Deterministic Pushdown Automata and Unary Languages

Giovanni Pighizzini

Dipartimento di Informatica e Comunicazione
Università degli Studi di Milano

Mathematical Institute
Slovak Academy of Sciences – Košice
November 13th, 2008

Outline of the talk

- Context-free grammars and pda's vs regular languages: some descriptional complexity results
- Exponential simulation of unary dpda's by dfa's
- Optimality of the simulation

Outline of the talk

- Context-free grammars and pda's vs regular languages: some descriptional complexity results
- Exponential simulation of unary dpda's by dfa's
- Optimality of the simulation
- Unary dpda's vs context-free grammars

Outline of the talk

- Context-free grammars and pda's vs regular languages: some descriptional complexity results
- Exponential simulation of unary dpda's by dfa's
- Optimality of the simulation
- Unary dpda's vs context-free grammars
- Simulation of unary dfa's by dpda's

Outline of the talk

- Context-free grammars and pda's vs regular languages: some descriptional complexity results
- Exponential simulation of unary dpda's by dfa's
- Optimality of the simulation
- Unary dpda's vs context-free grammars
- Simulation of unary dfa's by dpda's

Outline of the talk

- Context-free grammars and pda's vs regular languages: some descriptional complexity results
- Exponential simulation of unary dpda's by dfa's
- Optimality of the simulation
- Unary dpda's vs context-free grammars
- Simulation of unary dfa's by dpda's

Outline of the talk

- Context-free grammars and pda's vs regular languages: some descriptional complexity results
- Exponential simulation of unary dpda's by dfa's
- Optimality of the simulation
- Unary dpda's vs context-free grammars
- Simulation of unary dfa's by dpda's

Context-free vs regular: descriptional complexity

Given a context-free grammar (or a pushdown automaton) of size n, generating a regular language, how much is big an equivalent finite automaton, wrt n ?

Theorem ([Meyer and Fischer, 1971])

For any recursive function f and arbitrarily large integers n, there exists a cfg G of size n generating a regular language L, s.t. any dfa accepting L must have at least $f(n)$ states.

As a consequence, the trade-off between context-grammars and finite automata is not recursive.

However... The witness language L is defined over a binary alphabet.

Context-free vs regular: descriptional complexity

Given a context-free grammar (or a pushdown automaton) of size n, generating a regular language, how much is big an equivalent finite automaton, wrt n ?

Theorem ([Meyer and Fischer, 1971])

For any recursive function f and arbitrarily large integers n, there exists a cfg G of size n generating a regular language L, s.t. any dfa accepting L must have at least $f(n)$ states.

As a consequence, the trade-off between context-grammars and finite automata is not recursive.

However... The witness language L is defined over a binary alphabet.

What about languages over a one-letter alphabet?

Context-free vs regular: descriptional complexity

Given a context-free grammar (or a pushdown automaton) of size n, generating a regular language, how much is big an equivalent finite automaton, wrt n ?

Theorem ([Meyer and Fischer, 1971])

For any recursive function f and arbitrarily large integers n, there exists a cfg G of size n generating a regular language L, s.t. any dfa accepting L must have at least $f(n)$ states.

As a consequence, the trade-off between context-grammars and finite automata is not recursive.
However... The witness language L is defined over a binary alphabet.

What about languages over a one letter alphabet?

Context-free vs regular: descriptional complexity

Given a context-free grammar (or a pushdown automaton) of size n, generating a regular language, how much is big an equivalent finite automaton, wrt n ?

Theorem ([Meyer and Fischer, 1971])

For any recursive function f and arbitrarily large integers n, there exists a cfg G of size n generating a regular language L, s.t. any dfa accepting L must have at least $f(n)$ states.

As a consequence, the trade-off between context-grammars and finite automata is not recursive.
However... The witness language L is defined over a binary alphabet.

What about languages over a one letter alphabet?

Context-free vs regular: descriptional complexity

Given a context-free grammar (or a pushdown automaton) of size n, generating a regular language, how much is big an equivalent finite automaton, wrt n ?

Theorem ([Meyer and Fischer, 1971])

For any recursive function f and arbitrarily large integers n, there exists a cfg G of size n generating a regular language L, s.t. any dfa accepting L must have at least $f(n)$ states.

As a consequence, the trade-off between context-grammars and finite automata is not recursive.
However... The witness language L is defined over a binary alphabet.

What about languages over a one letter alphabet?

Unary languages

$\Sigma = \{a\}$

Theorem ([Ginsurg and Rice, 1962])

Every unary context-free language is regular.

Hence the classes of unary regular languages and unary *context–free* languages coincide!

Problem

Study the equivalence between unary context-free and regular languages from the descriptional complexity point of view.

Unary languages

$\Sigma = \{a\}$

Theorem ([Ginsurg and Rice, 1962])

Every unary context-free language is regular.

Hence the classes of unary regular languages and unary *context–free* languages coincide!

Problem

Study the equivalence between unary context-free and regular languages from the descriptional complexity point of view.

Context-free vs regular

Unary case [Pighizzini, Shallit, Wang, 2002]

Theorem

For any cfg in Chomsky normal form with h variables, generating a unary language, there exists an equivalent dfa with 2^{h^2} states. Furthermore, this bound is tight.

Corollary

Each unary pda with n states and m stack symbols, s.t. each push adds exactly one symbol, can be simulated by a dfa with $2^{O(n^4 m^2)}$ states.

What about the deterministic case?

Context-free vs regular

Unary case [Pighizzini, Shallit, Wang, 2002]

Theorem

For any cfg in Chomsky normal form with h variables, generating a unary language, there exists an equivalent dfa with 2^{h^2} states. Furthermore, this bound is tight.

Corollary

Each unary pda with n states and m stack symbols, s.t. each push adds exactly one symbol, can be simulated by a dfa with $2^{O(n^4 m^2)}$ states.

What about the deterministic case?

Context-free vs regular

Unary case [Pighizzini, Shallit, Wang, 2002]

Theorem

For any cfg in Chomsky normal form with h variables, generating a unary language, there exists an equivalent dfa with 2^{h^2} states. Furthermore, this bound is tight.

Corollary

Each unary pda with n states and m stack symbols, s.t. each push adds exactly one symbol, can be simulated by a dfa with $2^{O(n^4 m^2)}$ states.

What about the deterministic case?

Dpda's vs finite automata (general case)

- Each dpda of size s accepting a regular language can be simulated by a dfa with $2^{2^{2^s}}$ states. [Stearns, 1967]
- This upper bound was reduced to 2^{2^s} in [Valiant, 1975].
- It cannot be further reduced because a matching lower bound [Meyer and Fischer, 1971].
- However, in the *unary* case, it can be reduced to 2^s (tight bound) [illegible].

Dpda's vs finite automata (general case)

- Each dpda of size s accepting a regular language can be simulated by a dfa with $2^{2^{2^s}}$ states. [Stearns, 1967]
- This upper bound was reduced to 2^{2^s} in [Valiant, 1975].
- It cannot be further reduced because a matching lower bound [Meyer and Fischer, 1971].
- However, in the *unary* case, it can be reduced to 2^s (tight bound) [Pighizzini, 2008].

Dpda's vs finite automata (general case)

- Each dpda of size s accepting a regular language can be simulated by a dfa with $2^{2^{2^s}}$ states. [Stearns, 1967]
- This upper bound was reduced to 2^{2^s} in [Valiant, 1975].
- It cannot be further reduced because a matching lower bound [Meyer and Fischer, 1971].
- However, in the *unary* case, it can be reduced to 2^s (tight bound) [Pighizzini, 2008].

Dpda's vs finite automata (general case)

- Each dpda of size s accepting a regular language can be simulated by a dfa with $2^{2^{2^s}}$ states. [Stearns, 1967]
- This upper bound was reduced to 2^{2^s} in [Valiant, 1975].
- It cannot be further reduced because a matching lower bound [Meyer and Fischer, 1971].
- However, in the *unary* case, it can be reduced to 2^s (tight bound) [Pighizzini, 2008].

Size

- Size of a finite automaton:
  Number of its states
- Size of a pushdown automaton:
  Total number of symbols needed to write down its description.
  We have to keep into account:
  - the number of the states
  - the cardinality of the pushdown alphabet
  - the length of the strings that can be pushed in one move on the stack
  - the number of transitions

Size

- Size of a finite automaton:
  Number of its states
- Size of a pushdown automaton:
  Total number of symbols needed to write down its description.
  We have to keep into account:
  - the number of the states
  - the cardinality of the pushdown alphabet
  - the length of the strings that can be pushed in one move on the stack
  - the number of transitions

Size

- Size of a finite automaton:
  Number of its states
- Size of a pushdown automaton:
  Total number of symbols needed to write down its description.
  We have to keep into account:
  - the number of the states
  - the cardinality of the pushdown alphabet
  - the length of the strings that can be pushed in one move on the stack
  - the number of transitions

Size of dpda's

Normal form for pda's (some restrictions on the transitions)

- We can prove that each dpda of size s can be converted into an equivalent dpda in normal form such that the product of
  - the number of states
  - the cardinality of the pushdown alphabet

  is $O(s)$.

Hence, we can restrict our attention to:

- dpda's in normal form with
- size = # states × # pushdown alphabet

Size of dpda's

Normal form for pda's (some restrictions on the transitions)

- We can prove that each dpda of size s can be converted into an equivalent dpda in normal form such that the product of
  - the number of states
  - the cardinality of the pushdown alphabet

  is $O(s)$.

Hence, we can restrict our attention to:

- dpda's in normal form with
- size = $\#$ states $\times$ $\#$ pushdown alphabet

Size of dpda's

Normal form for pda's (some restrictions on the transitions)

- We can prove that each dpda of size s can be converted into an equivalent dpda in normal form such that the product of
  - the number of states
  - the cardinality of the pushdown alphabet

  is $O(s)$.

Hence, we can restrict our attention to:

- dpda's in normal form with
- size = $\#$ states $\times$ $\#$ pushdown alphabet

Input alphabet $\Sigma = \{a\}$

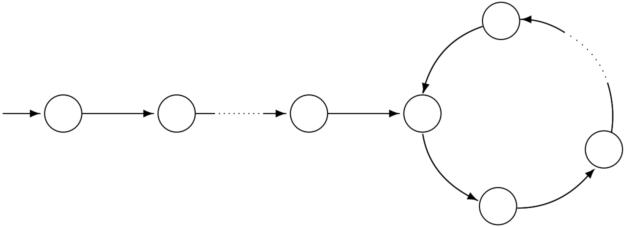
Theorem

$L \subseteq \{a\}^*$ *is regular iff* $\exists \mu \geq 0, \lambda \geq 1$ *s.t.*

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L.$$

Unary dfa's

Input alphabet $\Sigma = \{a\}$

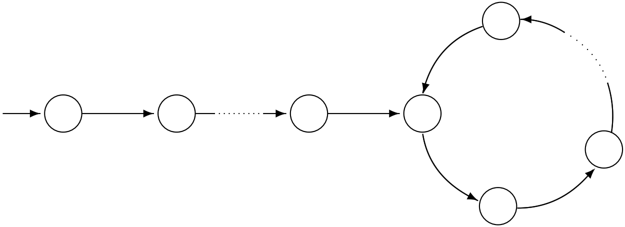


Theorem

$L \subseteq \{a\}^*$ is regular iff $\exists \mu \geq 0, \lambda \geq 1$ s.t.

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L.$$

Input alphabet $\Sigma = \{a\}$

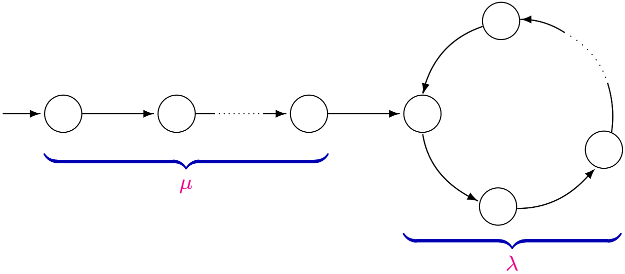


Theorem

$L \subseteq \{a\}^*$ *is regular iff* $\exists \mu \geq 0, \lambda \geq 1$ *s.t.*

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L.$$

Input alphabet $\Sigma = \{a\}$



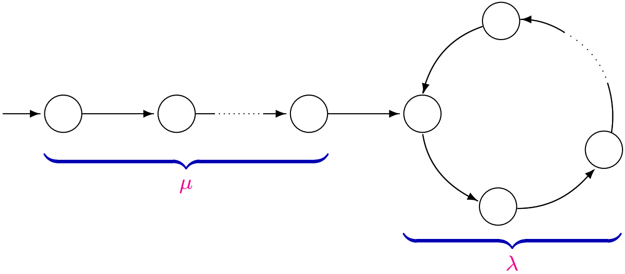
Theorem

$L \subseteq \{a\}^*$ is regular iff $\exists \mu \geq 0, \lambda \geq 1$ s.t.

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L.$$

Unary dfa's

Input alphabet $\Sigma = \{a\}$



Theorem

$L \subseteq \{a\}^*$ is regular iff $\exists \mu \geq 0, \lambda \geq 1$ s.t.

$$\forall n \geq \mu : a^n \in L \text{ iff } a^{n+\lambda} \in L.$$

Unary automata

The costs of the optimal simulations between automata are different in the unary and in the general case!

Unary case: [Chrobak 1986, Mereghetti and Pighizzini 2001]

dfa nfa

2dfa 2nfa

The costs of the optimal simulations between automata are different in the unary and in the general case!

Unary case: [Chrobak 1986, Mereghetti and Pighizzini 2001]

dfa nfa

2dfa 2nfa

The costs of the optimal simulations between automata are different in the unary and in the general case!

Unary case: [Chrobak 1986, Mereghetti and Pighizzini 2001]

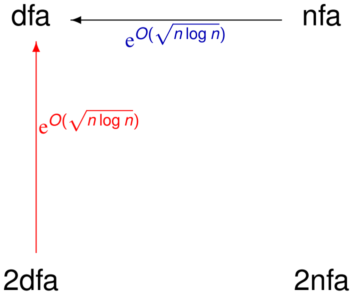
$$\text{dfa} \xleftarrow{e^{O(\sqrt{n \log n})}} \text{nfa}$$

2dfa 2nfa

Unary automata

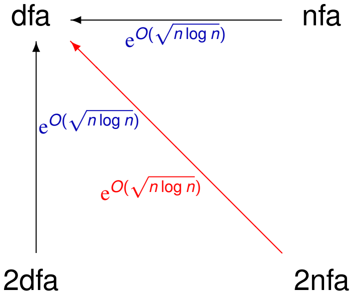
The costs of the optimal simulations between automata are different in the unary and in the general case!

Unary case: [Chrobak 1986, Mereghetti and Pighizzini 2001]



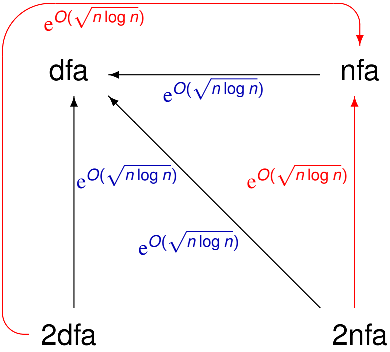
The costs of the optimal simulations between automata are different in the unary and in the general case!

Unary case: [Chrobak 1986, Mereghetti and Pighizzini 2001]



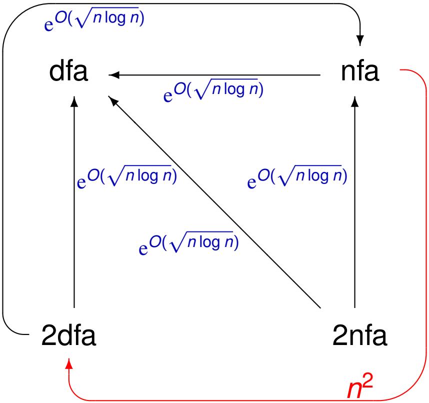
The costs of the optimal simulations between automata are different in the unary and in the general case!

Unary case: [Chrobak 1986, Mereghetti and Pighizzini 2001]



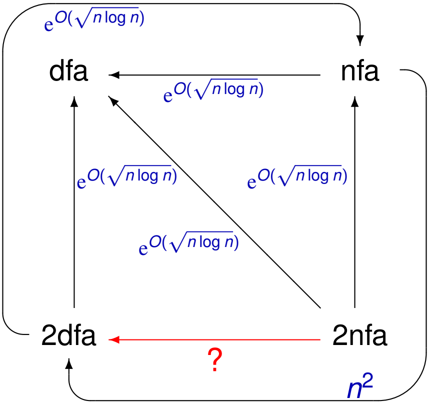
The costs of the optimal simulations between automata are different in the unary and in the general case!

Unary case: [Chrobak 1986, Mereghetti and Pighizzini 2001]



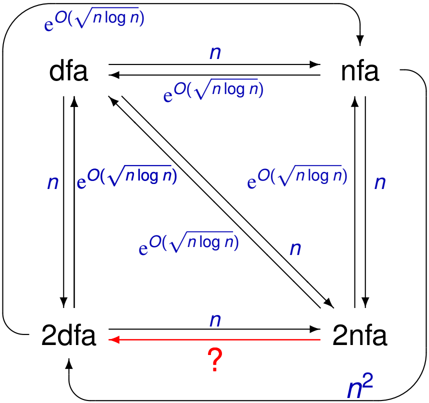
The costs of the optimal simulations between automata are different in the unary and in the general case!

Unary case: [Chrobak 1986, Mereghetti and Pighizzini 2001]



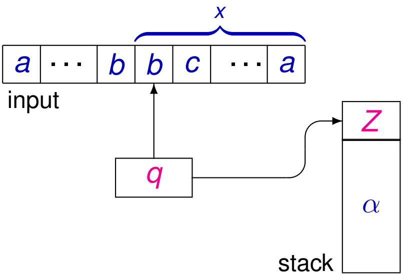
The costs of the optimal simulations between automata are different in the unary and in the general case!

Unary case: [Chrobak 1986, Mereghetti and Pighizzini 2001]



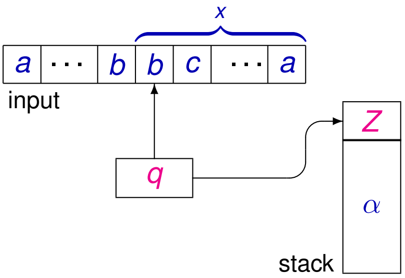
Pushdown automata

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$



- $(q, x, Z\alpha)$ configuration
- $[qZ]$ mode

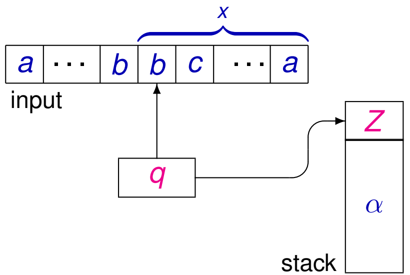
$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$



- $(q, x, Z\alpha)$ configuration
- $[qZ]$ mode

Pushdown automata

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$



- $(q, x, Z\alpha)$ configuration
- $[qZ]$ mode

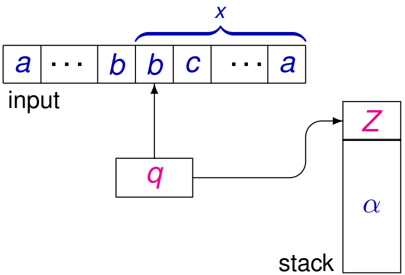
- M is *deterministic* iff $\forall q \in Q, Z \in \Gamma$:
  - if $\delta(q, \epsilon, Z) \neq \emptyset$ then $\delta(q, a, Z) = \emptyset$, for each $a \in \Sigma$
  - $\#\delta(q, \sigma, Z) \leq 1$, for each $\sigma \in \Sigma \cup \{\epsilon\}$.

- Deterministic cfl's: acceptance by *final states*

$$L(M) = \{x \in \Sigma^* \mid (q_0, x, Z_0) \overset{\star}{\vdash} (q, \epsilon, \gamma), q \in F, \gamma \in \Gamma^*\}$$

Pushdown automata

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$



- $(q, x, Z\alpha)$ configuration
- $[qZ]$ mode

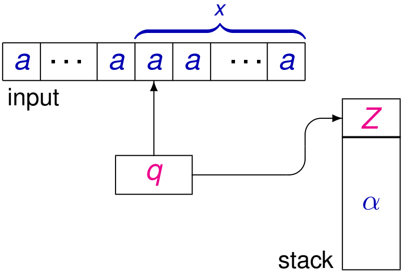
- M is *deterministic* iff $\forall q \in Q, Z \in \Gamma$:
  - if $\delta(q, \epsilon, Z) \neq \emptyset$ then $\delta(q, a, Z) = \emptyset$, for each $a \in \Sigma$
  - $\#\delta(q, \sigma, Z) \leq 1$, for each $\sigma \in \Sigma \cup \{\epsilon\}$.

- Deterministic cfl's: acceptance by *final states*

$$L(M) = \{x \in \Sigma^* \mid (q_0, x, Z_0) \overset{\star}{\vdash} (q, \epsilon, \gamma), q \in F, \gamma \in \Gamma^*\}$$

Pushdown automata

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$



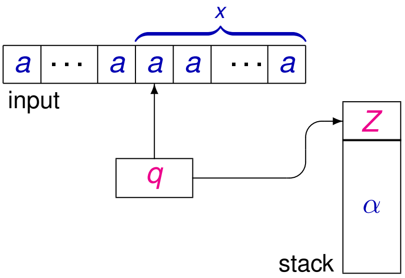
- $(q, x, Z\alpha)$ configuration
- $[qZ]$ mode

Unary deterministic pda's:

For each integer $t \geq 0$:

- if the computation does not stop before t steps then the configuration reached at the step t *does not depend* on the input length

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$



- $(q, x, Z\alpha)$ configuration
- $[qZ]$ mode

Unary deterministic pda's:

For each integer $t \geq 0$:

- if the computation does not stop before t steps then the configuration reached at the step t *does not depend* on the input length

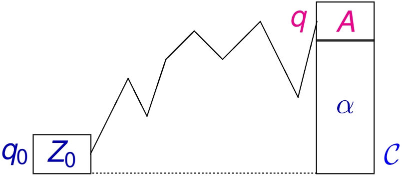
Modes

$[qA] \le [pB]$ iff all the following conditions hold:

1. A configuration C with mode $[qA]$ is reachable from the initial configuration
2. A configuration with mode $[pB]$ is reachable from the configuration with mode $[qA]$ and pushdown store containing only A
3. If a configuration C' with mode $[pB]$ is reachable before C, then

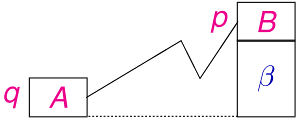
Modes

$[qA] \leq [pB]$ iff all the following conditions hold:



1. A configuration $\mathcal{C}$ with mode $[qA]$ is reachable from the initial configuration
2. A configuration with mode $[pB]$ is reachable from the configuration with mode $[qA]$ and pushdown store containing only A
3. If a configuration $\mathcal{C}'$ with mode $[pB]$ is reachable before $\mathcal{C}$, then the stack height in some configuration between $\mathcal{C}'$ and $\mathcal{C}$ must be less than in $\mathcal{C}'$.

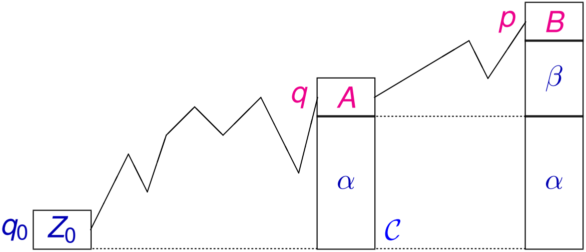
$[qA] \leq [pB]$ iff all the following conditions hold:



1. A configuration $\mathcal{C}$ with mode $[qA]$ is reachable from the initial configuration
2. A configuration with mode $[pB]$ is reachable from the configuration with mode $[qA]$ and pushdown store containing only A
3. If a configuration $\mathcal{C}'$ with mode $[pB]$ is reachable before $\mathcal{C}$, then the stack height in some configuration between $\mathcal{C}'$ and $\mathcal{C}$ must be less than in $\mathcal{C}'$.

Modes

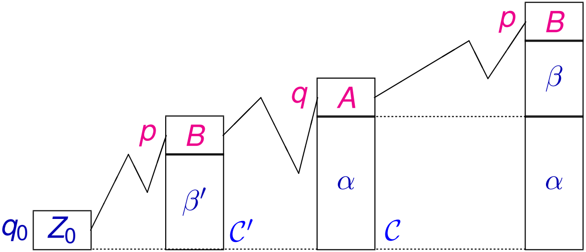
$[qA] \le [pB]$ iff all the following conditions hold:



1. A configuration $\mathcal{C}$ with mode $[qA]$ is reachable from the initial configuration
2. A configuration with mode $[pB]$ is reachable from the configuration with mode $[qA]$ and pushdown store containing only A
3. If a configuration $\mathcal{C}'$ with mode $[pB]$ is reachable before $\mathcal{C}$, then the stack height in some configuration between $\mathcal{C}'$ and $\mathcal{C}$ must be less than in $\mathcal{C}'$.

Modes

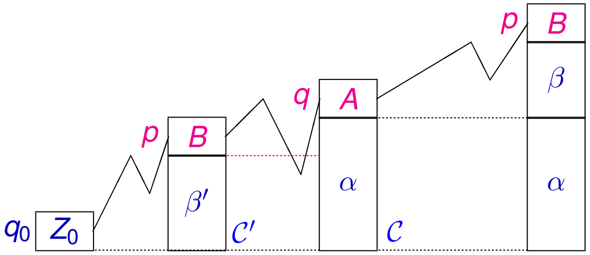
$[qA] \leq [pB]$ iff all the following conditions hold:



1. A configuration $\mathcal{C}$ with mode $[qA]$ is reachable from the initial configuration
2. A configuration with mode $[pB]$ is reachable from the configuration with mode $[qA]$ and pushdown store containing only A
3. If a configuration $\mathcal{C}'$ with mode $[pB]$ is reachable before $\mathcal{C}$, then the stack height in some configuration between $\mathcal{C}'$ and $\mathcal{C}$ must be less than in $\mathcal{C}'$.

Modes

$[qA] \leq [pB]$ iff all the following conditions hold:



1. A configuration $\mathcal{C}$ with mode $[qA]$ is reachable from the initial configuration
2. A configuration with mode $[pB]$ is reachable from the configuration with mode $[qA]$ and pushdown store containing only A
3. If a configuration $\mathcal{C}'$ with mode $[pB]$ is reachable before $\mathcal{C}$, then the stack height in some configuration between $\mathcal{C}'$ and $\mathcal{C}$ must be less than in $\mathcal{C}'$.

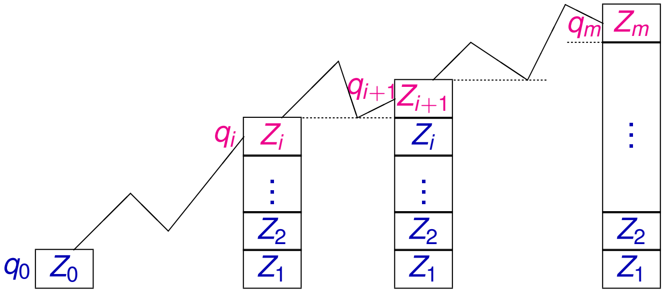
Modes

Lemma

The relation $\leq$ defines a partial order on the set of the modes.

h_t history at the time t

Stack content + state information



History h_t at the time t:
sequence of modes $[q_m Z_m] \ldots [q_1 Z_1]$ s.t.:

- $Z_m \ldots Z_1$ is the stack content after t computation steps
- for $i = 1, \ldots, m$, $[q_i Z_i]$ is the mode of the last visited configuration with stack height i

Histories and modes

For each step $t \geq 0$ we consider:

- h_t history
- m_t mode (leftmost element of h_t)

For the given dpda M we consider:

- $H = \{h_t \mid t \geq 0\}$, the set all *reachable histories*
- $(m_t)_{t \geq 0}$, the sequence of *reached modes*

Two possibilities:

1. Every history belonging to H does not contain a repeated mode
2. At least one history belonging to H contains a repetition

Histories and modes

For each step $t \geq 0$ we consider:

- h_t history
- m_t mode (leftmost element of h_t)

For the given dpda M we consider:

- $H = \{h_t \mid t \geq 0\}$, the set all *reachable histories*
- $(m_t)_{t \geq 0}$, the sequence of *reached modes*

Two possibilities:

1. Every history belonging to H does not contain a repeated mode
2. At least one history belonging to H contains a repetition

Histories and modes

For each step $t \geq 0$ we consider:

- h_t history
- m_t mode (leftmost element of h_t)

For the given dpda M we consider:

- $H = \{h_t \mid t \geq 0\}$, the set all *reachable histories*
- $(m_t)_{t \geq 0}$, the sequence of *reached modes*

Two possibilities:

1. Every history belonging to H does not contain a repeated mode
2. At least one history belonging to H contains a repetition

Histories

Case 1: Every history belonging to H does not contain a repeated mode

- H is finite
- The given dpda can be simulated by a deterministic automaton A whose set of states is H
- The number of the states of A is bounded by the number of histories without repetitions
- If an history $[q_m Z_m] \ldots [q_1 Z_1]$ does not contain any repetition, then $[q_1 Z_1] \leq [q_2 Z_2] \leq \ldots \leq [q_m Z_m]$
- Hence:

  the number of states of the deterministic automaton A is bounded by $2^{\#Q \cdot \#\Gamma}$

Case 1: Every history belonging to H does not contain a repeated mode

- H is finite
- The given dpda can be simulated by a deterministic automaton A whose set of states is H
- The number of the states of A is bounded by the number of histories without repetitions
- If an history $[q_m Z_m] \ldots [q_1 Z_1]$ does not contain any repetition, then $[q_1 Z_1] \leq [q_2 Z_2] \leq \ldots \leq [q_m Z_m]$
- Hence:

the number of states of the deterministic automaton A is bounded by $2^{\#Q \cdot \#\Gamma}$

Case 1: Every history belonging to H does not contain a repeated mode

- H is finite
- The given dpda can be simulated by a deterministic automaton A whose set of states is H
- The number of the states of A is bounded by the number of histories without repetitions
- If an history $[q_m Z_m] \ldots [q_1 Z_1]$ does not contain any repetition, then $[q_1 Z_1] \leq [q_2 Z_2] \leq \ldots \leq [q_m Z_m]$
- Hence:

the number of states of the deterministic automaton A is bounded by $2^{\#Q \cdot \#\Gamma}$

Histories

Case 1: Every history belonging to H does not contain a repeated mode

- H is finite
- The given dpda can be simulated by a deterministic automaton A whose set of states is H
- The number of the states of A is bounded by the number of histories without repetitions
- If an history $[q_m Z_m] \ldots [q_1 Z_1]$ does not contain any repetition, then $[q_1 Z_1] \leq [q_2 Z_2] \leq \ldots \leq [q_m Z_m]$
- Hence:

the number of states of the deterministic automaton A is bounded by $2^{\#Q \cdot \#\Gamma}$

Histories

Case 1: Every history belonging to H does not contain a repeated mode

- H is finite
- The given dpda can be simulated by a deterministic automaton A whose set of states is H
- The number of the states of A is bounded by the number of histories without repetitions
- If an history $[q_m Z_m] \ldots [q_1 Z_1]$ does not contain any repetition, then $[q_1 Z_1] \leq [q_2 Z_2] \leq \ldots \leq [q_m Z_m]$
- Hence:

the number of states of the deterministic automaton A is bounded by $2^{\#Q \cdot \#\Gamma}$

Case 1: Every history belonging to H does not contain a repeated mode

- H is finite
- The given dpda can be simulated by a deterministic automaton A whose set of states is H
- The number of the states of A is bounded by the number of histories without repetitions
- If an history $[q_m Z_m] \ldots [q_1 Z_1]$ does not contain any repetition, then $[q_1 Z_1] \leq [q_2 Z_2] \leq \ldots \leq [q_m Z_m]$
- Hence:

the number of states of the deterministic automaton A is bounded by $2^{\#Q \cdot \#\Gamma}$

Case 2: At least one history in H contains a repetition

- The histories in H grow in a periodic way, i.e.:
  $\exists \mu \geq 0, \lambda \geq 1$, $\exists$ sequences of modes $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_\lambda$ s.t. for $t \geq \mu$, the history at the step t is:
  $$h_t = \tilde{h}_{t \bmod \lambda} (\tilde{h}_\lambda)^{\lfloor \frac{t-\mu}{\lambda} \rfloor} h_\mu$$
- The sequence $(m_t)_{t \geq 0}$ is ultimately periodic (period λ, from $t \geq \mu$)
- The language can be accepted by a deterministic automaton A with at most $\lambda + \mu$ states
- $\lambda + \mu \leq 2^{\#Q \cdot \#\Gamma}$
- Hence:

  the given dpda can be simulated by a deterministic automaton A with at most $2^{\#Q \cdot \#\Gamma}$ states

Histories

Case 2: At least one history in H contains a repetition

- The histories in H grow in a periodic way, i.e.:
  $\exists \mu \geq 0, \lambda \geq 1$, $\exists$ sequences of modes $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_\lambda$ s.t. for $t \geq \mu$, the history at the step t is:
  $$h_t = \tilde{h}_{t \,\mathrm{MOD}\, \lambda} (\tilde{h}_\lambda)^{\lfloor \frac{t-\mu}{\lambda} \rfloor} h_\mu$$
- The sequence $(m_t)_{t \geq 0}$ is ultimately periodic (period λ, from $t \geq \mu$)
- The language can be accepted by a deterministic automaton A with at most $\lambda + \mu$ states
- $\lambda + \mu \leq 2^{\#Q \cdot \#\Gamma}$
- Hence:

  the given dpda can be simulated by a deterministic automaton A with at most $2^{\#Q \cdot \#\Gamma}$ states

Histories

Case 2: At least one history in H contains a repetition

- The histories in H grow in a periodic way, i.e.:
  $\exists \mu \geq 0, \lambda \geq 1$, $\exists$ sequences of modes $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_\lambda$ s.t. for $t \geq \mu$, the history at the step t is:

$$h_t = \tilde{h}_{t \,\mathrm{MOD}\, \lambda} (\tilde{h}_\lambda)^{\lfloor \frac{t-\mu}{\lambda} \rfloor} h_\mu$$

- The sequence $(m_t)_{t \geq 0}$ is ultimately periodic (period λ, from $t \geq \mu$)
- The language can be accepted by a deterministic automaton A with at most $\lambda + \mu$ states
- $\lambda + \mu \leq 2^{\#Q \cdot \#\Gamma}$
- Hence:

the given dpda can be simulated by a deterministic automaton A with at most $2^{\#Q \cdot \#\Gamma}$ states

Case 2: At least one history in H contains a repetition

- The histories in H grow in a periodic way, i.e.:
  $\exists \mu \geq 0, \lambda \geq 1$, $\exists$ sequences of modes $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_\lambda$ s.t. for $t \geq \mu$, the history at the step t is:
  $$h_t = \tilde{h}_{t \,\mathrm{MOD}\, \lambda} (\tilde{h}_\lambda)^{\lfloor \frac{t-\mu}{\lambda} \rfloor} h_\mu$$
- The sequence $(m_t)_{t \geq 0}$ is ultimately periodic (period λ, from $t \geq \mu$)
- The language can be accepted by a deterministic automaton A with at most $\lambda + \mu$ states
- $\lambda + \mu \leq 2^{\#Q \cdot \#\Gamma}$
- Hence:

  the given dpda can be simulated by a deterministic automaton A with at most $2^{\#Q \cdot \#\Gamma}$ states

Case 2: At least one history in H contains a repetition

- The histories in H grow in a periodic way, i.e.:
  $\exists \mu \geq 0, \lambda \geq 1$, $\exists$ sequences of modes $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_\lambda$ s.t. for $t \geq \mu$, the history at the step t is:
  $$h_t = \tilde{h}_{t \,\mathrm{MOD}\, \lambda} (\tilde{h}_\lambda)^{\lfloor \frac{t-\mu}{\lambda} \rfloor} h_\mu$$
- The sequence $(m_t)_{t \geq 0}$ is ultimately periodic (period λ, from $t \geq \mu$)
- The language can be accepted by a deterministic automaton A with at most $\lambda + \mu$ states
- $\lambda + \mu \leq 2^{\#Q \cdot \#\Gamma}$
- Hence:

  the given dpda can be simulated by a deterministic automaton A with at most $2^{\#Q \cdot \#\Gamma}$ states

Case 2: At least one history in H contains a repetition

- The histories in H grow in a periodic way, i.e.:
  $\exists \mu \geq 0, \lambda \geq 1$, $\exists$ sequences of modes $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_\lambda$ s.t. for $t \geq \mu$, the history at the step t is:
  $$h_t = \tilde{h}_{t \,\mathrm{MOD}\, \lambda} (\tilde{h}_\lambda)^{\lfloor \frac{t-\mu}{\lambda} \rfloor} h_\mu$$
- The sequence $(m_t)_{t \geq 0}$ is ultimately periodic (period λ, from $t \geq \mu$)
- The language can be accepted by a deterministic automaton A with at most $\lambda + \mu$ states
- $\lambda + \mu \leq 2^{\#Q \cdot \#\Gamma}$
- Hence:

  the given dpda can be simulated by a deterministic automaton A with at most $2^{\#Q \cdot \#\Gamma}$ states

Case 2: At least one history in H contains a repetition

- The histories in H grow in a periodic way, i.e.: $\exists \mu \geq 0, \lambda \geq 1$, $\exists$ sequences of modes $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_\lambda$ s.t. for $t \geq \mu$, the history at the step t is:

$$h_t = \tilde{h}_{t \bmod \lambda} (\tilde{h}_\lambda)^{\lfloor \frac{t-\mu}{\lambda} \rfloor} h_\mu$$

- The sequence $(m_t)_{t \geq 0}$ is ultimately periodic (period λ, from $t \geq \mu$)
- The language can be accepted by a deterministic automaton A with at most $\lambda + \mu$ states
- $\lambda + \mu \leq 2^{\#Q \cdot \#\Gamma}$
- Hence:

the given dpda can be simulated by a deterministic automaton A with at most $2^{\#Q \cdot \#\Gamma}$ states

Unary dpda's vs dfa's

As a consequence:

Theorem

Each unary dpda of size s can be simulated by a dfa with $2^{O(s)}$ states.

What about the optimality of this simulation?

Unary dpda's vs dfa's

As a consequence:

Theorem

Each unary dpda of size s can be simulated by a dfa with $2^{O(s)}$ states.

What about the optimality of this simulation?

Unary dpda's vs dfa's: lower bound

Given $s > 0$ consider $L_s = (a^{2^s})^*$.

We can prove that:

- There exists a dpda of size $8s + 4$ accepting L_s.
- Each dfa accepting L_s must have at least 2^s states.

Hence our simulation is optimal!

Problem: Does it is possible to reduce the cost of the simulation of unary dpda's, by using *nondeterministic* or *two-way* finite automata?

Unary dpda's vs dfa's: lower bound

Given $s > 0$ consider $L_s = (a^{2^s})^*$.

We can prove that:

- There exists a dpda of size $8s + 4$ accepting L_s.
- Each dfa accepting L_s must have at least 2^s states.

Hence our simulation is optimal!

Problem: Does it is possible to reduce the cost of the simulation of unary dpda's, by using *nondeterministic* or *two-way* finite automata?

Unary dpda's vs dfa's: lower bound

Given $s > 0$ consider $L_s = (a^{2^s})^*$.

We can prove that:

- There exists a dpda of size $8s + 4$ accepting L_s.
- Each dfa accepting L_s must have at least 2^s states.

Hence our simulation is optimal!

Problem: Does it is possible to reduce the cost of the simulation of unary dpda's, by using *nondeterministic* or *two-way* finite automata?

Unary dpda's vs dfa's: lower bound

Given $s > 0$ consider $L_s = (a^{2^s})^*$.

We can prove that:

- There exists a dpda of size $8s + 4$ accepting L_s.
- Each dfa accepting L_s must have at least 2^s states.

Hence our simulation is optimal!

Problem: Does it is possible to reduce the cost of the simulation of unary dpda's, by using *nondeterministic* or *two-way* finite automata?

Unary dpda's vs dfa's: lower bound

Given $s > 0$ consider $L_s = (a^{2^s})^*$.

We can prove that:

- There exists a dpda of size $8s + 4$ accepting L_s.
- Each dfa accepting L_s must have at least 2^s states.

Hence our simulation is optimal!

Problem: Does it is possible to reduce the cost of the simulation of unary dpda's, by using *nondeterministic* or *two-way* finite automata?

Unary dpda's vs 2nfa's

We recall the following result:

Theorem ([Mereghetti, Pighizzini, 2000])

$L \subseteq a^*$ ultimately properly λ-cyclic, with $\lambda = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s}$, for primes $p_1, \ldots, p_s$, integers $k_1, \ldots, k_s \geq 1$.
Then each 2nfa accepting L must have at least $p_1^{k_1} + p_2^{k_2} + \cdots + p_s^{k_s}$ states in its cycles.

- We consider again $L_s = (a^{2^s})^*$, $s > 0$
- L_s is properly 2^s-cyclic
- Hence, each two-way nondeterministic automaton accepting L_s needs 2^s states

Since L_s is accepted by a dpda with $O(s)$ states, we get that:

Even the cost of the optimal simulation of unary dpda's by 2nfa's is exponential!

Unary dpda's vs 2nfa's

We recall the following result:

Theorem ([Mereghetti, Pighizzini, 2000])

$L \subseteq a^*$ ultimately properly λ-cyclic, with $\lambda = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s}$, for primes $p_1, \ldots, p_s$, integers $k_1, \ldots, k_s \geq 1$.
Then each 2nfa accepting L must have at least $p_1^{k_1} + p_2^{k_2} + \cdots + p_s^{k_s}$ states in its cycles.

- We consider again $L_s = (a^{2^s})^*$, $s > 0$
- L_s is properly 2^s-cyclic
- Hence, each two-way nondeterministic automaton accepting L_s needs 2^s states

Since L_s is accepted by a dpda with $O(s)$ states, we get that:

Even the cost of the optimal simulation of unary dpda's by 2nfa's is exponential!

Unary dpda's vs 2nfa's

We recall the following result:

Theorem ([Mereghetti, Pighizzini, 2000])

$L \subseteq a^*$ ultimately properly λ-cyclic, with $\lambda = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s}$, for primes $p_1, \ldots, p_s$, integers $k_1, \ldots, k_s \geq 1$.
Then each 2nfa accepting L must have at least $p_1^{k_1} + p_2^{k_2} + \cdots + p_s^{k_s}$ states in its cycles.

- We consider again $L_s = (a^{2^s})^*$, $s > 0$
- L_s is properly 2^s-cyclic
- Hence, each two-way nondeterministic automaton accepting L_s needs 2^s states

Since L_s is accepted by a dpda with $O(s)$ states, we get that:

Even the cost of the optimal simulation of unary dpda's by 2nfa's is exponential!

Unary dpda's vs 2nfa's

We recall the following result:

Theorem ([Mereghetti, Pighizzini, 2000])

$L \subseteq a^*$ ultimately properly λ-cyclic, with $\lambda = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s}$, for primes $p_1, \ldots, p_s$, integers $k_1, \ldots, k_s \geq 1$.
Then each 2nfa accepting L must have at least $p_1^{k_1} + p_2^{k_2} + \cdots + p_s^{k_s}$ states in its cycles.

- We consider again $L_s = (a^{2^s})^*$, $s > 0$
- L_s is properly 2^s-cyclic
- Hence, each two-way nondeterministic automaton accepting L_s needs 2^s states

Since L_s is accepted by a dpda with $O(s)$ states, we get that:

Even the cost of the optimal simulation of unary dpda's by 2nfa's is exponential!

Unary dpda's vs 2nfa's

We recall the following result:

Theorem ([Mereghetti, Pighizzini, 2000])

$L \subseteq a^*$ ultimately properly λ-cyclic, with $\lambda = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s}$, for primes $p_1, \ldots, p_s$, integers $k_1, \ldots, k_s \geq 1$.
Then each 2nfa accepting L must have at least $p_1^{k_1} + p_2^{k_2} + \cdots + p_s^{k_s}$ states in its cycles.

- We consider again $L_s = (a^{2^s})^*$, $s > 0$
- L_s is properly 2^s-cyclic
- Hence, each two-way nondeterministic automaton accepting L_s needs 2^s states

Since L_s is accepted by a dpda with $O(s)$ states, we get that:

Even the cost of the optimal simulation of unary dpda's by 2nfa's is exponential!

How to transform pda's into equivalent cfg's?

The standard construction uses the following set of variables:

$$\{S\} \cup \{[qAp] \mid q, p \in Q, A \in V\}$$

with a set of productions such that

$$[qAp] \stackrel{\star}{\Rightarrow} w \text{ iff } (q, w, A) \vdash^{\star} (p, \epsilon)$$

Hence, the total number of variables is $(\#Q)^2 \cdot \#\Gamma + 1$.

This number cannot be reduced, even if the given pda is deterministic [Goldstine, Price, Wotschke, 1982].

However, if the dpda is *unary* we can do better!

This number can be reduced to $2 \cdot \#Q \cdot \#\Gamma$

How to transform pda's into equivalent cfg's?

The standard construction uses the following set of variables:

$$\{S\} \cup \{[qAp] \mid q, p \in Q, A \in V\}$$

with a set of productions such that

$$[qAp] \overset{\star}{\Rightarrow} w \text{ iff } (q, w, A) \vdash^{\star} (p, \epsilon)$$

Hence, the total number of variables is $(\#Q)^2 \cdot \#\Gamma + 1$.

This number cannot be reduced, even if the given pda is deterministic [Goldstine, Price, Wotschke, 1982].

However, if the dpda is *unary* we can do better!

This number can be reduced to $2 \cdot \#Q \cdot \#\Gamma$

How to transform pda's into equivalent cfg's?

The standard construction uses the following set of variables:

$$\{S\} \cup \{[qAp] \mid q, p \in Q, A \in V\}$$

with a set of productions such that

$$[qAp] \overset{\star}{\Rightarrow} w \text{ iff } (q, w, A) \overset{\star}{\vdash} (p, \epsilon)$$

Hence, the total number of variables is $(\#Q)^2 \cdot \#\Gamma + 1$.

This number cannot be reduced, even if the given pda is deterministic [Goldstine, Price, Wotschke, 1982].

However, if the dpda is *unary* we can do better!

This number can be reduced to $2 \cdot \#Q \cdot \#\Gamma$

How to transform pda's into equivalent cfg's?

The standard construction uses the following set of variables:

$$\{S\} \cup \{[qAp] \mid q, p \in Q, A \in V\}$$

with a set of productions such that

$$[qAp] \overset{\star}{\Rightarrow} w \text{ iff } (q, w, A) \overset{\star}{\vdash} (p, \epsilon)$$

Hence, the total number of variables is $(\#Q)^2 \cdot \#\Gamma + 1$.

This number cannot be reduced, even if the given pda is deterministic [Goldstine, Price, Wotschke, 1982].

However, if the dpda is *unary* we can do better!

This number can be reduced to $2 \cdot \#Q \cdot \#\Gamma$

How to transform pda's into equivalent cfg's?

The standard construction uses the following set of variables:

$$\{S\} \cup \{[qAp] \mid q, p \in Q, A \in V\}$$

with a set of productions such that

$$[qAp] \stackrel{\star}{\Rightarrow} w \text{ iff } (q, w, A) \vdash^{\star} (p, \epsilon)$$

Hence, the total number of variables is $(\#Q)^2 \cdot \#\Gamma + 1$.

This number cannot be reduced, even if the given pda is deterministic [Goldstine, Price, Wotschke, 1982].

However, if the dpda is *unary* we can do better!

This number can be reduced to $2 \cdot \#Q \cdot \#\Gamma$

How to transform pda's into equivalent cfg's?

The standard construction uses the following set of variables:

$$\{S\} \cup \{[qAp] \mid q, p \in Q, A \in V\}$$

with a set of productions such that

$$[qAp] \overset{\star}{\Rightarrow} w \text{ iff } (q, w, A) \overset{\star}{\vdash} (p, \epsilon)$$

Hence, the total number of variables is $(\#Q)^2 \cdot \#\Gamma + 1$.

This number cannot be reduced, even if the given pda is deterministic [Goldstine, Price, Wotschke, 1982].

However, if the dpda is *unary* we can do better!

This number can be reduced to $2 \cdot \#Q \cdot \#\Gamma$

From unary dpda's to cfg's (outline)

- For each mode $[qA]$ there exists at most one state p and one string $x \in a^*$ such that $(q, x, A) \vdash^{\star} (p, \epsilon, \epsilon)$
- Given $[qA]$ we consider two variables:
  - $[qA]_0$ is used to generate the only string, if any, accepted by *empty stack* starting from the mode $[qA]$, i.e.,

$$[qA]_0 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^{\star} (p, \epsilon, \epsilon), p \in Q$$

  - $[qA]_1$ is used to generate all strings accepted by *final states* starting from the mode $[qA]$, i.e.,

$$[qA]_1 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^{\star} (p, \epsilon, \gamma),\ p \in F, \gamma \in \Gamma^+$$

- Hence, $[q_0 S] \stackrel{+}{\Rightarrow} x$ iff x is accepted by the given pda.

Each *unary deterministic* pda's in normal form with n states and m pushdown symbols can be converted into an equivalent cfg with $2mn$ variables.

From unary dpda's to cfg's (outline)

- For each mode $[qA]$ there exists at most one state p and one string $x \in a^*$ such that $(q, x, A) \vdash^\star (p, \epsilon, \epsilon)$
- Given $[qA]$ we consider two variables:
  - $[qA]_0$ is used to generate the only string, if any, accepted by *empty stack* starting from the mode $[qA]$, i.e.,

$$[qA]_0 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^\star (p, \epsilon, \epsilon), p \in Q$$

  - $[qA]_1$ is used to generate all strings accepted by *final states* starting from the mode $[qA]$, i.e.,

$$[qA]_1 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^\star (p, \epsilon, \gamma),\ p \in F, \gamma \in \Gamma^+$$

- Hence, $[q_0 S] \stackrel{+}{\Rightarrow} x$ iff x is accepted by the given pda.

Each *unary deterministic* pda's in normal form with n states and m pushdown symbols can be converted into an equivalent cfg with $2mn$ variables.

From unary dpda's to cfg's (outline)

- For each mode $[qA]$ there exists at most one state p and one string $x \in a^*$ such that $(q, x, A) \vdash^{\star} (p, \epsilon, \epsilon)$
- Given $[qA]$ we consider two variables:
  - $[qA]_0$ is used to generate the only string, if any, accepted by *empty stack* starting from the mode $[qA]$, i.e.,

$$[qA]_0 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^{\star} (p, \epsilon, \epsilon), p \in Q$$

  - $[qA]_1$ is used to generate all strings accepted by *final states* starting from the mode $[qA]$, i.e.,

$$[qA]_1 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^{\star} (p, \epsilon, \gamma),\ p \in F, \gamma \in \Gamma^+$$

- Hence, $[q_0 S] \stackrel{+}{\Rightarrow} x$ iff x is accepted by the given pda.

Each *unary deterministic* pda's in normal form with n states and m pushdown symbols can be converted into an equivalent cfg with $2mn$ variables.

From unary dpda's to cfg's (outline)

- For each mode $[qA]$ there exists at most one state p and one string $x \in a^*$ such that $(q, x, A) \vdash^\star (p, \epsilon, \epsilon)$
- Given $[qA]$ we consider two variables:
  - $[qA]_0$ is used to generate the only string, if any, accepted by *empty stack* starting from the mode $[qA]$, i.e.,

$$[qA]_0 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^\star (p, \epsilon, \epsilon), p \in Q$$

  - $[qA]_1$ is used to generate all strings accepted by *final states* starting from the mode $[qA]$, i.e.,

$$[qA]_1 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^\star (p, \epsilon, \gamma),\ p \in F, \gamma \in \Gamma^+$$

- Hence, $[q_0 S] \stackrel{+}{\Rightarrow} x$ iff x is accepted by the given pda.

Each *unary deterministic* pda's in normal form with n states and m pushdown symbols can be converted into an equivalent cfg with $2mn$ variables.

From unary dpda's to cfg's (outline)

- For each mode $[qA]$ there exists at most one state p and one string $x \in a^*$ such that $(q, x, A) \vdash^\star (p, \epsilon, \epsilon)$
- Given $[qA]$ we consider two variables:
  - $[qA]_0$ is used to generate the only string, if any, accepted by *empty stack* starting from the mode $[qA]$, i.e.,

$$[qA]_0 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^\star (p, \epsilon, \epsilon), p \in Q$$

  - $[qA]_1$ is used to generate all strings accepted by *final states* starting from the mode $[qA]$, i.e.,

$$[qA]_1 \stackrel{+}{\Rightarrow} x \text{ iff } (q, x, A) \vdash^\star (p, \epsilon, \gamma),\ p \in F, \gamma \in \Gamma^+$$

- Hence, $[q_0 S] \stackrel{+}{\Rightarrow} x$ iff x is accepted by the given pda.

Each *unary deterministic* pda's in normal form with n states and m pushdown symbols can be converted into an equivalent cfg with $2mn$ variables.

Languages with "complex" dpda's

Unary dpda's can be exponentially more succinct than dfa's.
Does this is true for *each* unary regular language?

Problem

For $m \geq 0$, let $L_m \subseteq a^*$ be a language accepted by a dfa with 2^m states.
Does there exists an equivalent dpda with $O(m)$ states?

The answer to this question is negative:

For each $m > 0$ there exists a language $L_m \subseteq a^*$ s.t.:

- L_m is accepted by a dfa with 2^m states.
- The size of any dpda accepting L_m is at least $d\frac{2^m}{m^2}$, for a constant d.

Languages with “complex” dpda’s

Unary dpda’s can be exponentially more succinct than dfa’s.
Does this is true for *each* unary regular language?

Problem

For $m \geq 0$, let $L_m \subseteq a^*$ be a language accepted by a dfa with 2^m states.
Does there exists an equivalent dpda with $O(m)$ states?

The answer to this question is negative:

For each $m > 0$ there exists a language $L_m \subseteq a^*$ s.t.:

- L_m is accepted by a dfa with 2^m states.
- The size of any dpda accepting L_m is at least $d\frac{2^m}{m^2}$, for a constant d.

Languages with “complex” dpda’s

Unary dpda’s can be exponentially more succinct than dfa’s.
Does this is true for *each* unary regular language?

Problem

For $m \geq 0$, let $L_m \subseteq a^*$ be a language accepted by a dfa with 2^m states.
Does there exists an equivalent dpda with $O(m)$ states?

The answer to this question is negative:

For each $m > 0$ there exists a language $L_m \subseteq a^*$ s.t.:

- L_m is accepted by a dfa with 2^m states.
- The size of any dpda accepting L_m is at least $d\frac{2^m}{m^2}$, for a constant d.

Languages with “complex” dpda’s

w_m de Bruijn word of order m on $\{0, 1\}$:

- $|w_m| = 2^m + m - 1$
- each string of length m is a factor of w_m, occurring in w_m exactly one time
- the suffix and the prefix of length $m - 1$ of w_m coincide.

$L_m = \{a^k \mid \text{the letter of } w_m \text{ in position } k \text{ MOD}' 2^m \text{ is } 1\}$,
where $x \text{ MOD}' y = x \text{ MOD } y$, if $x \text{ MOD } y > 0$, y otherwise

The language L_m is periodic, with period 2^m. Hence:

L_m is accepted by a dfa with 2^m states

Languages with “complex” dpda’s

w_m de Bruijn word of order m on $\{0,1\}$:

- $|w_m| = 2^m + m - 1$
- each string of length m is a factor of w_m, occurring in w_m exactly one time
- the suffix and the prefix of length $m-1$ of w_m coincide.

$L_m = \{a^k \mid \text{the letter of } w_m \text{ in position } k \text{ MOD}' 2^m \text{ is } 1\}$,
where $x \text{ MOD}' y = x \text{ MOD } y$, if $x \text{ MOD } y > 0$, y otherwise

The language L_m is periodic, with period 2^m. Hence:

L_m is accepted by a dfa with 2^m states

w_m de Bruijn word of order m on $\{0, 1\}$:

- $|w_m| = 2^m + m - 1$
- each string of length m is a factor of w_m, occurring in w_m exactly one time
- the suffix and the prefix of length $m - 1$ of w_m coincide.

$L_m = \{a^k \mid \text{the letter of } w_m \text{ in position } k \text{ MOD}' 2^m \text{ is } 1\}$,
where $x \text{ MOD}' y = x \text{ MOD } y$, if $x \text{ MOD } y > 0$, y otherwise

The language L_m is periodic, with period 2^m. Hence:

L_m is accepted by a dfa with 2^m states

Languages with “complex” dpda’s

w_m de Bruijn word of order m on $\{0, 1\}$:

- $|w_m| = 2^m + m - 1$
- each string of length m is a factor of w_m, occurring in w_m exactly one time
- the suffix and the prefix of length $m - 1$ of w_m coincide.

$L_m = \{a^k \mid \text{the letter of } w_m \text{ in position } k \bmod{}' 2^m \text{ is } 1\}$,
where $x \bmod{}' y = x \bmod y$, if $x \bmod y > 0$, y otherwise

The language L_m is periodic, with period 2^m. Hence:

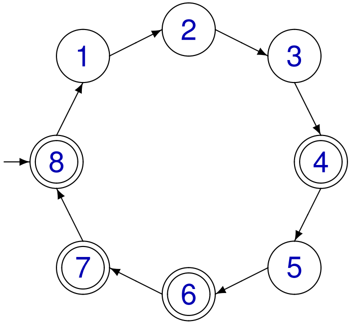
L_m is accepted by a dfa with 2^m states

Example: $m = 3$

de Bruijn word $w_3 = 0001011100$

$L_3 = \{a^0, a^4, a^6, a^7\}\{a^8\}^*$

The language L_3 is accepted by the following automaton:



Languages with “complex” dpda’s

- M: a dpda of size s accepting L_m
- M': M extended with an output tape to generate the de Bruijn word
- A: a dfa with $m+1$ states, input alphabet $\{0,1\}$, ending and accepting when the last m input symbols coincide with the suffix of length m of w_m
- M'': a dpda of size $O(ms)$, composition of M' and A, accepting $\{a^{2^m+m-1}\}$
- G: cfg grammar of size $O(ms)$, obtained from M'', generating $\{w_m\}$

Lemma ([Domaratzki, Pighizzini, Shallit, 2002])

The size of each grammar G generating $\{w_m\}$ must be at least $c\frac{2^m}{m}$, for some constant c.

Hence $s \geq d\frac{2^m}{m^2}$, for some $d > 0$.

Languages with “complex” dpda’s

- M: a dpda of size s accepting L_m
- M': M extended with an output tape to generate the de Bruijn word
- A: a dfa with $m+1$ states, input alphabet $\{0,1\}$, ending and accepting when the last m input symbols coincide with the suffix of length m of w_m
- M'': a dpda of size $O(ms)$, composition of M' and A, accepting $\{a^{2^m+m-1}\}$
- G: cfg grammar of size $O(ms)$, obtained from M'', generating $\{w_m\}$

Lemma ([Domaratzki, Pighizzini, Shallit, 2002])

The size of each grammar G generating $\{w_m\}$ must be at least $c\frac{2^m}{m}$, for some constant c.

Hence $s \geq d\frac{2^m}{m^2}$, for some $d > 0$.

Languages with “complex” dpda’s

- M: a dpda of size s accepting L_m
- M': M extended with an output tape to generate the de Bruijn word
- A: a dfa with $m+1$ states, input alphabet $\{0,1\}$, ending and accepting when the last m input symbols coincide with the suffix of length m of w_m
- M'': a dpda of size $O(ms)$, composition of M' and A, accepting $\{a^{2^m+m-1}\}$
- G: cfg grammar of size $O(ms)$, obtained from M'', generating $\{w_m\}$

Lemma ([Domaratzki, Pighizzini, Shallit, 2002])

The size of each grammar G generating $\{w_m\}$ must be at least $c\frac{2^m}{m}$, for some constant c.

Hence $s \geq d\frac{2^m}{m^2}$, for some $d > 0$.

Languages with “complex” dpda’s

- M: a dpda of size s accepting L_m
- M': M extended with an output tape to generate the de Bruijn word
- A: a dfa with $m+1$ states, input alphabet $\{0,1\}$, ending and accepting when the last m input symbols coincide with the suffix of length m of w_m
- M'': a dpda of size $O(ms)$, composition of M' and A, accepting $\{a^{2^m+m-1}\}$
- G: cfg grammar of size $O(ms)$, obtained from M'', generating $\{w_m\}$

Lemma ([Domaratzki, Pighizzini, Shallit, 2002])

The size of each grammar G generating $\{w_m\}$ must be at least $c\frac{2^m}{m}$, for some constant c.

Hence $s \geq d\frac{2^m}{m^2}$, for some $d > 0$.

Languages with “complex” dpda’s

- M: a dpda of size s accepting L_m
- M': M extended with an output tape to generate the de Bruijn word
- A: a dfa with $m+1$ states, input alphabet $\{0,1\}$, ending and accepting when the last m input symbols coincide with the suffix of length m of w_m
- M'': a dpda of size $O(ms)$, composition of M' and A, accepting $\{a^{2^m+m-1}\}$
- G: cfg grammar of size $O(ms)$, obtained from M'', generating $\{w_m\}$

Lemma ([Domaratzki, Pighizzini, Shallit, 2002])

The size of each grammar G generating $\{w_m\}$ must be at least $c^{\frac{2^m}{m}}$, for some constant c.

Hence $s \geq d^{\frac{2^m}{m^2}}$, for some $d > 0$.

Languages with "complex" dpda's

- M: a dpda of size s accepting L_m
- M': M extended with an output tape to generate the de Bruijn word
- A: a dfa with $m+1$ states, input alphabet $\{0,1\}$, ending and accepting when the last m input symbols coincide with the suffix of length m of w_m
- M'': a dpda of size $O(ms)$, composition of M' and A, accepting $\{a^{2^m+m-1}\}$
- G: cfg grammar of size $O(ms)$, obtained from M'', generating $\{w_m\}$

Lemma ([Domaratzki, Pighizzini, Shallit, 2002])

The size of each grammar G generating $\{w_m\}$ must be at least $c^{\frac{2^m}{m}}$, for some constant c.

Hence $s \geq d^{\frac{2^m}{m^2}}$, for some $d > 0$.

Languages with “complex” dpda’s

- M: a dpda of size s accepting L_m
- M': M extended with an output tape to generate the de Bruijn word
- A: a dfa with $m+1$ states, input alphabet $\{0,1\}$, ending and accepting when the last m input symbols coincide with the suffix of length m of w_m
- M'': a dpda of size $O(ms)$, composition of M' and A, accepting $\{a^{2^m+m-1}\}$
- G: cfg grammar of size $O(ms)$, obtained from M'', generating $\{w_m\}$

Lemma ([Domaratzki, Pighizzini, Shallit, 2002])

The size of each grammar G generating $\{w_m\}$ must be at least $c^{\frac{2^m}{m}}$, for some constant c.

Hence $s \geq d^{\frac{2^m}{m^2}}$, for some $d > 0$.

Simulation of unary dfa's by dpda's

As a consequence we get the following *lower bound*:

Corollary

There exists a constant $K > 0$ *such that the conversion of unary* n*-state dfa's into equivalent dpda's produces dpda's of size at least* $K\frac{n}{\log^2 n}$*, for infinitely many* n*'s.*

Related questions and results

Bounded languages:
Subsets of $w_1^* w_2^* \ldots w_n^*$, for given words $w_1, \ldots, w_n$.

Extend the investigation to *bounded deterministic context-free* languages:

- Simulation of dpda's accepting *bounded regular* languages, by finite automata.
- Simulation of dpda's accepting *bounded (context-free)* languages, by finite-turn pushdown automata.

In the nondeterministic case we have the following:

Theorem ([Malcher, Pighizzini 2007])

Each bounded context-free language generated by a cfg with h variables in Chomsky normal form is accepted by a finite-turn pda with 2^h and $O(1)$ stack symbols.

Related questions and results

Bounded languages:
Subsets of $w_1^* w_2^* \ldots w_n^*$, for given words $w_1, \ldots, w_n$.

Extend the investigation to *bounded deterministic context-free* languages:

- Simulation of dpda's accepting *bounded regular* languages, by finite automata.
- Simulation of dpda's accepting *bounded (context-free)* languages, by finite-turn pushdown automata.

In the nondeterministic case we have the following:

Theorem ([Malcher, Pighizzini, 2007])

Each bounded context-free language generated by a cfg with h variables in Chomsky normal form is accepted by a finite-turn pda with 2^h and $O(1)$ stack symbols.

Related questions and results

Bounded languages:
Subsets of $w_1^* w_2^* \dots w_n^*$, for given words $w_1, \dots, w_n$.

Extend the investigation to *bounded deterministic context-free* languages:

- Simulation of dpda's accepting *bounded regular* languages, by finite automata.
- Simulation of dpda's accepting *bounded (context-free)* languages, by finite-turn pushdown automata.

In the nondeterministic case we have the following:

Theorem ([Malcher, Pighizzini, 2007])

Each bounded context-free language generated by a cfg with h variables in Chomsky normal form is accepted by a finite-turn pda with 2^h and $O(1)$ stack symbols.

Related questions and results

Bounded languages:
Subsets of $w_1^* w_2^* \ldots w_n^*$, for given words $w_1, \ldots, w_n$.

Extend the investigation to *bounded deterministic context-free* languages:

- Simulation of dpda's accepting *bounded regular* languages, by finite automata.
- Simulation of dpda's accepting *bounded (context-free)* languages, by finite-turn pushdown automata.

In the nondeterministic case we have the following:

Theorem ([Malcher, Pighizzini, 2007])

Each bounded context-free language generated by a cfg with h variables in Chomsky normal form is accepted by a finite-turn pda with 2^h and $O(1)$ stack symbols.

Bounded languages:
Subsets of $w_1^* w_2^* \ldots w_n^*$, for given words $w_1, \ldots, w_n$.

Extend the investigation to *bounded deterministic context-free* languages:

- Simulation of dpda's accepting *bounded regular* languages, by finite automata.
- Simulation of dpda's accepting *bounded (context-free)* languages, by finite-turn pushdown automata.

In the nondeterministic case we have the following:

Theorem ([Malcher, Pighizzini, 2007])

Each bounded context-free language generated by a cfg with h variables in Chomsky normal form is accepted by a finite-turn pda with 2^h and $O(1)$ stack symbols.

What about *two-way* dpda's?

- 2dpda's are very powerful, even in the unary case: e.g., the unary versions of all languages in the class P are accepted by 2dpda's [Monien 1984].
- However, every unary 2dpda making $O(1)$ *input head reversals* accepts a regular language [Chrobak 1984].
- The same does not hold for nonunary languages: e.g., consider $\{a^n b^n a^n \mid n \geq 1\}$ which is not a cfl.

Problems:

- Input head reversals are useful to reduce the size of dpda's?
- Given a unary 2dpda of size s making at most r reversals on each input string, how many state should have a finite automaton simulating it (wrt s and r)?

What about *two-way* dpda's?

- 2dpda's are very powerful, even in the unary case: e.g., the unary versions of all languages in the class P are accepted by 2dpda's [Monien 1984].
- However, every unary 2dpda making $O(1)$ *input head reversals* accepts a regular language [Chrobak 1984].
- The same does not hold for nonunary languages: e.g., consider $\{a^n b^n a^n \mid n \geq 1\}$ which is not a cfl.

Problems:

- Input head reversals are useful to reduce the size of dpda's?
- Given a unary 2dpda of size s making at most r reversals on each input string, how many state should have a finite automaton simulating it (wrt s and r)?

What about *two-way* dpda's?

- 2dpda's are very powerful, even in the unary case: e.g., the unary versions of all languages in the class P are accepted by 2dpda's [Monien 1984].
- However, every unary 2dpda making $O(1)$ *input head reversals* accepts a regular language [Chrobak 1984].
- The same does not hold for nonunary languages: e.g., consider $\{a^n b^n a^n \mid n \geq 1\}$ which is not a cfl.

Problems:

- Input head reversals are useful to reduce the size of dpda's?
- Given a unary 2dpda of size s making at most r reversals on each input string, how many state should have a finite automaton simulating it (wrt s and r)?

What about *two-way* dpda's?

- 2dpda's are very powerful, even in the unary case: e.g., the unary versions of all languages in the class P are accepted by 2dpda's [Monien 1984].
- However, every unary 2dpda making $O(1)$ *input head reversals* accepts a regular language [Chrobak 1984].
- The same does not hold for nonunary languages: e.g., consider $\{a^n b^n a^n \mid n \geq 1\}$ which is not a cfl.

Problems:

- Input head reversals are useful to reduce the size of dpda's?
- Given a unary 2dpda of size s making at most r reversals on each input string, how many state should have a finite automaton simulating it (wrt s and r)?

What about *two-way* dpda's?

- 2dpda's are very powerful, even in the unary case: e.g., the unary versions of all languages in the class P are accepted by 2dpda's [Monien 1984].
- However, every unary 2dpda making $O(1)$ *input head reversals* accepts a regular language [Chrobak 1984].
- The same does not hold for nonunary languages: e.g., consider $\{a^n b^n a^n \mid n \geq 1\}$ which is not a cfl.

Problems:

- Input head reversals are useful to reduce the size of dpda's?
- Given a unary 2dpda of size s making at most r reversals on each input string, how many state should have a finite automaton simulating it (wrt s and r)?

Number of reversal vs nonregular acceptance

- There is a 2dpda accepting the powers of 2 in unary notation

$$\{a^{2^k} \mid k \geq 0\}$$

  using $O(\log n)$ input head reversals

- What is the minumum amount of head reversals needed by a 2dpda in order to recognize a unary nonregular language?

- I conjecture that with $o(\log n)$ reversals only regular languages can be accepted.

Number of reversal vs nonregular acceptance

- There is a 2dpda accepting the powers of 2 in unary notation
$$\{a^{2^k} \mid k \geq 0\}$$
using $O(\log n)$ input head reversals
- What is the minumum amount of head reversals needed by a 2dpda in order to recognize a unary nonregular language?
- I conjecture that with $o(\log n)$ reversals only regular languages can be accepted.

Number of reversal vs nonregular acceptance

- There is a 2dpda accepting the powers of 2 in unary notation
$$\{a^{2^k} \mid k \geq 0\}$$
using $O(\log n)$ input head reversals
- What is the minumum amount of head reversals needed by a 2dpda in order to recognize a unary nonregular language?
- I conjecture that with $o(\log n)$ reversals only regular languages can be accepted.

Number of reversal vs nonregular acceptance

- There is a 2dpda accepting the powers of 2 in unary notation
$$\{a^{2^k} \mid k \geq 0\}$$
using $O(\log n)$ input head reversals
- What is the minumum amount of head reversals needed by a 2dpda in order to recognize a unary nonregular language?
- I conjecture that with $o(\log n)$ reversals only regular languages can be accepted.