Deterministic Pushdown Automata and Unary Languages

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- Context-free grammars and pda's vs regular languages: some descriptional complexity results
- Exponential simulation of unary dpda's by dfa's
- Optimality of the simulation

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Context-free vs regular: descriptional complexity

Given a context-free grammar (or a pushdown automaton) of size n, generating a regular language, how much is big an equivalent finite automaton, wrt n ?

Theorem ([Meyer and Fischer, 1971])

For any recursive function f and arbitrarily large integers n, there exists a cfg G of size n generating a regular language L, s.t. any dfa accepting L must have at least $f(n)$ states.

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Every unary context-free language is regular.

Hence the classes of unary regular languages and unary *context–free* languages coincide!

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Unary case [Pighizzini, Shallit, Wang, 2002]

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For any cfg in Chomsky normal form with h variables, generating a unary language, there exists an equivalent dfa with 2^{h^2} states. Furthermore, this bound is tight.

Corollary

Each unary pda with n states and m stack symbols, s.t. each push adds exactly one symbol, can be simulated by a dfa with $2^{O(n^4 m^2)}$ states.

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- Each dpda of size s accepting a regular language can be simulated by a dfa with $2^{2^{2^s}}$ states. [Stearns, 1967]
- This upper bound was reduced to 2^{2^s} in [Valiant, 1975].
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  Number of its states
- Size of a pushdown automaton:
  Total number of symbols needed to write down its description.
  We have to keep into account:
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Unary dfa's

Input alphabet $\Sigma = \{a\}$

Theorem

$L \subseteq \{a\}^*$ is regular iff $\exists \mu \geq 0, \lambda \geq 1$ *s.t.*

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L.$$

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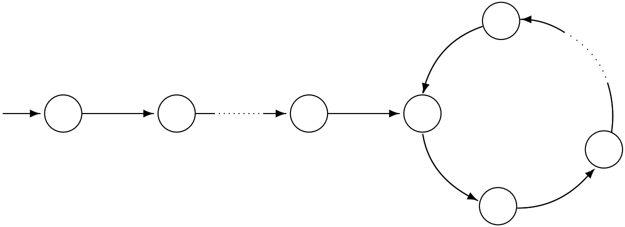

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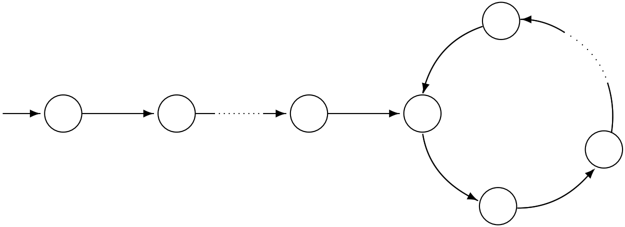

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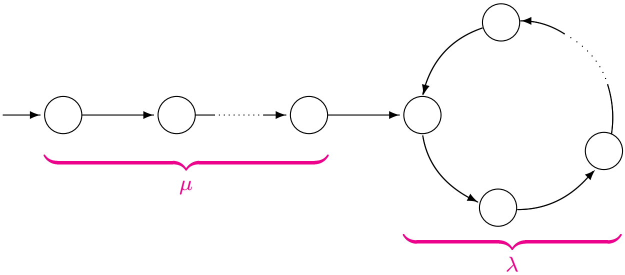

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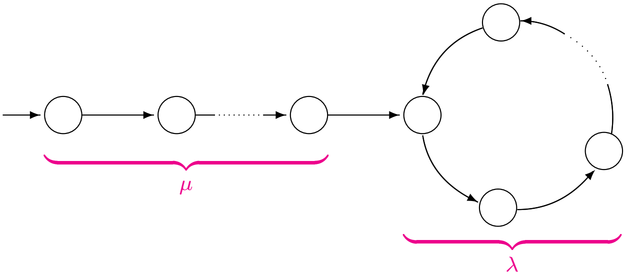

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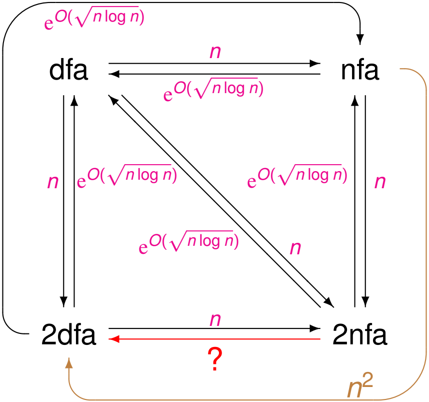
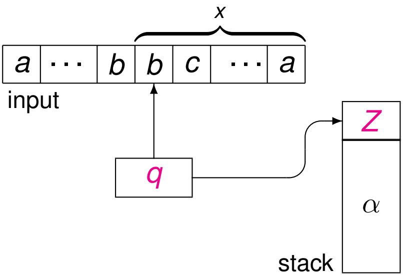


Pushdown automata

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$



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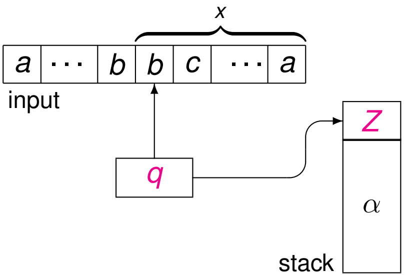

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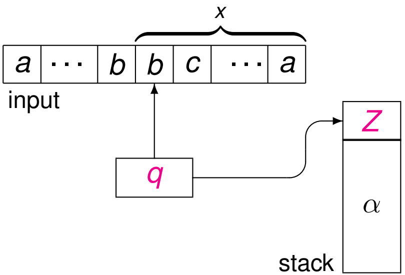

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- Deterministic cfl's: acceptance by *final states*

$$L(M) = \{x \in \Sigma^* \mid (q_0, x, Z_0) \overset{*}{\vdash} (q, \epsilon, \gamma), q \in F, \gamma \in \Gamma^*\}$$

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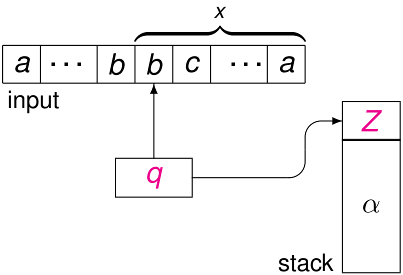

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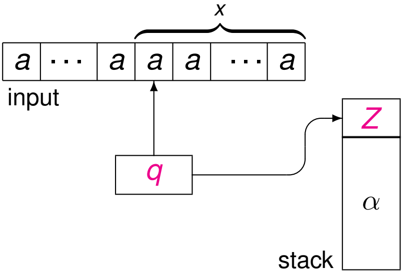

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For each integer $t \geq 0$:

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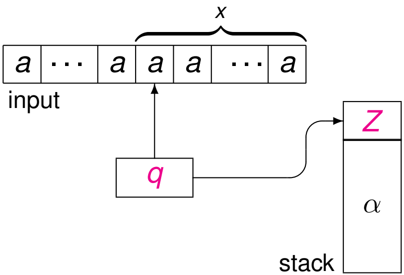

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$[qA] \leq [pB]$ iff all the following conditions hold:

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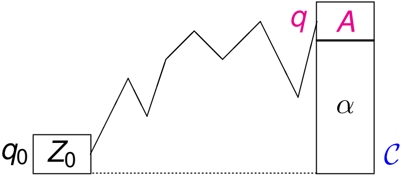

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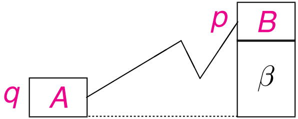

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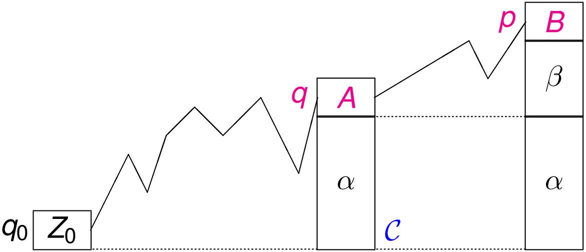

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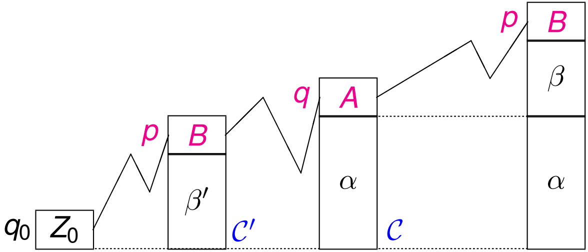

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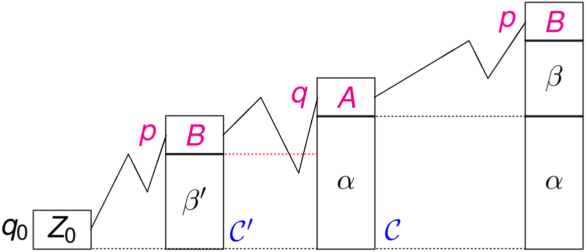

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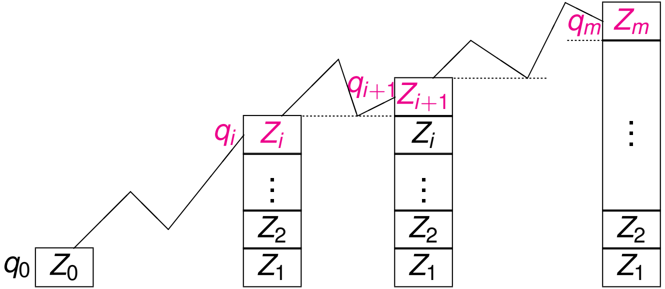
Lemma

The relation $\leq$ defines a partial order on the set of the modes.

Stack content + state information



History h_t at the time t:
sequence of modes $[q_m Z_m] \ldots [q_1 Z_1]$ s.t.:

- $Z_m \ldots Z_1$ is the stack content after t computation steps
- for $i = 1, \ldots, m$, $[q_i Z_i]$ is the mode of the last visited configuration with stack height i

Histories and modes

For each step $t \geq 0$ we consider:

- h_t history
- m_t mode (leftmost element of h_t)

For the given dpda M we consider:

- $H = \{h_t \mid t \geq 0\}$, the set all *reachable histories*
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Two possibilities:

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- H is finite
- The given dpda can be simulated by a deterministic automaton A whose set of states is H
- The number of the states of A is bounded by the number of histories without repetitions
- If an history $[q_m Z_m] \ldots [q_1 Z_1]$ does not contain any repetition, then $[q_1 Z_1] \leq [q_2 Z_2] \leq \ldots \leq [q_m Z_m]$
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- The number of the states of A is bounded by the number of histories without repetitions
- If an history $[q_m Z_m] \ldots [q_1 Z_1]$ does not contain any repetition, then $[q_1 Z_1] \leq [q_2 Z_2] \leq \ldots \leq [q_m Z_m]$
- Hence:

  the number of states of the deterministic automaton A is bounded by $2^{\#Q \cdot \#\Gamma}$

Case 2: At least one history in H contains a repetition

- The histories in H grow in a periodic way, i.e.:
  $\exists \mu \geq 0, \lambda \geq 1$, $\exists$ sequences of modes $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_\lambda$ s.t. for $t \geq \mu$, the history at the step t is:

$$h_t = \tilde{h}_{t \bmod \lambda} (\tilde{h}_\lambda)^{\lfloor \frac{t-\mu}{\lambda} \rfloor} h_\mu$$

- The sequence $(m_t)_{t \geq 0}$ is ultimately periodic (period λ, from $t \geq \mu$)
- The language can be accepted by a deterministic automaton A with at most $\lambda + \mu$ states
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Unary dpda's vs dfa's

As a consequence:

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Each unary dpda of size s can be simulated by a dfa with $2^{O(s)}$ states.

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Unary dpda's vs dfa's: lower bound

Given $s > 0$ consider $L_s = (a^{2^s})^*$.

We can prove that:

- There exists a dpda of size $8s + 4$ accepting L_s.
- Each dfa accepting L_s must have at least 2^s states.

Hence our simulation is optimal!

Problem: Does it is possible to reduce the cost of the simulation of unary dpda's, by using *nondeterministic* or *two-way* finite automata?

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Languages with “complex” dpda’s

Unary dpda’s can be exponentially more succinct than dfa’s.
Does this is true for *each* unary regular language?

Problem

For $m \geq 0$, let $L_m \subseteq a^*$ be a language accepted by a dfa with 2^m states.
Does there exists an equivalent dpda with $O(m)$ states?

The answer to this question is negative:

For each $m > 0$ there exists a language $L_m \subseteq a^*$ s.t.:

- L_m is accepted by a dfa with 2^m states.
- The size of any dpda accepting L_m is at least $d\frac{2^m}{m^2}$, for a constant d.

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Languages with “complex” dpda’s

w_m **de Bruijn word** of order m on $\{0, 1\}$:

- $|w_m| = 2^m + m - 1$
- each string of length m is a factor of w_m, occurring in w_m exactly one time
- the suffix and the prefix of length $m - 1$ of w_m coincide.

Example: $w_3 = 0001011100$

$L_m = \{a^k \mid \text{the letter of } w_m \text{ in position } k \text{ MOD}' 2^m \text{ is } 1\}$,
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Languages with “complex” dpda’s

- M: a dpda of size s accepting L_m
- M': M extended with an output tape to generate the de Bruijn word
- A: a dfa with $m+1$ states, input alphabet $\{0,1\}$, ending and accepting when the last m input symbols coincide with the suffix of length m of w_m
- M'': a dpda of size $O(ms)$, composition of M' and A, accepting $\{a^{2^m+m-1}\}$
- G: cfg grammar of size $O(ms)$, obtained from M'', generating $\{w_m\}$

Lemma ([Domaratzki, Pighizzini, Shallit, 2002])

The size of each grammar G generating $\{w_m\}$ must be at least $c\frac{2^m}{m}$, for some constant c.

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Simulation of unary dfa's by dpda's

As a consequence we get the following *lower bound*:

Corollary

There exists a constant $K > 0$ such that the conversion of unary n-state dfa's into equivalent dpda's produces dpda's of size at least $K\frac{n}{\log^2 n}$, for infinitely many n's.

Pda's vs cfg's

To prove the last result we have investigated the transformation of unary dpda's into context-free grammars.

- Each pda can be transformed into an equivalent cfg with $(\#Q)^2 \cdot \#\Gamma + 1$ variables.
- This number cannot be reduced, even if the given pda is deterministic [Goldstine, Price, Wotschke, 1982].

However, we proved that:

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Bounded languages:

Subsets of $w_1^* w_2^* \ldots w_n^*$, for given words $w_1, \ldots, w_n$.

Extend the investigation to *bounded deterministic context-free* languages:

- Simulation of dpda's accepting *bounded regular* languages, by finite automata.
- Simulation of dpda's accepting *bounded (context-free)* languages, by finite-turn pushdown automata.

In the nondeterministic case we have the following:

Theorem ([Malcher, Pighizzini, 2007])

Each bounded context-free language generated by a cfg with h variables in Chomsky normal form is accepted by a finite-turn pda with 2^h and $O(1)$ stack symbols.

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