# Converting Self-Verifying Automata into Deterministic Automata

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# Self-verifying machines

Standard machines (e.g. finite automata, pushdown automata, Turing machines) with nondeterministic transitions

The state set is partitioned in three groups:

- *accepting* states ("yes")
- *rejecting* states ("no")
- *neutral* states ("I do not know")

For each input word $x$ the following conditions must be satisfyied:

- At least one computation on input $x$ ends either in an accepting or in a rejecting state
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# Self-verifying machines

Some references:

- Ďuriš, Hromkovič, Rolim, and Schnitger (STACS 1997)
  Definition of the model in connection with the study of Las Vegas automata.

- Hromkovič and Schnitger (Information and Comp. 2001)
  Hromkovič and Schnitger (SIAM J. Comp. 2003)
  Further investigations in connection with Las Vegas computations and also *per se*.

- Assent and Seibert (RAIRO-ITA 2007)
  Simulation of self-verifying automata by deterministic automata.

# Basic properties

- Trivial complementation
- Given nondeterministic machines $M'$ and $M''$ for $L$ and $L^c$, we can build a self-verifying machine $M$ for $L$ as the "union" of $M'$ and $M''$, with a new initial state:




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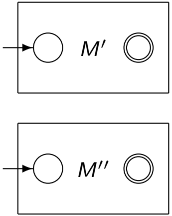


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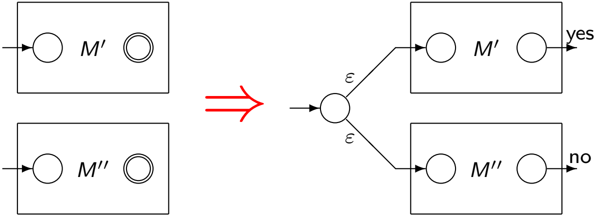


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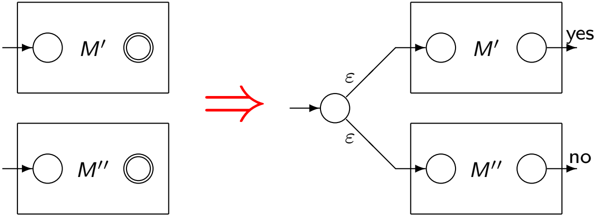


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$A = (Q, \Sigma, \delta, q_0, F^a, F^r)$ where:

- $Q$ is the finite set of states
- $\Sigma$ is the input alphabet
- $q_0 \in Q$ is the initial state
- $\delta : Q \times \Sigma \to 2^Q$ is the transition function
- $F^a \subseteq Q$ is the *set of accepting states*
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  namely, for each string there exists at least one accepting computation or one rejecting computation
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# Languages of svfa's

We associate with an svfa $A$ the following languages:

- The set of strings *accepted* by $A$:

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*First question*

What is the class of languages accepted by svfa's?

The answer to this question is easy:

- Each svfa is a nondeterministic automaton
- Each deterministic automaton is also an svfa

Hence:

Svfa's characterize the class of *regular languages*

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In this work we further investigate this problem:

- We reduce the upper bound to a function $g(n)$ which grows like $3^{\frac{n}{3}}$
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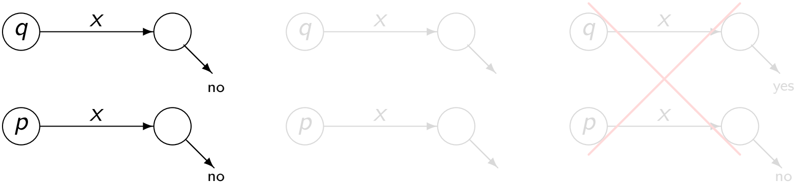

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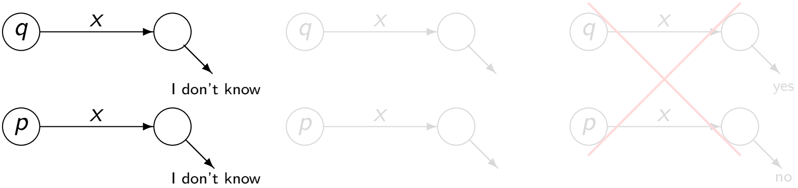

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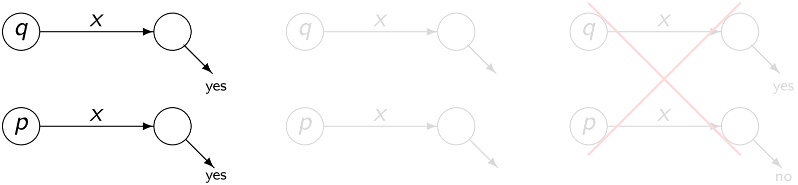

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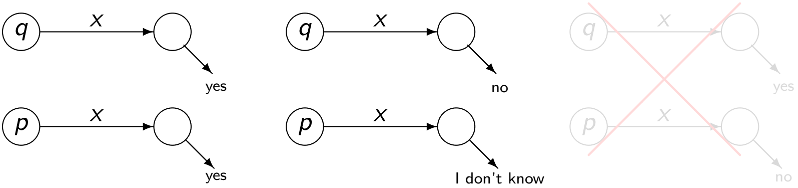

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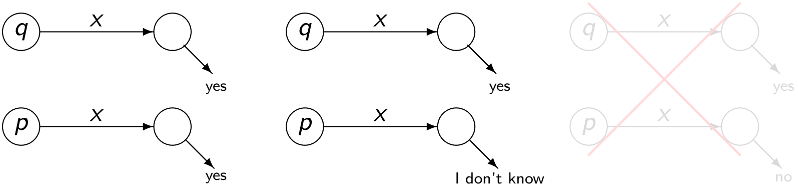

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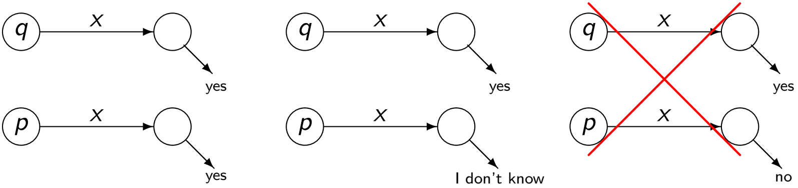

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# Properties of the subset automaton

Let $\alpha$ be a state of the subset automaton $A_{sub}$. Then:

Each two states $q, p \in \alpha$ are compatible

*Proof*
If $q, p \in \alpha$ are not compatible then:




Given a string $y$ s.t. $\alpha$ is reached on $y$, the original svfa on $yx$ should give contradictory answers!

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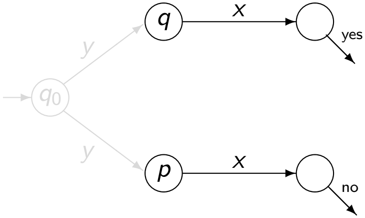


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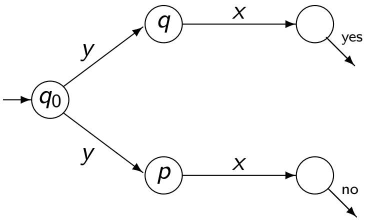


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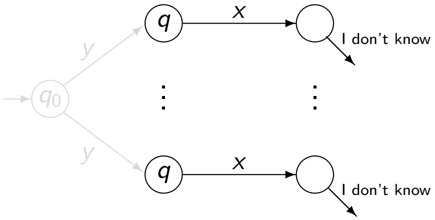


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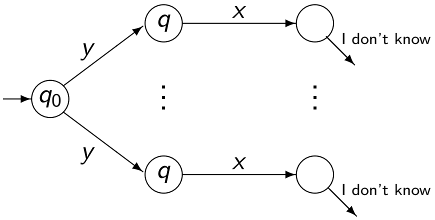


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# Compatibility graph

We define the following *compatibility graph* $G$, associated with the given svfa $A$:

- The nodes of $G$ are the states of $A$
- Two states $q, p$ are connected by an edge iff $q$ and $p$ are compatible

Hence:

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If $\alpha \cup \beta$ is a clique of $G$ then $\alpha$ and $\beta$ are equivalent

*Proof*

By contradiction, let $x$ be a string distinguishing $\alpha$ and $\beta$:




subset automaton $A_{sub}$

Then $\exists q \in \alpha$, $p \in \beta$ s.t.:




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This should imply that $q$ and $p$ are not compatible.

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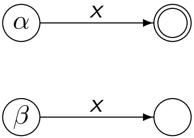


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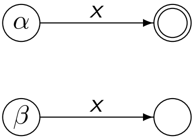


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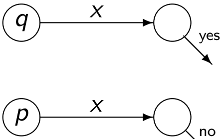


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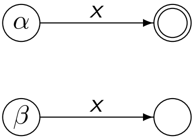


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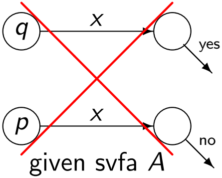


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# Subset automatom

By the previous properties:

- Each state of $A_{sub}$ corresponds to a clique of the compatibility graph $G$
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How many maximal cliques can a graph with $n$ nodes have?

This question was answered by Moon and Moser (1965).

They proved the following *exact bound* $f(n)$ for the maximum number of maximal cliques in a graph with $n$ nodes:

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Using the result of Moon and Moser, we can prove that

Each $n$-state svfa's can be simulated by a dfa with at most $g(n) = 1 + f(n-1)$ states

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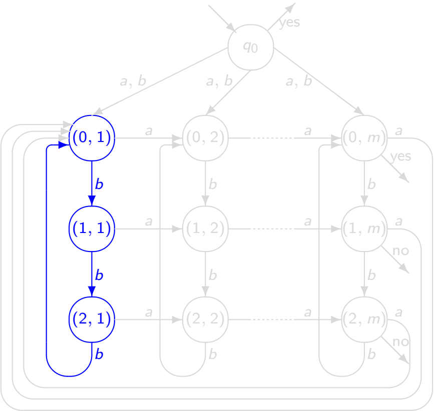

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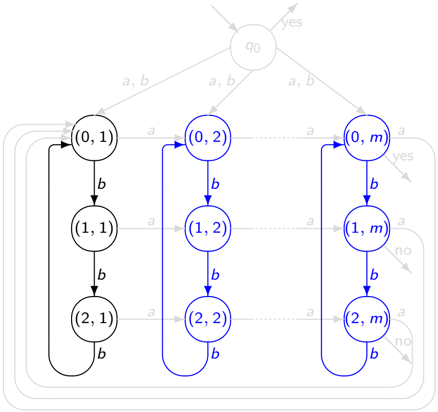

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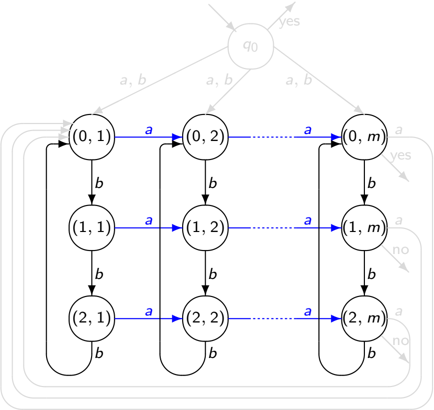

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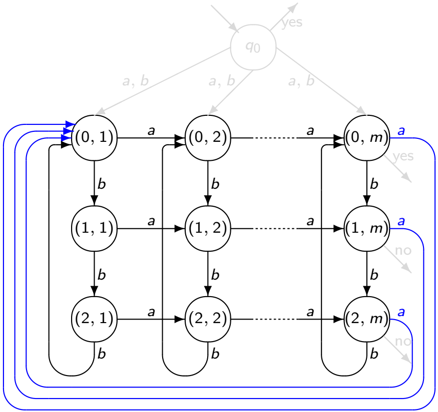

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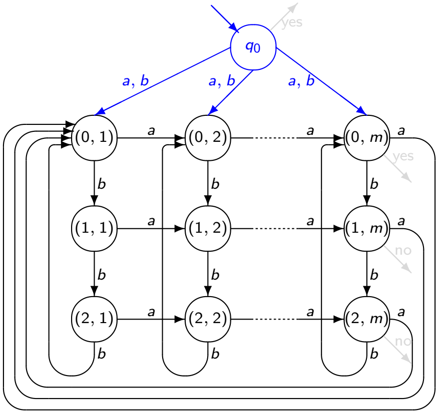

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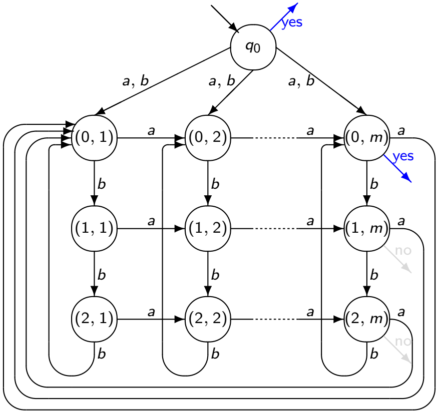

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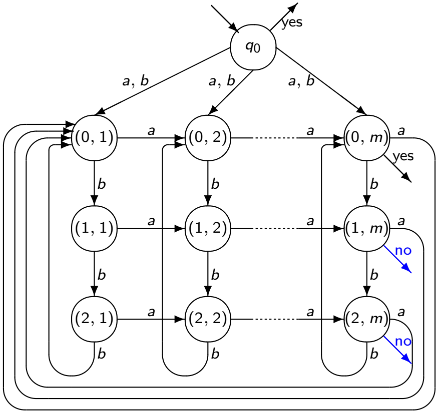

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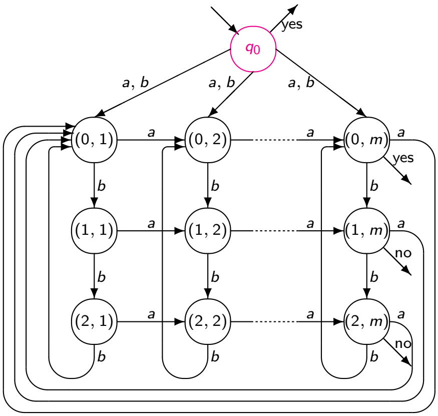

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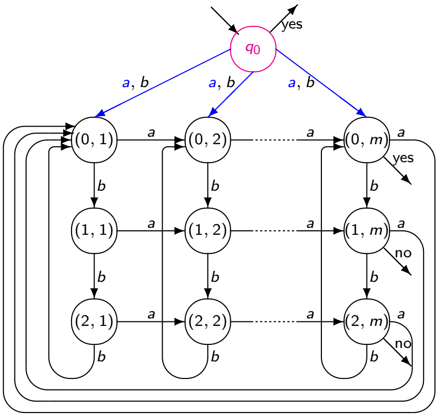


*a*

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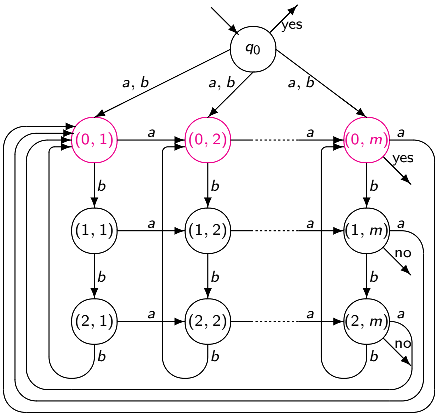


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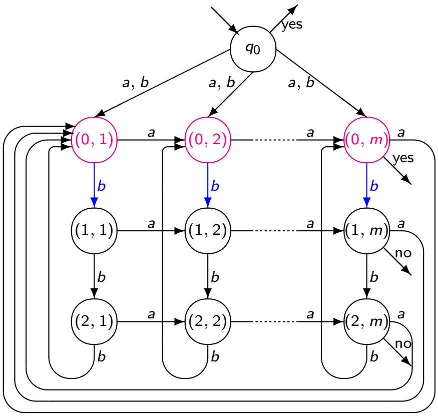


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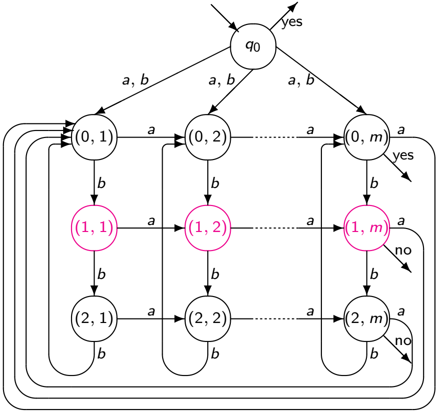


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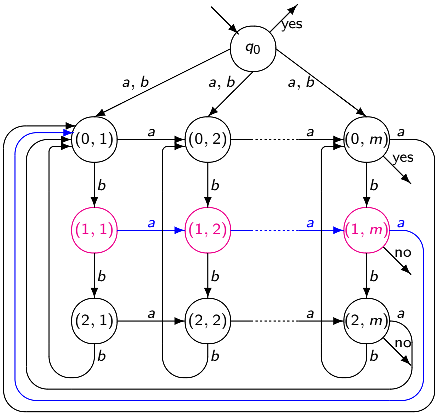


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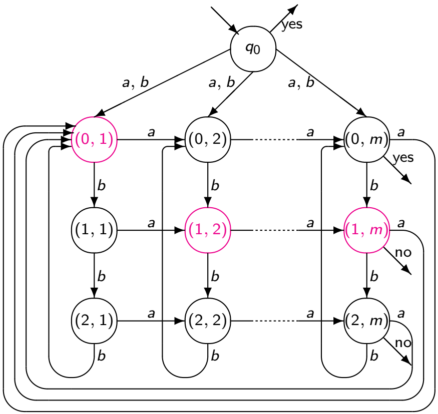


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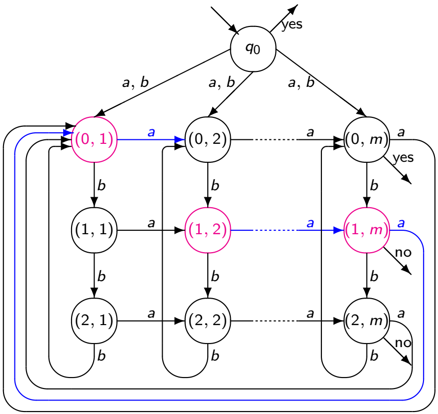

yes
$q_0$
a, b
a, b
a, b
(0, 1)
a
(0, 2)
a
(0, m)
a
yes
b
b
b
(1, 1)
a
(1, 2)
a
(1, m)
a
no
b
b
b
(2, 1)
a
(2, 2)
a
(2, m)
a
no
b
b
b


$abaa$

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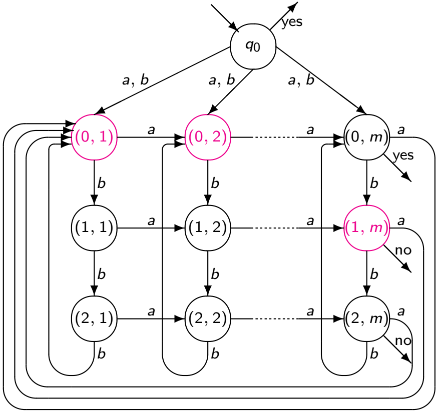


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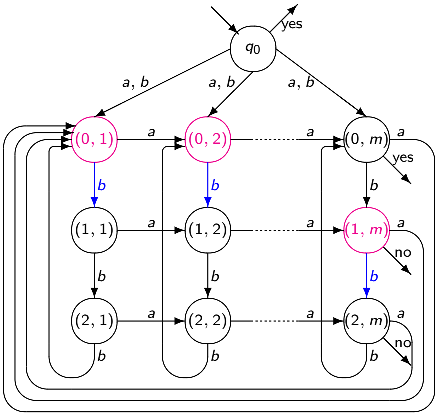


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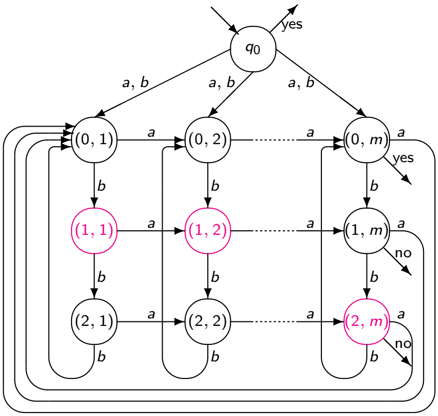


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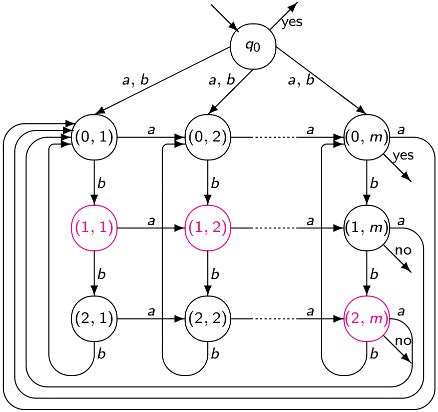
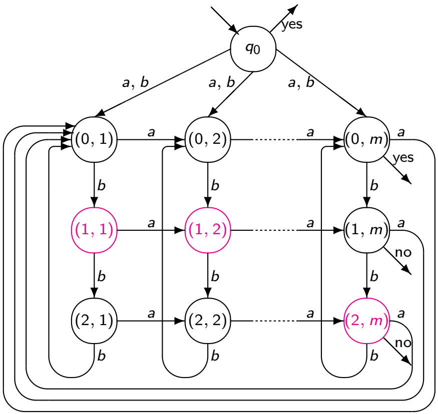



The reachable states of the subset automaton $A_{sub}$ are:

- $\{q_0\}$
- the $3^m$ subsets obtained by taking one state from each column in the "grid part" (hence $A_n$ is an svfa!)




- We can verify that each two states of $A_{sub}$ are distinguishable

# Properties of $A_n$

Summing up:

- The subset automaton $A_{sub}$ has exactly $g(n) = 1 + 3^{\frac{n-1}{3}}$ states
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- Hence, it is the minimal dfa equivalent to $A_n$
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Hence:

the *exact cost* for the conversion of $n$-state svfa's into equivalent dfa's is:

$$g(n) = \begin{cases} 1 + 3^{\frac{n-1}{3}} & \text{if } n \equiv 1 \pmod 3 \text{ and } n \geq 4 \\ 1 + 4 \cdot 3^{\frac{n-2}{3}-1} & \text{if } n \equiv 2 \pmod 3 \text{ and } n \geq 5 \\ 1 + 2 \cdot 3^{\frac{n}{3}-1} & \text{if } n \equiv 0 \pmod 3 \text{ and } n \geq 3 \\ n & \text{if } n \leq 2 \end{cases}$$

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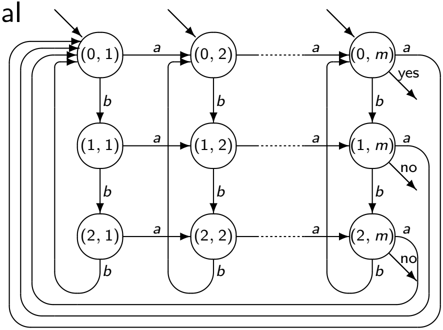

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**What about the optimality in the unary case?**

- We proved the optimality using automata over a binary alphabet
- The cost of the conversion of unary nfa's into dfa's is $F(n) = e^{O(\sqrt{n \log n})}$ (Chrobak, 1986)
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- Each $n$-state svfa can be converted into an equivalent dfa with $g(n)$ states:
  - We found the value of $g(n)$, which grows like $3^{\frac{n}{3}}$
  - *The bound is exact:*
    for each integer $n$, there exists an svfa $A_n$ with $n$ states and an input alphabet of two letters such that the minimal equivalent dfa has $g(n)$ states.

- Each $n$-state svfa with multiple initial states can be converted into an equivalent dfa with $f(n) = g(n+1) - 1$ states.
  *Also this bound is exact.*

- In the unary case, a better upper bound is given by the function $F(n) = e^{O(\sqrt{n \ln n})}$.
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