Simulating Unary Context-Free Grammars and Pushdown Automata with Finite Automata

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Outline of the talk

1. Introduction
2. Simulation of unary cfg's by nfa's
3. Simulation of unary cfg's by dfa's
4.

The results presented at points 2, 3, 4, and 5 are from [Pighizzini, Shallit, Wang, 2002].

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Descriptional complexity

Given

- $\mathcal{C}$, a class of languages
- $\mathcal{S}$, a formal system (e.g., class of devices, class of grammars,..) able to represent all the languages in $\mathcal{C}$

What is the *size* of the representations of the languages in $\mathcal{C}$ by the system $\mathcal{S}$?

Usually, descriptional complexity compares different description for a same class of languages:

- given $\mathcal{S}'$, another formal system able to represent all the languages in $\mathcal{C}$

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- Formal language point of view:

  nondeterministic finite automata are as powerful as deterministic finite automata

- Descriptional complexity point of view:

  Each n-state nfa can be simulated by a 2^n state dfa (upper bound)

  For each integer n there exists a language accepted by an n-state nfa such that the minimun equivalent dfa requires 2^n states

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- Different variant of finite automata (one-way/two-way, deterministic/nondeterministic/alternating, ...).
- All of them characterize regular languages
- Many results in the literature compare these models from the descriptional point of view.
- However, we can describe regular languages using more powerful devices or formalisms, as context-free grammars and pushdown automata.

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Context-free vs regular: descriptional complexity

Given a context-free grammar (or a pushdown automaton) of size n, generating a regular language, how much is big an equivalent finite automaton, wrt n ?

Theorem ([Meyer and Fischer, 1971])

For any recursive function f and arbitrarily large integers n, there exists a cfg G of size n generating a regular language L, s.t. any dfa accepting L must have at least $f(n)$ states.

As a consequence, the trade-off between context-grammars and finite automata is not recursive.

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Unary languages

$\Sigma = \{a\}$

Theorem ([Ginsurg and Rice, 1962])

Every unary context-free language is regular.

Hence the classes of unary regular languages and unary *context-free* languages coincide!

Problem

Study the equivalence between unary context-free and regular languages from the descriptional complexity point of view.

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- Deterministic automata (dfa):
  number of states.
- Nondeterministic automata (nfa):
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- Context-free grammars:
  number of variables?

  For $n \geq 1$, consider the language $L_n = (a^n)^*$:
  - L_n requires n states to be accepted by dfa or nfa
  - L_n is generated by the grammar with one variable S and the productions

$$S \to a^n \qquad S \to a^n S \qquad S \to \epsilon$$

Thus, the number of variables cannot be a descriptional complexity measure for context-free grammars.
However, for grammars in *Chomsky Normal Form* the number of variables is a "reasonable" measure of complexity [Gruska, 1973].

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- Pushdown automata:
  W.l.o.g., we can consider only pda's s.t. push operations add exactly one symbol on the pushdown store.

  The total size of the description of a pda satisfying this restriction is a polynomial function of two parameters:
  - the number of the states
  - the cardinality of the pushdown alphabet.

  Strong relationships have been discovered between descriptional complexities of cfg's and pda's [Goldstine, Price, Wotschke, 1982], e.g.:

Theorem

For any pda satisfying the above restriction, with n states and m pushdown symbols, there exists an equivalent cfg in Chomsky normal form, with $n^2m + 1$ variables.

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- $G = (V, \{a\}, P, S)$
  a cfg grammar in Chomsky normal form
- h
  the number of variables in G
- $T : A \stackrel{\star}{\Rightarrow} \alpha$
  a *parse tree* for the derivation $A \stackrel{\star}{\Rightarrow} \alpha$, with $A \in V$, $\alpha \in (V \cup \{a\})^*$.
- $\nu(T)$
  the set of variables which appear as labels of some nodes in T

Unary cfg $\longrightarrow$ nfa

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Lemma (padding)

Given

- $T : S \stackrel{\star}{\Rightarrow} a^\ell$
- $T' : A \stackrel{+}{\Rightarrow} a^i A a^j$, *with* $A \in \nu(T)$

there exists

- $T'' : S \stackrel{\star}{\Rightarrow} a^{\ell+i+j}$, *with* $\nu(T'') = \nu(T) \cup \nu(T')$

i.e., T can be "padded" with T' in order to get T''.

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Lemma (decomposition)

If $T : S \overset{\star}{\Rightarrow} a^\ell$ *and* $\ell > 2^{h-1}$*, then there exist:*

- *a tree* $T_1 : S \overset{\star}{\Rightarrow} a^s$
- *a tree* $T_2 : A \overset{+}{\Rightarrow} a^i A a^j$*, with* $A \in \nu(T_1)$

such that:

- $\nu(T) = \nu(T_1) \cup \nu(T_2)$
- $\ell = s + i + j$*,* $s > 0$ *and* $0 < i + j < 2^h$*.*

In other words, a tree $T : S \overset{\star}{\Rightarrow} a^\ell$ of a "long" string ($\ell > 2^{h-1}$) can be obtained by padding

- a tree $T_1 : S \overset{\star}{\Rightarrow} a^s$ of shortest string
- with a "small" tree $T_2 : A \overset{+}{\Rightarrow} a^i A a^j$, where $A \in \nu(T_1)$ ("small" means $0 < i + j < 2^h$)

Notice that if $s > 2^{h-1}$ then the tree T_1 can be obtained in a similar way.

Unary cfg $\longrightarrow$ nfa

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Unary cfg $\rightarrow$ nfa

MAIN IDEA:

we can generate all strings belonging to $L(G)$ by using “small” trees corresponding to derivations of the forms

- $S \overset{\star}{\Rightarrow} a^{\ell}$ and
- $A \overset{+}{\Rightarrow} a^i A a^j$,

where $\ell \leq 2^{h-1}$ and $0 < i + j < 2^h$.

Generation procedure for $L(G)$

nondeterministically select a tree $T_1 : S \overset{\star}{\Rightarrow} a^\ell$, with $\ell \leq 2^{h-1}$
enabled $\leftarrow \nu(T_1)$
iterate $\leftarrow$ nondeterministically choose *true* or *false*
while *iterate* **do**
    nondeterministically select a tree $T_2 : A \overset{+}{\Rightarrow} a^i A a^j$,
        with $0 < i + j < 2^h$ and $A \in$*enabled*
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output a^ℓ

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Unary cfg $\to$ nfa

- The strategy of the generation procedure can be implemented by a nfa with ϵ-moves A.
- The states of A are pairs (α, s), where:
  - $\alpha \subseteq V$ represents the variable *enabled*
  - $s < 2^h$ is used to count input factors
- After some simplifications, the number of the states of such an automaton can be reduced to $2^{2h-1} + 1$.

Hence

Theorem (upper bound)

For any unary cfg in Chomsky normal form with h variables, there exists an equivalent nfa with at most $2^{2h-1} + 1$ states.

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Unary cfg → nfa (lower bound)

Theorem (lower bound)

For any integer $h \geq 1$, there exists a unary cfg in Chomsky normal form with h variables, such that any equivalent nfa must have at least $2^{h-1} + 1$ states.

(Sketch of the proof)

For $h > 1$, consider variables $A_0, \ldots, A_{h-1}$ and productions:

- $A_0 \rightarrow a$
- $A_j \rightarrow A_{j-1}A_{j-1}$, for $j = 1, \ldots, h-2$
- $A_{h-1} \rightarrow A_{h-2}A_{h-2} \mid A_{h-1}A_{h-1}$

Then, for $j = 0, \ldots, h-2$: $A_j \overset{\star}{\Rightarrow} a^x$ iff $x = 2^j$.

If A_{h-1} is the start symbol, the language is $(a^{2^{h-1}})^+$

This language needs $2^{h-1} + 1$ states to be accepted by an nfa.

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Let G be a unary cfg in Chomsky normal form with h variables.

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- Using the subset construction, we can get an equivalent dfa with $2^{2^{O(h)}}$ states.

...we can do better!

Theorem

$L(G)$ is accepted by a dfa with 2^{h^2} states.

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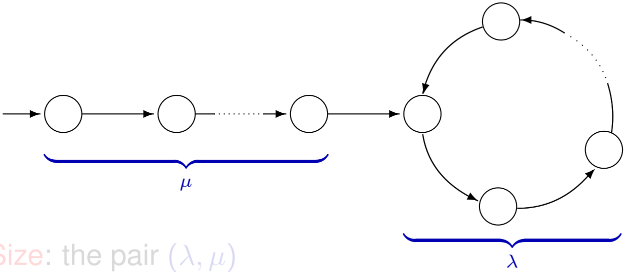
Unary deterministic automata



Size: the pair (λ, μ)

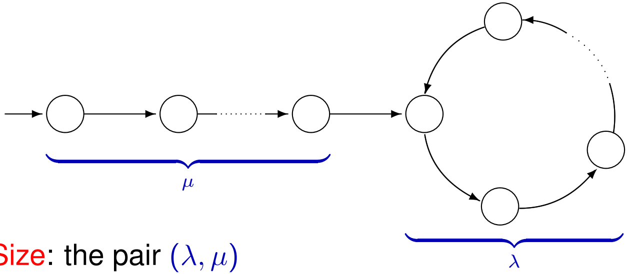
Theorem ([Pighizzini and Shallit, 2002])

- $L_1, \ldots, L_k$ *unary regular languages*
- L_i *accepted by a dfa of size* (λ_i, μ_i), $i = 1, \ldots, k$.

Then $\bigcup_{i=1}^{k} L_i$ *is accepted by a dfa of size*

$$(lcm(\lambda_1, \ldots, \lambda_k), \max(\mu_1, \ldots, \mu_k))$$

Unary deterministic automata



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- $L_1, \ldots, L_k$ *unary regular languages*
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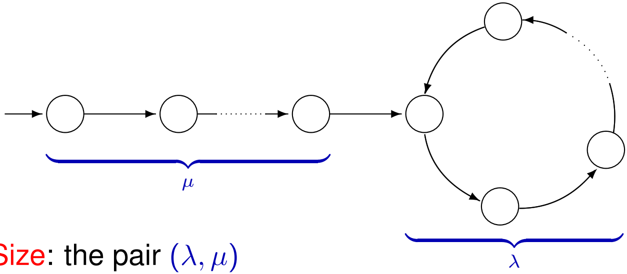

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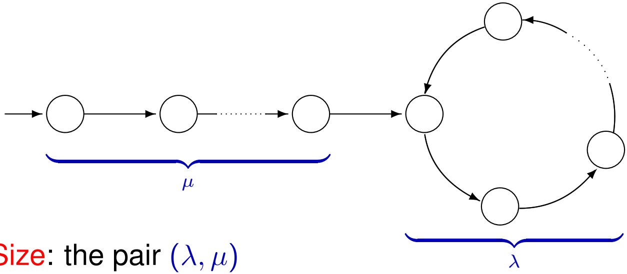

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Unary cfg $\longrightarrow$ dfa (1)

Given a variable $A \in V$:

- Let a L_A be the set of strings in L generated using A
- A is said to be *cyclic* iff $A \stackrel{+}{\Rightarrow} a^i A a^j$, for some i, j s.t. $0 < i + j < 2^h$,
- If A is cyclic:
  - we set $\lambda_A = i + j$, for an arbitrary chosen pair of integers (i, j) satisfying the above condition.
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Notice that $L = L_S$. Hence, *if S is cyclic:*

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If S is not cyclic, then we decompose L as:

$$L = L^{\leq 2^{h-1}} \cup \bigcup_{A \in V_p} L_A$$

where V_p denotes the set of cyclic variables.

- $L^{\leq 2^{h-1}}$ is accepted by a dfa of size $(1, 2^{h-1} + 1)$.
- Hence, L is accepted by a dfa of size (λ, μ), where
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Summarizing:

Theorem

For any unary cfg in Chomsky normal form with $h \geq 2$ variables, there exists an equivalent dfa with less than 2^{h^2} states.

The upper bound is tight!!!

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There is a constant $c > 0$ s.t., for infinitely many integers $h > 0$, there exists a unary cfg in Chomsky normal form with h variables, s.t. any equivalent dfa must have 2^{ch^2} states.

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For any unary pushdown automaton with

- n *states*
- m *pushdown symbols*

s.t. each push add one symbol on the stack, there exist

- *an equivalent nfa with at most* $2^{2n^2m+1}+1$ *states*
- *an equivalent dfa with less than* $2^{n^4m^2+2n^2m+1}$ *states.*

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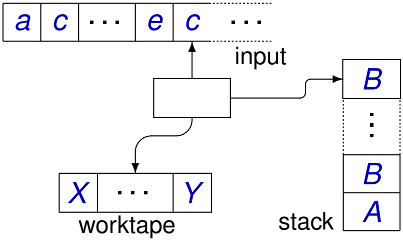
Auxiliary pushdown automata (auxpda)

Turing machines augmented with a pushdown store

or, equivalently

(2way) pda augmented with an auxiliary worktape



"SPACE" $\equiv$ worktape

Theorem ([Cook 1971])

Given $L \subseteq \Sigma^*$, $s(n) \geq \log n$, the following statements are equivalent:

1. *L is accepted in $s(n)$ space by a nondeterministic auxpda*
2. *L is accepted in $s(n)$ space by a deterministic auxpda*
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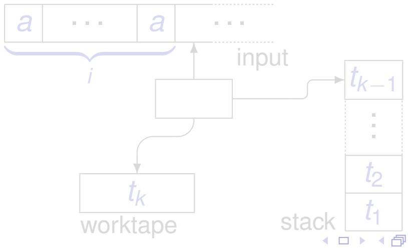
The input head can be moved only to the right

Example: how to count the input length.

- Given $i \in \mathbf{N}$ consider its binary representation.
- Let $t_1 > \ldots > t_k$ be the sequence of the positions of digits 1, i.e., $i = 2^{t_1} + 2^{t_2} + \ldots + 2^{t_k}$.
- The auxdpa can store i (the length of the scanned input prefix) as follows:



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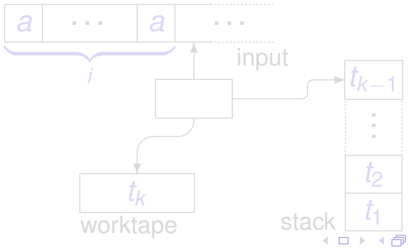

One-way auxiliary pda (1auxpda) [Brandenburg 1977]

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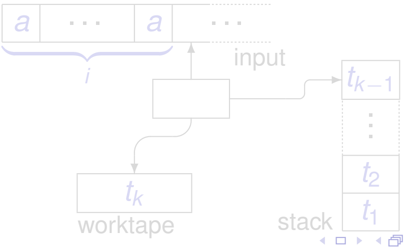

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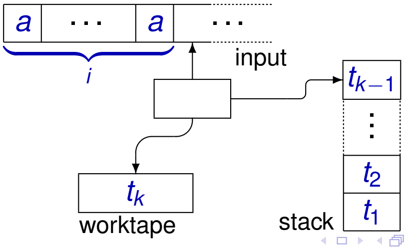

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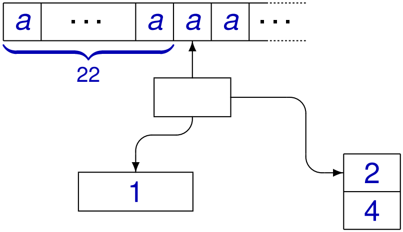

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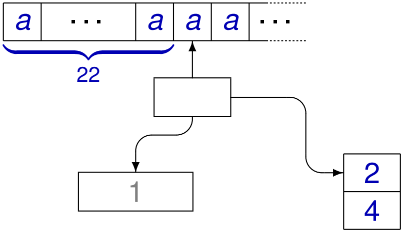

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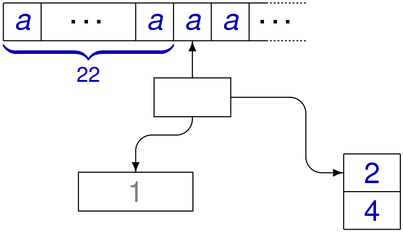

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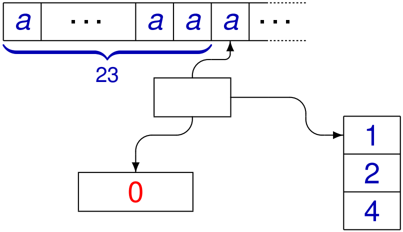

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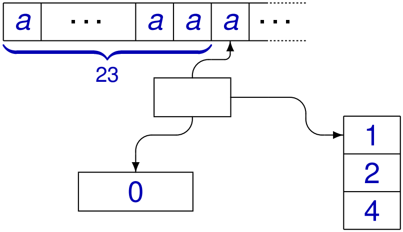

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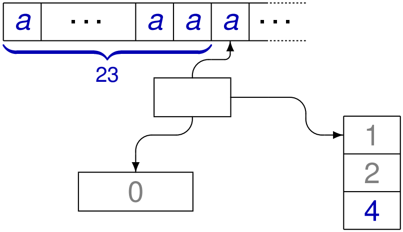

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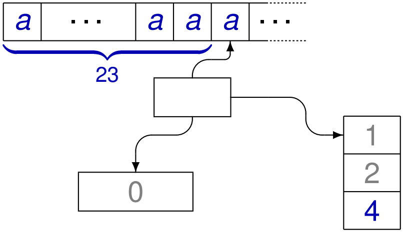

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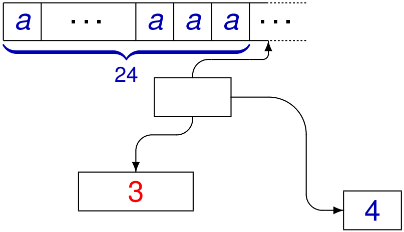

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Let $\mathcal{L} = \{a^{2^k} \mid k \geq 0\}$.

We define a 1auxpda M working as follows:

- M scans the input tape, counting the input length.
- When the end of the input is reached, M accepts if and only if the pushdown store is empty.
- If n is the input length, then the largest integer stored on the worktape is $\lfloor \log_2 n \rfloor$.
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One-way auxiliary pda (1auxpda)

Problem ([Brandenburg 1977])

What is the minimum amount of space $s(n)$ s.t. 1auxpda working in $s(n)$ space are able to accept noncontext-free languages?

space $s(n)$:

STRONG: *any computation on each input* of length n uses no more than $s(n)$ worktape cells.

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Theorem ([Brandenburg 1977])

$L \subseteq \Sigma^*$ noncontext-free language accepted by a 1auxpda in strong $s(n)$ space.
Then there exists $c > 0$ such that

$$s(n) \geq c \log \log n$$

infinitely often.

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For each accepted input, at least one accepting computation satisfies the bound.

Theorem ([Chytil 1986])

1. *For each integer $k \geq 2$ there is a language L_k such that*
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space bounds for 1auxpda

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What about the unary case?

Automaticity [Shallit and Breitbart,1996]

- $A_L : \mathbf{N} \to \mathbf{N}$ *automaticity* of a language $L \subseteq \Sigma^*$:

$$A_L(n) = \text{minimum number of states of a dfa accepting a language } L', \text{ s.t. } L^{\leq n} = L'^{\leq n}$$

i.e., L and L' agree on strings of length $\leq n$.

- If L is regular the $A_L(n)$ is a constant (the size of the minimal dfa accepting L).

What about the automaticity of nonregular languages?

Theorem ([Karp, 1971])

Let $L \subseteq \Sigma^*$ be a nonregular language. Then:

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- M: unary 1auxpda accepting in weak $s(n)$ space a nonregular language L.
- M_n: a pda whose states encode the configurations of M using $s(n)$ space, for a given $n \geq 1$.
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Let M be a unary auxpda accepting a non-context-free language L in weak $s(n)$ space. Then $s(n) \notin o(\log \log n)$.

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Bounded languages:
Subsets of $w_1^* w_2^* \dots w_n^*$, for given words $w_1, \dots, w_n$
(*letter bounded* if $w_1, \dots, w_n \in \Sigma$).

- The class bounded regular languages is *properly* included in that of bounded cfl's, e.g., $\{a^n b^n \mid n \geq 0\}$.
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- The *generation procedure* used to simulate unary cfg's with nfa's can be extended in order to simulate cfg's generating *letter bounded* languages with finite turn pda's.
- Using a suitable homomorphism such a simulation can be extended also to the case of *bounded* languages

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Even this upper bound in tight.

In particular:

For all integers $m \geq 1, h \geq 1$ there exists a language $L_{m,h} \subseteq a_1^* a_2^* \ldots a_m^*$ s.t.:

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Further investigations: dpda's vs finite automata

Let M be a unary pda with n states and m stack symbols, s.t. each push adds exactly one symbol.

We proved that M can be simulated by a dfa with with $2^{O(n^4 m^2)}$ states.

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- 1auxpda's with an input alphabet of at least two symbols can recognize noncontext-free languages using very slowly increasing (but nonconstant) weak space.
- Unary 1auxpda's must use weak space growing at least as $\log\log n$ to recognize noncontext-free languages.

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