Nondeterministic One-Tape Off-Line Turing Machines

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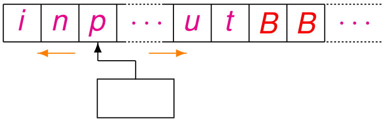
Machine model



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- *Semi-infinite tape* which contains, at the beginning of the computation:
  - the input string, on its part of the tape
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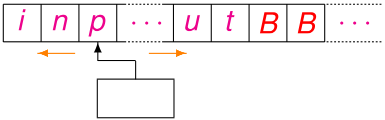

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  - changes its internal state
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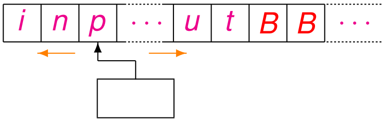

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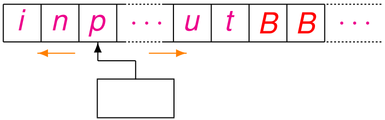

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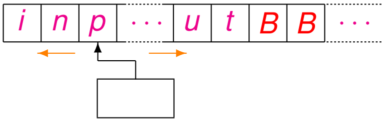

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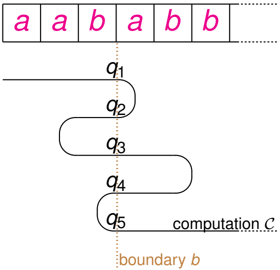
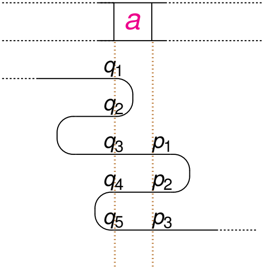
Crossing sequences



The *crossing sequence* of a computation $\mathcal{C}$ at a boundary b between two tape squares is the sequence of the states $(q_1, \ldots, q_k)$ s.t. q_i is the state when the boundary b is crossed for the ith time.

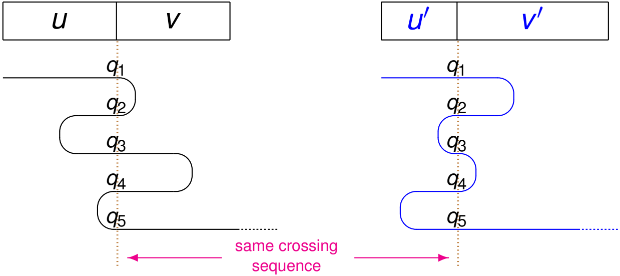
Crossing sequences: compatibility



Given two finite crossing sequences $(q_1, \ldots, q_k)$ and $(p_1, \ldots, p_h)$, it is possible to verify whether or not they are *compatible* with respect to an input symbol a,
i.e., $(q_1, \ldots, q_k)$ and $(p_1, \ldots, p_h)$ can be, in some computation, the crossing sequence at the left boundary and at the right boundary of a tape square which initially contains the symbol a.

Crossing sequences: “cut–and–paste”

Given:



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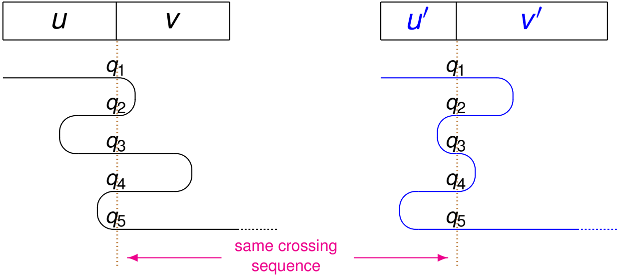

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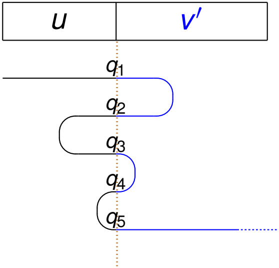

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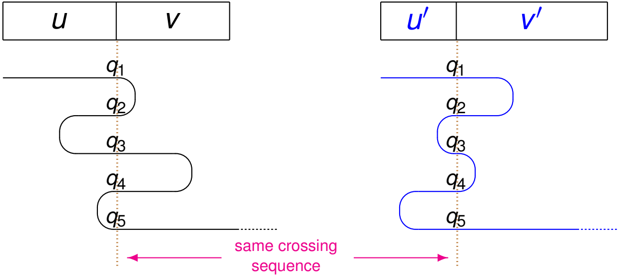

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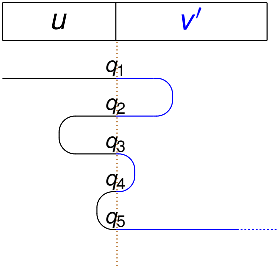

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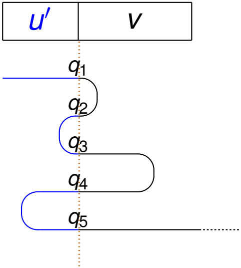

Complexity measures

Let $\mathcal{C}$ be a computation of a TM. We consider:

- The *time* $t(\mathcal{C})$, namely the number of moves in $\mathcal{C}$.
- The *length of the crossing sequences* $c(\mathcal{C})$, namely the maximal length of the crossing sequences defined by $\mathcal{C}$.

nTMs can have many different computations for a same input string

How to define

- $t(x)$ and $c(x)$ for an input x and,
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Complexity measures: strong vs. weak measures

Let $r \in \{t, c\}$ (time or length of crossing sequences)

- **strong measure:** costs of *all computations* on x

$$r(x) = \max\{r(\mathcal{C}) \mid \mathcal{C} \text{ is a computation on } x\}$$

- **weak measure:** *minimum* cost of *accepting* x

$$r(x) = \begin{cases} \min\{r(\mathcal{C}) \mid \mathcal{C} \text{ is accepting on } x\} & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases}$$

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Problem:

Find tight lower bounds for

- the minimum amount of time $t(n)$
- the length of crossing sequences $c(n)$

for nonregular language recognition.

One-Tape Off-Line TMs: Simple bounds

For the *length of the crossing sequences*, the following result can be easily proved:

Theorem

If L is accepted by a nTM such that $c(n) = O(1)$*, under the* weak *measure, then L is regular.*

Idea of the proof:

- Let K be such that $c(n) \leq K$.
- Define a nfa A accepting L s.t.:
  - the states of A are the crossing sequences of length at most K
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Does it is possible to improve the lower bounds on $c(n)$ and $t(n)$ for nonregular language recognition given in the previous results?

Different bounds have found depending

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- Hennie (1965) proved that

  one-tape off-line *deterministic* machines working in linear time accept regular languages.

  Furthermore, in order to accept nonregular languages $c(n)$ must grow at least as $\log n$.

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For the accept measure we proved [Pighizzini, 2009]:

Let M be a nTM accepting a language L using

- crossing sequences of length bounded by $c(n)$
- time $t(n)$

under the accept measure. Then:

- If $c(n) = o(\log n)$ then $c(n) = O(1)$ and then L is regular
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- If $c(n) = o(\log n)$ then $c(n) = O(1)$ and then L is regular
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Lower bounds for the accept measure

For the accept measure we proved [Pighizzini, 2009]:

Let M be a nTM accepting a language L using

- crossing sequences of length bounded by $c(n)$
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  - L is regular

The proof uses counting arguments based on the following lemma:

Lemma

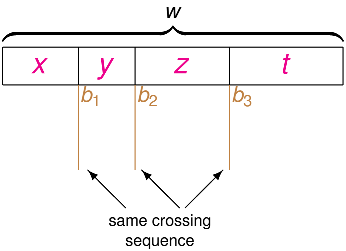
Given

- *an integer k*
- *an input string w accepted by a computation $\mathcal{C}$ with $c(\mathcal{C}) = k$.*

If a same crossing sequence occurs in $\mathcal{C}$ at three different boundaries of the input zone of the tape, then there is another string w' s.t.

- $|w| < |w'|$
- *w' is accepted by a computation $\mathcal{C}'$ with $c(\mathcal{C}') = k$*

Sketch of the proof



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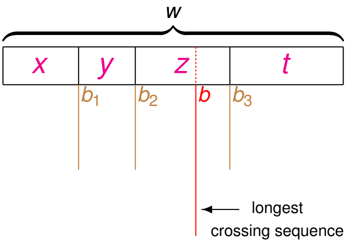
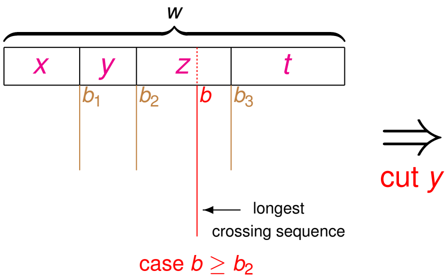


case $b \geq b_2$

Sketch of the proof



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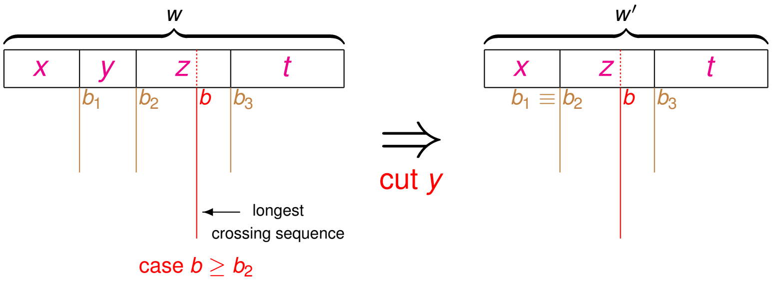

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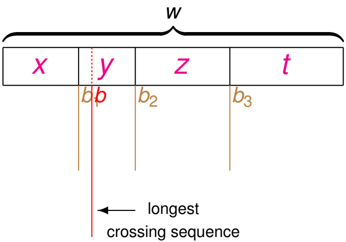

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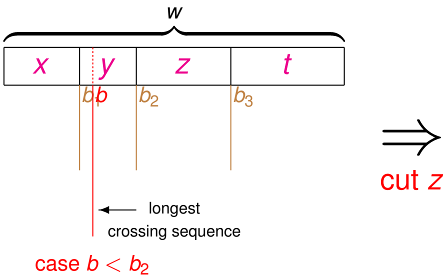

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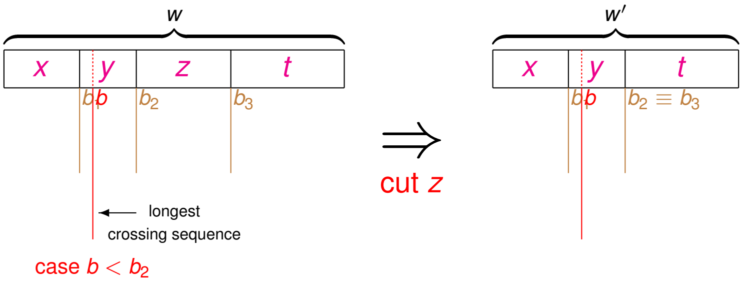

Does it is possible to extend the lower bound for the accept measure to the weak one?

- By the above mentioned result of Michel (1991), for the time the answer is negative
- For the length of the crossing sequences a $\log \log n$ lower bound has been proved

The weak measure

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$c(n)$ grows at least as $\log\log n$, weak measure

Sketch of the proof

- $L :=$ language accepted by the given machine M
- $q :=$ number of states of M
- for $n \geq 1$:
  $N_n :=$ nfa whose states are the crossing sequences of length $\leq c(n)$ and whose transitions are defined according to the "compatibility" relation
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- Hence $2^{q^{c(n)+1}} \geq \frac{n+3}{2}$, i.o., implying that

$$c(n) \geq d \log\log n$$

for some constant d and infinitely many n

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Summary of the lower bounds

		strong	accept	weak
dTM	$t(n)$			
	$c(n)$			
nTM	$t(n)$			
	$c(n)$			

Summary of the lower bounds

		strong	accept	weak
dTM	$t(n)$	$n \log n$		
	$c(n)$	$\log n$		
nTM	$t(n)$			
	$c(n)$			

Trakhtenbrot (1964) and Hartmanis (1968)
Hennie (1965) for $c(n)$

Summary of the lower bounds

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Tadaki, Yamakami, and Lin (2004)

Summary of the lower bounds

		strong	accept	weak
dTM	$t(n)$	$n \log n$		
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Pighizzini (2009)

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		strong	accept	weak
dTM	$t(n)$	$n \log n$	$n \log n$	
	$c(n)$	$\log n$	$\log n$	
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	$c(n)$	$\log n$	$\log n$	

Consequence of accept nTM

Summary of the lower bounds

		strong	accept	weak
dTM	$t(n)$	$n \log n$	$n \log n$	$n \log n$
	$c(n)$	$\log n$	$\log n$	$\log n$
nTM	$t(n)$	$n \log n$	$n \log n$	
	$c(n)$	$\log n$	$\log n$	

For dTM, accept and weak is the same

Summary of the lower bounds

		strong	accept	weak
dTM	$t(n)$	$n \log n$	$n \log n$	$n \log n$
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$t(n)$: simple bound
$c(n)$: Pighizzini (2009)

Summary of the lower bounds

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Optimality of the bounds

Consider the following unary language (Hartmanis, 1968)

$$L = \{a^{2^m} \mid m \geq 0\}$$

We can build a dTM M accepting L which works as follows:

- At the beginning all the input cells are "unmarked"
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a	a	a	a	a	a	a	a	a	a	a	a	input a^{12}

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- *M* accepts if and only if all the input segment is marked.

a	a	a	a	a	a	a	a	a	a	a	a

$$\Downarrow$$

X	a	X	a	X	a	X	a	X	a	X	a

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X	X	X	a	X	X	X	a	X	X	X	a

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X	X	X	X	X	X	X	a	X	X	X	X

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X	X	X	X	X	X	X	a	X	X	X	X

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X	X	X	X	X	X	X	a	X	X	X	X	reject!

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a	a	a	a	a	a	a	a

input a^8

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a	a	a	a	a	a	a	a

$$\Downarrow$$

X	a	X	a	X	a	X	a

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X	a	X	a	X	a	X	a

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X	a	X	a	X	a	X	a

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X	X	X	X	X	X	X	X

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X	X	X	X	X	X	X	X

accept!

- On input a^n, the machine M makes $O(\log n)$ sweeps of the part of the tape which contains the input. Hence:

$$c(n) = O(\log n) \text{ and } t(n) = O(n \log n).$$

- M is *deterministic* and the previous bounds hold for accepting and rejecting computations: strong measure.

- This gives the optimality of all the lower bounds in the table, with the only exception of those for nTMs under the weak measure:

		strong	accept	weak
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Complexity

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	$c(n)$	$\log n$	$\log n$	$\log n$
nTM	$t(n)$	$n \log n$	$n \log n$	n
	$c(n)$	$\log n$	$\log n$	$\log \log n$

- The optimality was proved by using a *unary language*.

nTMs and weak measure

What about the bounds for the weak measure for nTMs?

- There are nonregular languages accepted in time $O(n)$.
- Hence, in this case there is no a "gap" between regular and nonregular languages.
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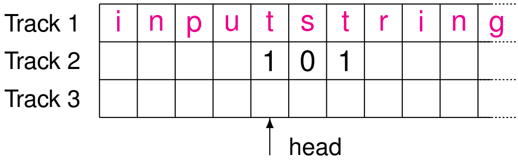
Basic techniques

- We can consider a tape divided in a fixed number of tracks.
- The input is written on the first track.

Track 1	i	n	p	u	t	s	t	r	i	n	g
Track 2	m	e	m	o	r	y	s	p	a	c	e
Track 3	m	e	m	o	r	y	s	p	a	c	e

How to count input symbols



- The counter is kept on track 2, *starting from the position scanned by the tape head*
- When the head must be moved to the right, counting one more input position, the counter is incremented and shifted to the right
- This is done in $O(\log j)$ steps (where j is the value of the counter) using track 3 as an auxiliary variable

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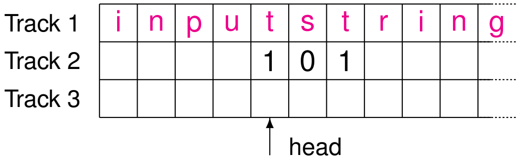

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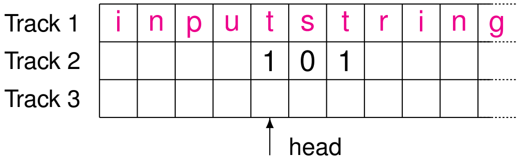

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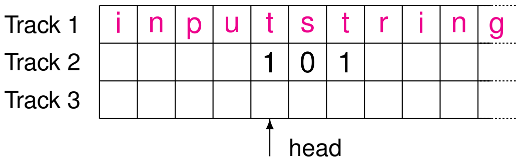

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How to count input symbols

Track 1	i	n	p	u	t	s	t	r	i	n	g
Track 2					1	0	1				
Track 3											

head

- The counter is kept on track 2, *starting from the position scanned by the tape head*
- When the head must be moved to the right, counting one more input position, the counter is incremented and shifted to the right
- This is done in $O(\log j)$ steps (where j is the value of the counter) using track 3 as an auxiliary variable.
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How to compute n mod k

($n =$ input length, $k =$ an integer written somewhere)

We adapt the previous technique:

- The counter on track 2 is reset each time it becomes equal to k.
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- To implement the comparison between the counter and k:
  - The value of k is kept on one extra track (track 4)
- The total time is $O(n \log k)$

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A unary language accepted in weak time $O(n \log \log n)$

- For each integer n let

$$q(n) := \text{ the smallest integer that does not divide } n$$

- We consider the language

$$L = \{a^n \mid q(n) \text{ is not a power of } 2\}$$

- L can be recognized using the following nondeterministic algorithm (Mereghetti, 2008):

input a^n
guess an integer s, $s > 1$
guess an integer t, $2^s < t < 2^{s+1}$
if $n \bmod 2^s = 0$ **and** $n \bmod t \neq 0$ **then accept**
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A unary language accepted in weak time $O(n \log \log n)$

Hence, we have proved the following:

Theorem ([Pighizzini, 2009])

The language L is accepted by a nTM with

- $t(n) = O(n \log \log n)$
- $c(n) = O(\log \log n)$

under the weak *measure*

More on L

The language L and its complement have been widely studied in the literature. These are some results:

- L^c is accepted by a dTM with a separate worktape, using the minimum amount of space $O(\log \log n)$ [Alt and Mehlhron, 1975]
- The same space complexity can be achieved using the smallest possible number of input head reversals $O(\frac{\log n}{\log \log n})$ [Bertoni, Mereghetti, and Pighizzini, 1994]
- For L we can even do better: L is accepted by a one-way nTM with a separate worktape, using the minimum amount of space $O(\log \log n)$, under the weak measure

Hence, L seems to be a good example of nonregular language with “low” complexity.

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Final remarks

We considered the “border” between regular and nonregular languages, wrt to the time $t(n)$ and the length of crossing sequences $c(n)$.

Similar investigations can be (or have been) done (even for different classes of languages) wrt other resources:

- Space (e.g., [Szepietowski, 1994], [Mereghetti, 2008])
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We used of tape tracks, namely large alphabets, big-Oh notation, etc.

It is interesting to investigate what happens when we put stronger restrictions on the resources.

An interesting result in this line has been recently proved by Hemaspaandra, Mukherji and Tantau (2005):

> *Context-free languages are accepted by Turing machines with absolutely no space overhead*

The work space of the machine is:

- the finite state control
- the space that initially contains the input, with the restriction that *only a binary alphabet can be used on it*

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