 2DFAs
  [Chrobak '86]

Sakoda and Sipser Question

- Very difficult in its general form
- Not very encouraging obtained results:

  Lower and upper bounds too far
  (Polynomial vs exponential)

- Hence:

  Try to attack restricted versions of the problem!

NFAs vs 2DFAs: Restricted Versions

(i) Restrictions on the resulting machines (2DFAs)

- sweeping automata [Sipser '80]
- oblivious automata [Hromkovič&Schnitger '03]
- "few reversal" automata [Kapoutsis '11]

(ii) Restrictions on the languages

- unary regular languages [Geffert Mereghetti&P '03]

(iii) Restrictions on the starting machines (2NFAs)

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Naïf algorithm: compare input positions i and $i+n$, $i = 1, 2, \ldots$

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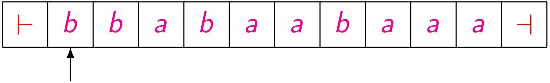

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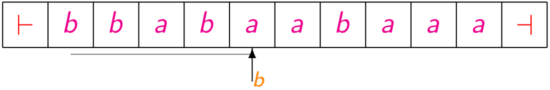

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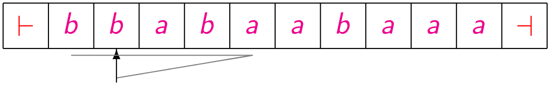

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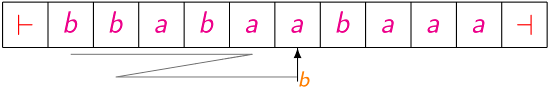

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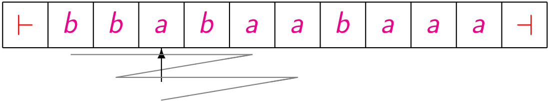

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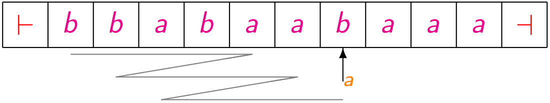

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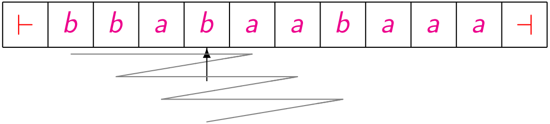

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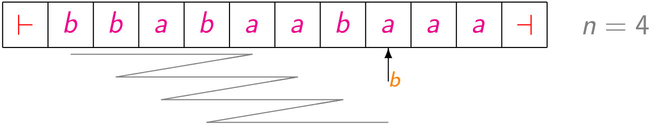

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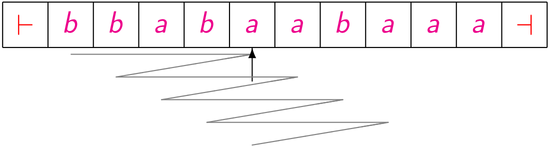

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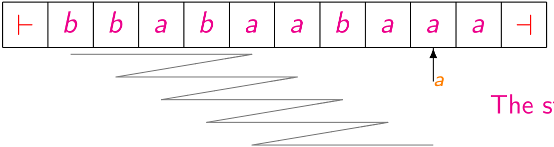

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The string can be accepted!

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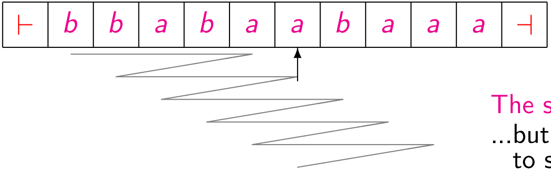

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...but our automaton continues to scan

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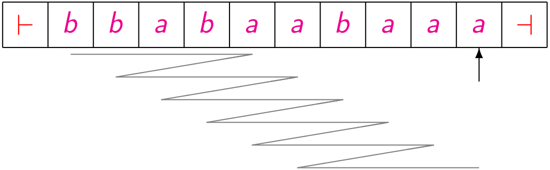

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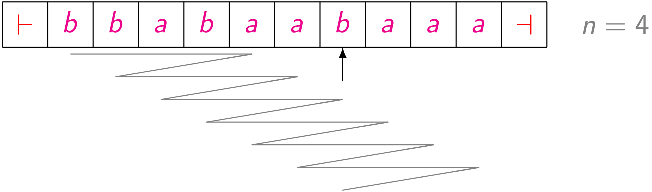

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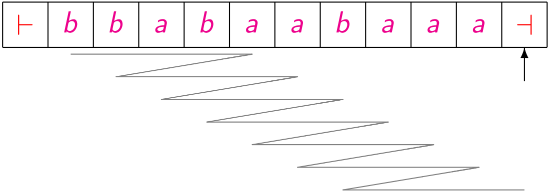

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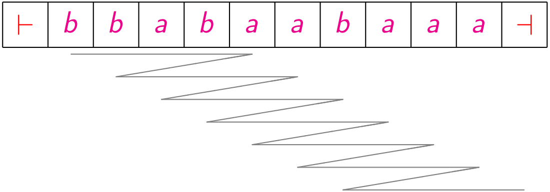

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Even in this case $O(n)$ states!

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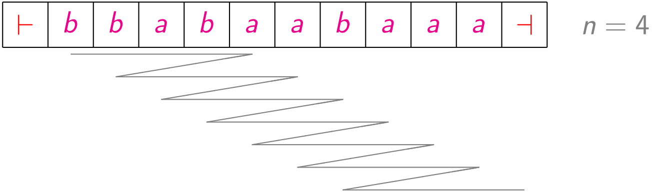

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Oblivious Automata:

- Deterministic transitions
- Same "trajectory" on all inputs of the same length

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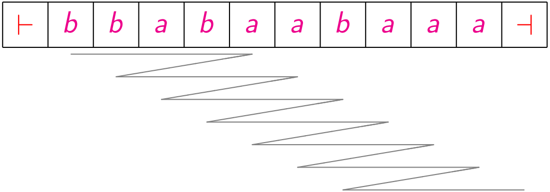

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Number of head reversals:
On input of length m:

- This technique uses about $2m$ reversals, a *linear number* in the input length
- The "sweeping" algorithm uses about $2n$ reversals, a *constant number* in the input length

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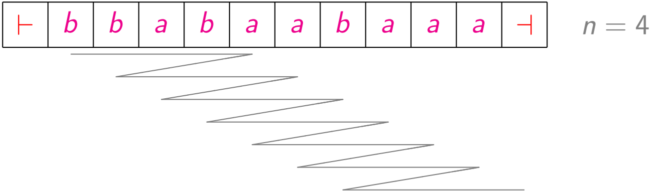

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Another Restricted Model

"Few Reversal" Automata [Kapoutsis '11]:

- On input of length m the number of reversals is $o(m)$, i.e., sublinear
- We consider only the *deterministic case*

Theorem ([Kapoutsis&P '12])

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Restricted Models: Separations

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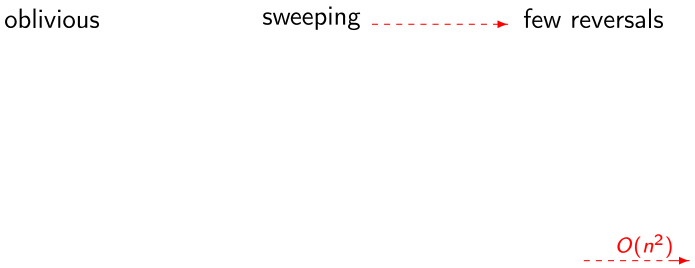
Restricted Models: Separations

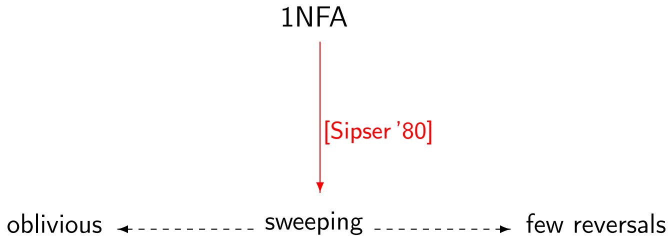


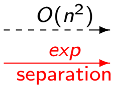
Restricted Models: Separations

oblivious ← sweeping → few reversals

$O(n^2)$ →

Restricted Models: Separations





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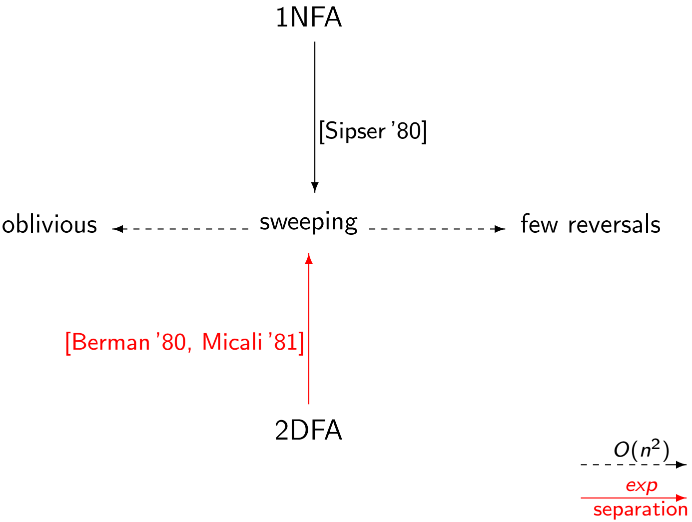

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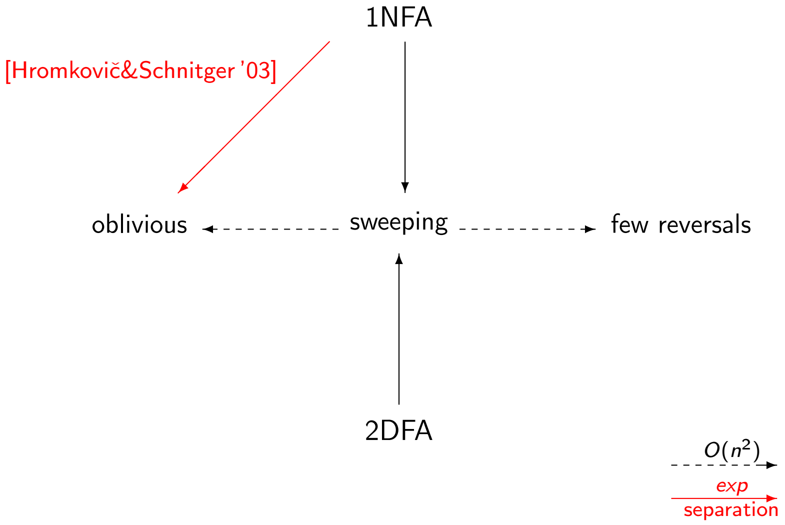

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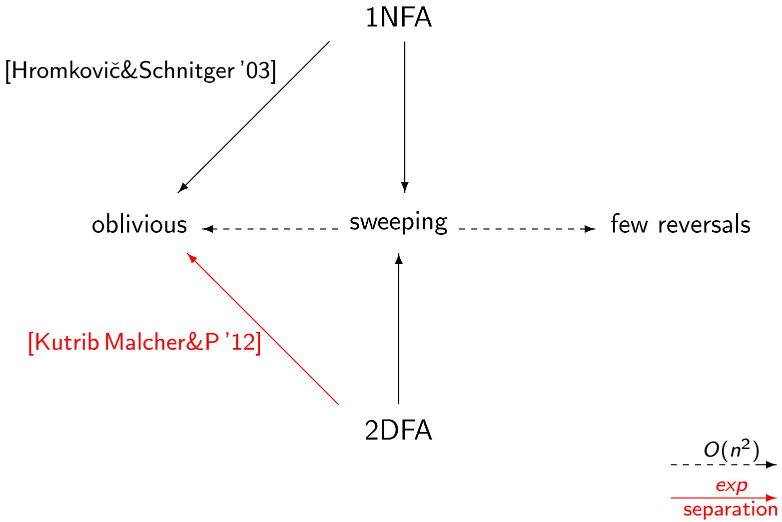

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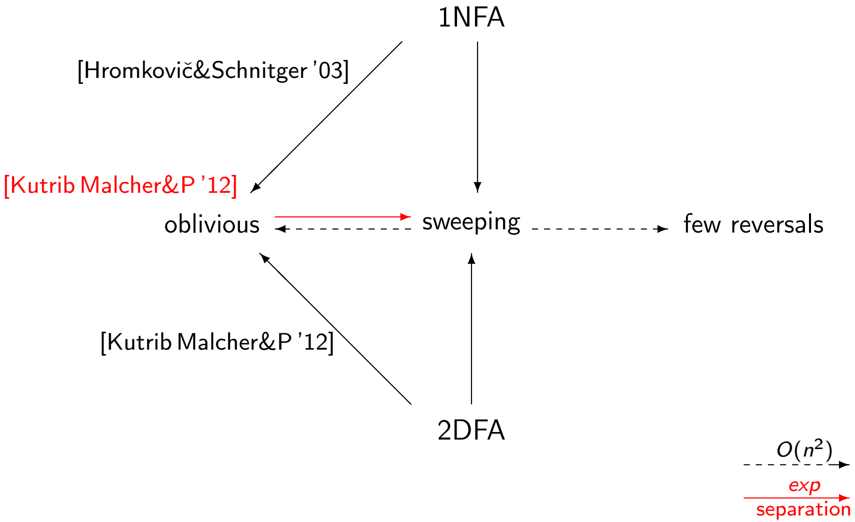

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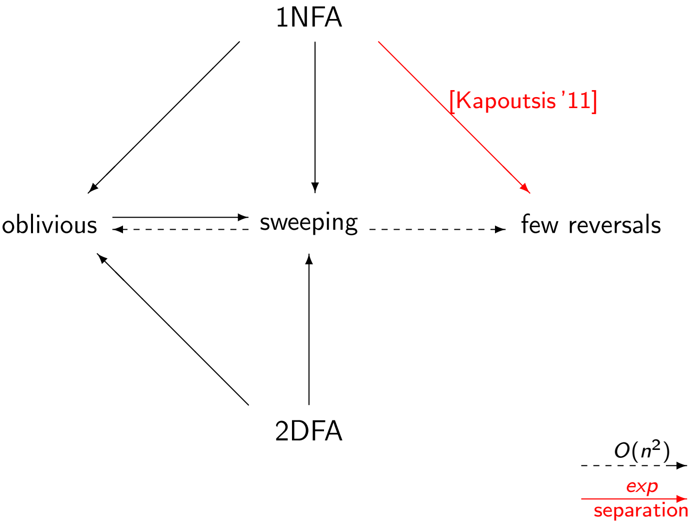

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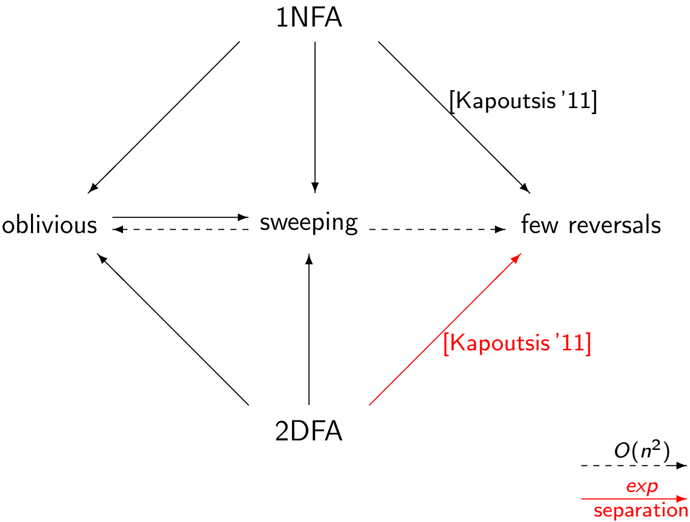

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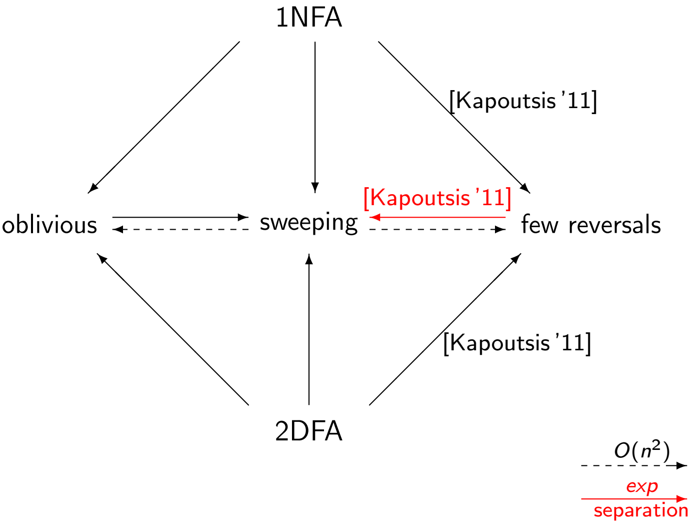

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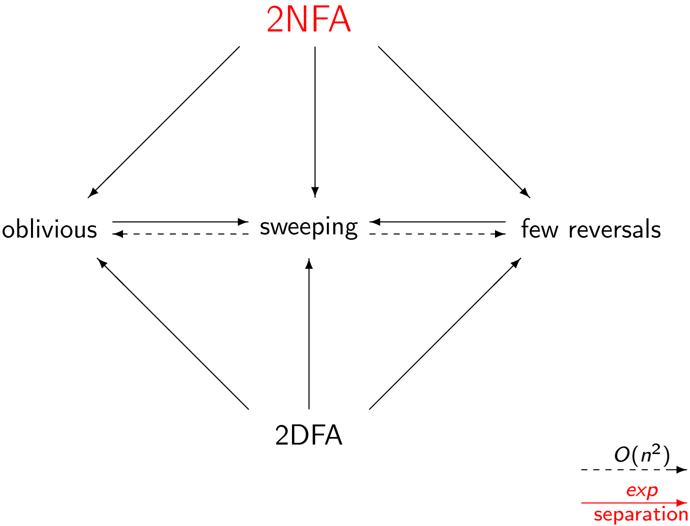
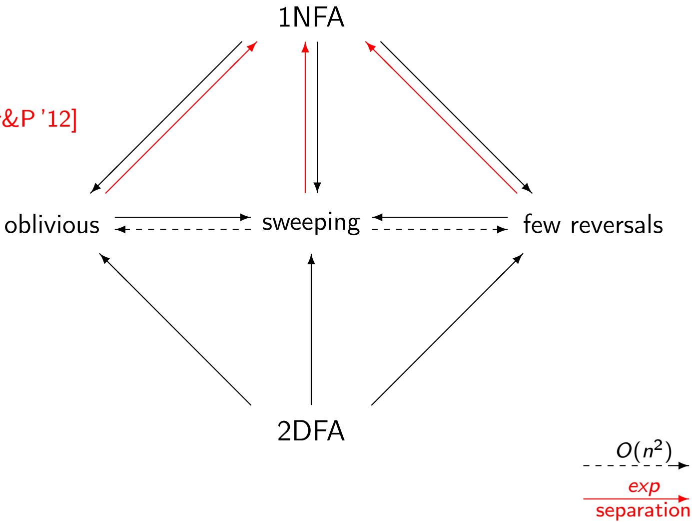


Restricted Models: Separations

[Kutrib Malcher&P '12]



Sakoda&Sipser Question

Problem ([Sakoda&Sipser '78])

Do there exist polynomial simulations of

- *1NFAs by 2DFAs*
- *2NFAs by 2DFAs ?*

Another possible restriction:

The unary case $\#\Sigma = 1$

Optimal Simulation Between Unary Automata

The costs of the optimal simulations between automata are different in the unary and in the general case

1DFA 1NFA

2DFA 2NFA

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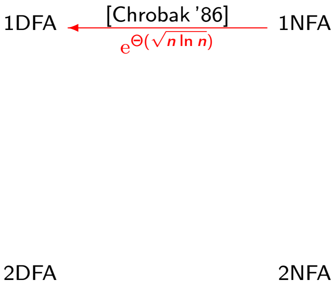

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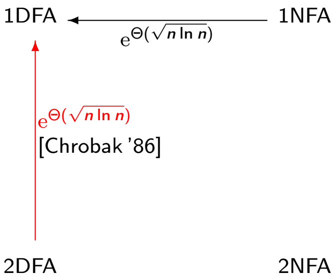

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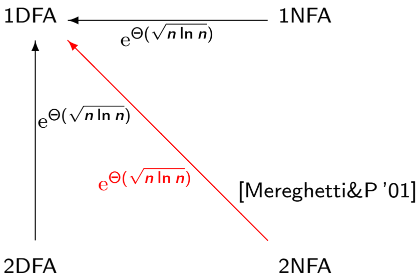

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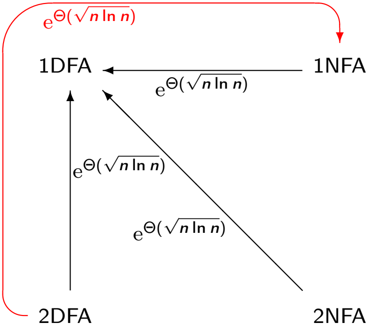

follows from 2DFA $\rightarrow$ 1DFA

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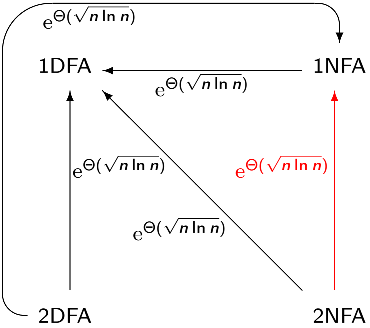

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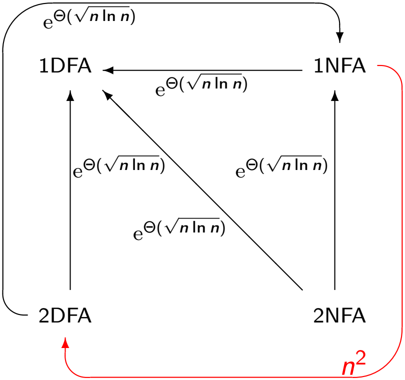

[Chrobak '86]

1NFA → 2DFA
In the unary case
this question is solved!
(polynomial conversion)

Optimal Simulation Between Unary Automata

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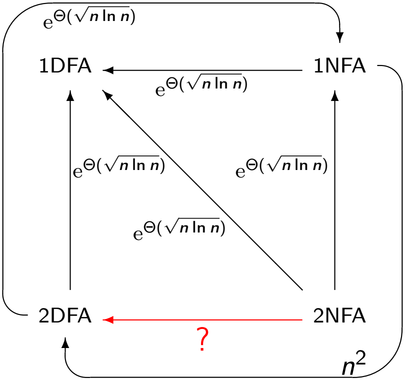

2NFA → 2DFA
Even in the unary case
this question is open!

- $e^{\Theta(\sqrt{n \ln n})}$ upper bound (from 2NFA → 1DFA)
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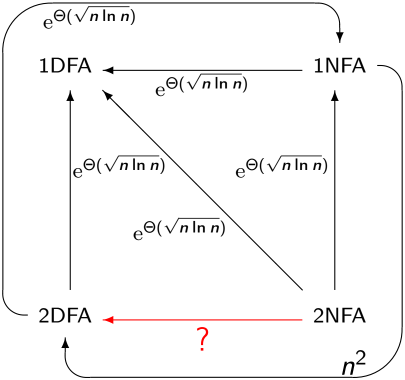

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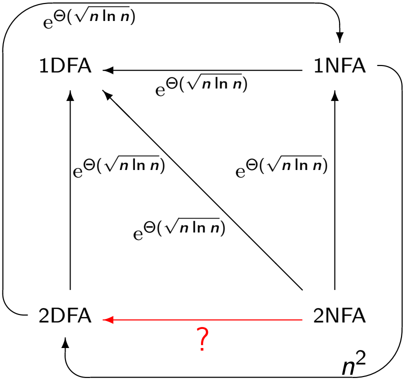

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A better upper bound $e^{O(\ln^2 n)}$
has been proved!

A Normal Form for Unary 2NFAs

[Geffert Mereghetti&P '03]

Quasi Sweeping Automata (qsNFA):

- *nondeterministic choices* and
- *head reversals*

are *possible only* when the head is visiting the *endmarkers*

Theorem (Quasi Sweeping Simulation)

Each n-state unary 2NFA A can be transformed into a 2NFA M s.t.

- *M is quasi sweeping*
- *M has at most $N \leq 2n + 2$ states*
- *M and A are "almost equivalent"*
  (possible differences only for inputs of length $\leq 5n^2$)

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From Unary qsNFAs to 2DFAs

[Geffert Mereghetti&P '03]

- M a fixed qsNFA with N states
- An input w is accepted iff there is an accepting computation visiting the left endmarker $\leq N$ times
- For $p, q \in Q$, $k \geq 1$, we define the predicate

  reachable$(p, q, k) \equiv \exists$*computation path on w which*
  - *starts in the state p on the left endmarker*
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  - *visits the left endmarker $\leq k$ more times*
- Assuming acceptance on the left endmarker in state q_f:

  $w \in L(M)$ iff *reachable*(q_0, q_f, N) is true

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How to Evaluate *reachable*?

Divide–and–conquer technique

function *reachable*(p, q, k)
if $k = 1$ **then return** *reach1*(p, q) `//direct simulation`
else begin
  for each state $r \in Q$ **do**
    if *reachable*(p, r, $\lfloor k/2 \rfloor$) **and** *reachable*(r, q, $\lceil k/2 \rceil$)
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This strategy can be implemented by a 2DFA with $e^{O(\ln^2 N)}$ states
in order to compute *reachable*(q_0, q_f, N),
i.e., to decide if the input $w \in L(M)$

From Unary 2NFAs by 2DFAs

A	given unary 2NFA	n states
$\Downarrow$		
M	almost equivalent qsNFA	$N \leq 2n+2$ states
$\Downarrow$		
B	2DFA equivalent to M	$e^{O(\ln^2 N)}$ states
$\Downarrow$		
C	2DFA equivalent to A	$e^{O(\ln^2 n)}$ states

Theorem ([Geffert Mereghetti&P '03])

Each unary n-state 2NFA can be simulated by a 2DFA with $e^{O(\ln^2 n)}$ states

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Using quasi sweeping simulation of unary 2NFAs several results have been discovered:

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(iii) Polynomial simulation of unary 2NFAs by 2DFAs
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Outer Nondeterministic Automata (OFAs) [Guillon Geffert&P '12]:

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Hence:

- No restrictions on the *input alphabet*
- No restrictions on *head reversals*
- *Deterministic transitions* on "real" input symbols

Outer Nondeterministic Automata (OFAs)

The results we obtained for the unary case
can be extended to 2OFAs: [Guillon Geffert&P '12]

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While in the unary case all the proofs rely
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- Checks the existence of a computation segment
  - from the left endmarker in the state p
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  - not visiting the left endmarker in between
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  - Modification of a technique for the complementation of 2DFAs [Geffert Mereghetti&P '07], which refines a construction for space bounded TM [Sipser '80]

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Loops involving endmarkers are also possible

- They can be avoided by observing that for each accepting computation visiting one endmarkers more than $|Q|$ times there exists a shorter accepting computation

Sakoda&Sipser Question: Current Knowledge

- Upper bounds

	1NFA→2DFA	2NFA→2DFA
unary case and OFAs	$O(n^2)$ optimal	$e^{O(\ln^2 n)}$
general case	exponential	exponential

Unary case [Chrobak '86, Geffert Mereghetti&P '03]
OFAs [Guillon Geffert&P '12]

- Lower Bounds
In all the cases, the best known lower bound is $\Omega(n^2)$
[Chrobak '86]

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In both cases:

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Minicomplexity

- Complexity theory of two-way finite automata

[Kapoutsis, DCFS 2012]

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- In the investigation of restricted versions many interesting and not artificial models have been considered
- The results obtained under restrictions, even if not solving the full problem, are not trivial and, in many cases, very deep
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Thank you for your attention!