Noetherianity and Combination Problems

Silvio Ghilardi[1], Enrica Nicolini[2], Silvio Ranise[2], and Daniele Zucchelli[1,2]

[1] Dipartimento di Informatica, Università degli Studi di Milano (Italia)
[2] LORIA & INRIA-Lorraine, Nancy (France)

Abstract. In abstract algebra, a structure is said to be Noetherian if it does not admit infinite strictly ascending chains of congruences. In this paper, we adapt this notion to first-order logic by defining the class of Noetherian theories. Examples of theories in this class are Linear Arithmetics without ordering and the empty theory containing only a unary function symbol. Interestingly, it is possible to design a non-disjoint combination method for extensions of Noetherian theories. We investigate sufficient conditions for adding a temporal dimension to such theories in such a way that the decidability of the satisfiability problem for the quantifier-free fragment of the resulting temporal logic is guaranteed. This problem is firstly investigated for the case of Linear time Temporal Logic and then generalized to arbitrary modal/temporal logics whose propositional relativized satisfiability problem is decidable.

1 Introduction

Since full first-order temporal logics are known to be highly undecidable, researchers concentrated on finding fragments having good computational properties, such as the decidable monodic fragments investigated in, e.g., [17,9,12]. Although such fragments may also be used in verification, widely adopted formalisms for the specification of reactive or distributed systems (e.g., the one proposed by Manna and Pnueli [23] or the Temporal Logic of Actions by Lamport [19]) are such that the temporal part, used to describe the dynamic behavior of the systems, is parametric with respect to the underlying language of first-order logic, used to formalize the data structures manipulated by the systems. While the expressiveness of these formalisms helps in writing concise and abstract specifications, it is not clear how these can be amenable to automated analysis. The work presented in this paper contributes towards the solution of this problem, by analyzing what happens when we "add a temporal dimension" (in a sense similar to that investigated in [11]) to a decidable fragment of a *first-order theory* T with identity. By doing this, the hope is to transfer the decidability of the theory T to its "temporalized" version. This point of view has been pioneered by Plaisted in [29], where he further refined the semantics of the "temporalized T" by partitioning the symbols of the signature of T in *rigid* (whose interpretation is time-independent) and *flexible* (whose interpretation is time-dependent). This facilitates the expression of properties of both open and closed systems (see, e.g., [11] for more on this issue).

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In [14], we have presented a uniform framework where the approach in [29] has been clarified and extended. In particular, we have obtained undecidability and decidability results for quantifier-free satisfiability and model-checking problems in a temporal logic obtained by extending a decidable theory T with the operators of Linear time Temporal Logic (LTL). The key to obtain the results in [14] is a reduction of satisfiability and model-checking to the combination of (infinitely many) *partially renamed copies* of T (the symbols that are not renamed are those belonging to the rigid sub-signature Σ_r). The viewpoint of combination helps clarifying both decidability and undecidability issues. In fact, it is not always possible to transfer the decidability of the quantifier-free fragment of T to its "temporalized" version as shown by a simple reduction to known undecidable combination problems [5], even when the rigid subsignature Σ_r is empty. Fortunately, it is possible to use combination methods for non-disjoint theories in first-order logic [13] and find suitable requirements on the theory T to derive the decidability of both the satisfiability and the model-checking problem for the quantifier-free formulae of the "temporalized" version of T. The key ingredients are two. First (for correctness), it is assumed that T has a decidable universal fragment and is T_r-compatible [13], where T_r is the Σ_r-reduct of the universal fragment of T. Second (for termination), T_r is assumed to be locally finite [13]. Under these hypotheses, a (non-deterministic) combination schema can be obtained by using guessings over the finitely many (because of local finiteness) literals in the shared theory. This also simplifies the proof of correctness.

In this paper, we weaken the requirement of local finiteness to that of Noetherianity (cf. Section 3), and we focus our attention to the satisfiability problem, since model-checking is easily shown to be undecidable when considering Noetherian theories [15]. The *first contribution* of this paper is to show that our combinability requirements related to Noetherianity are met by any extension with a free unary function symbol of a stably infinite theory (cf. Section 3.2). The *second contribution* is to derive an amalgamation lemma (cf. Lemma 3.7) for combinations of (infinitely many) theories sharing a Noetherian theory (cf. Section 3.1). The combination procedure is more complex than in the locally finite case, since the exhaustive enumeration of guessings can no more be used to abstract away the exchange of now (possibly) infinitely many literals between the component theories and the combination results in [13,14] do not apply. The exchange mechanism is formalized by *residue enumerators*, i.e. computable functions returning entailed positive clauses in the shared theory. The *third contribution* of the paper is the application of the amalgamation lemma to show the decidability of the satisfiability problem for quantifier-free LTL formulae modulo a first order theory T, when T is an effectively Noetherian and T_r-compatible extension of T_r (cf. Section 4). Finally, the decidability result is extended to any modal/temporal logic whose propositional relativized satisfiability problem is decidable (cf. Section 5). For lack of space, the proofs of all results are included in the Appendix.

2 Formal Preliminaries

We adopt the usual first-order syntactic notions of signature, term, position, atom, (ground) formula, sentence, and so on. Let Σ be a first-order signature; we assume the binary equality predicate symbol '=' to be in any signature (so, if $\Sigma = \emptyset$, then Σ does not contain other symbols than equality). The signature obtained from Σ by adding it a set $\underline{a}$ of new constants (i.e., 0-ary function symbols) is denoted by $\Sigma^{\underline{a}}$. A *positive clause* is a disjunction of atoms. A *constraint* is a conjunctions of literals. A *Σ-theory* T is a set of sentences (called the axioms of T) in the signature Σ and it is *universal* iff it has universal closures of open formulae as axioms.

We also assume the usual first-order notion of interpretation and truth of a formula, with the proviso that the equality predicate $=$ is always interpreted as the identity relation. We let $\bot$ denote an arbitrary formula which is true in no structure. A formula φ is *satisfiable* in $\mathcal{M}$ iff its *existential* closure is true in $\mathcal{M}$. A Σ-structure $\mathcal{M}$ is a *model* of a Σ-theory T (in symbols $\mathcal{M} \models T$) iff all the sentences of T are true in $\mathcal{M}$. If φ is a formula, $T \models \varphi$ ('*φ is a logical consequence of T*') means that the universal closure of φ is true in all the models of T. A Σ-theory T is *complete* iff for every Σ-sentence φ, either φ or $\neg\varphi$ is a logical consequence of T. T admits *quantifier elimination* iff for every formula $\varphi(\underline{x})$ there is a quantifier-free formula $\varphi'(\underline{x})$ such that $T \models \varphi(\underline{x}) \leftrightarrow \varphi'(\underline{x})$ (notations like $\varphi(\underline{x})$ mean that φ contains free variables only among the tuple $\underline{x}$). T is *consistent* iff it has a model, i.e., if $T \not\models \bot$. A sentence φ is T-consistent iff $T \cup \{\varphi\}$ is consistent.

The *constraint satisfiability problem* for the constraint theory T is the problem of deciding whether a Σ-constraint is satisfiable in a model of T (or, equivalently, T-satisfiable). In the following, we use free constants instead of variables in constraint satisfiability problems, so that we (equivalently) redefine a constraint satisfiability problem for the theory T as the problem of *establishing the consistency of $T \cup \Gamma$ for a finite set Γ of ground $\Sigma^{\underline{a}}$-literals* (where $\underline{a}$ is a finite set of new constants). For the same reason, we abbreviate 'ground $\Sigma^{\underline{a}}$-constraint' with 'Σ-constraint,' when $\underline{a}$ is clear from the context.

If $\Sigma_0 \subseteq \Sigma$ is a subsignature of Σ and if $\mathcal{M}$ is a Σ-structure, the *Σ_0-reduct* of $\mathcal{M}$ is the Σ_0-structure $\mathcal{M}_{|\Sigma_0}$ obtained from $\mathcal{M}$ by forgetting the interpretation of function and predicate symbols from $\Sigma \setminus \Sigma_0$. A *$\Sigma$-embedding* (or, simply, an embedding) between two Σ-structures $\mathcal{M} = (M, \mathcal{I})$ and $\mathcal{N} = (N, \mathcal{J})$ is any mapping $\mu : M \longrightarrow N$ among the corresponding support sets satisfying the condition

$$\mathcal{M} \models \varphi \quad \text{iff} \quad \mathcal{N} \models \varphi \tag{1}$$

for all Σ^M-atoms φ (here $\mathcal{M}$ is regarded as a Σ^M-structure, by interpreting each additional constant $a \in M$ into itself and $\mathcal{N}$ is regarded as a Σ^M-structure by interpreting each additional constant $a \in M$ into $\mu(a)$). If $M \subseteq N$ and if the embedding $\mu : \mathcal{M} \longrightarrow \mathcal{N}$ is just the identity inclusion $M \subseteq N$, we say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ or that $\mathcal{N}$ is an *extension* of $\mathcal{M}$. In case condition (1) holds for all first order formulae, the embedding μ is said to be *elementary*.

3 Noetherian Theories

In abstract algebra, the adjective Noetherian is used to describe structures that satisfy an ascending chain condition on congruences (see, e.g., [22]): since congruences can have special representations, Noetherianity concerns, e.g., chains of ideals in the case of rings and chains of submodules in the case of modules. Although this is somewhat non-standard, we may take a more abstract view and say that a *variety* (i.e. an equational class of structures) is Noetherian iff finitely generated free algebras satisfy the ascending chain condition for congruences or, equivalently, iff finitely generated algebras are finitely presented. Now, congruences over finitely generated free algebras may be represented as sets of equations among terms. This allows us to equivalently re-state the Noetherianity of varieties as "there are no infinite ascending chains of sets of equations modulo logical consequence". This observation was the basis for the abstract notion of Noetherian Fragment introduced in [16], here adapted for an arbitrary first-order theory.

Definition 3.1 (Noetherian Theory). *A Σ_0-theory T_0 is* Noetherian *if and only if for every* finite *set of free constants $\underline{a}$, every infinite ascending chain*

$$\Theta_1 \subseteq \Theta_2 \subseteq \cdots \subseteq \Theta_n \subseteq \cdots$$

of sets of ground $\Sigma_0^{\underline{a}}$-atoms is eventually constant modulo T_0, i.e. there is an n such that $T_0 \cup \Theta_n \models A$, for every natural number m and atom $A \in \Theta_m$.

Natural examples of Noetherian theories are the first-order axiomatization (in equational logic) of varieties like K-algebras, K-vector spaces, and R-modules, where K is a field and R is a Noetherian ring (see [22] for further details). *Abelian semigroups* are also Noetherian (cf. Theorem 3.11 in [8]). Notice that, since any extension (in the same signature) of a Noetherian theory is also Noetherian, any theory extending the theory of a single Associative-Commutative symbol is Noetherian. This shows that the family of Noetherian theories is important for verification because theories axiomatizing *integer addition* or *multiset union* formalize crucial aspects of a system to be verified (e.g., multisets may be used to check that the result of some operations like sorting on a collection of objects yields a permutation of the initial collection). More examples will be considered below.

Before being able to describe our new combination method, we need to introduce some preliminary notions. In the remaining of this section, *we fix two theories $T_0 \subseteq T$ in their respective signatures $\Sigma_0 \subseteq \Sigma$.*

Definition 3.2 (T_0-basis). *Given a finite set Θ of ground clauses (built out of symbols from Σ and possibly further free constants) and a finite set of free constants $\underline{a}$, a* T_0-basis *for Θ w.r.t. $\underline{a}$ is a set Δ of* positive *ground $\Sigma_0^{\underline{a}}$-clauses such that*

(i) *$T \cup \Theta \models C$, for all $C \in \Delta$ and*
(ii) *if $T \cup \Theta \models C$ then $T_0 \cup \Delta \models C$, for every positive ground $\Sigma_0^{\underline{a}}$-clause C.*

Notice that only constants in $\underline{a}$ may occur in a T_0-basis for Θ w.r.t. $\underline{a}$, although Θ may contain constants not in $\underline{a}$.

Definition 3.3 (Residue Enumerator). *Given a finite set $\underline{a}$ of free constants, a T-residue enumerator for T_0 w.r.t. $\underline{a}$ is a computable function $Res_T^{\underline{a}}(\Gamma)$ mapping a Σ-constraint Γ to a finite T_0-basis of Γ w.r.t. $\underline{a}$.*

If Γ is T-unsatisfiable, then a residue enumerator can always return the singleton set containing the empty clause. The concept of (Noetherian) residue enumerator is inspired by the work on partial theory reasoning (see, e.g., [3]) and generalizes the notion of deduction complete procedure of [18]. Given a residue enumerator for constraints (cf. Definition 3.3), it is always possible to build one for clauses (this will be useful for the combination method, see below).

Lemma 3.4. *Given a finite set $\underline{a}$ of free constants and a T-residue enumerator for T_0 w.r.t. $\underline{a}$, there exists a computable function $Res_T^{\underline{a}}(\Theta)$ mapping a finite set of ground clauses Θ to a finite T_0-basis of Θ w.r.t. $\underline{a}$.*

If T_0 is Noetherian, then it is possible to show that a finite T_0-basis for Γ w.r.t. $\underline{a}$ always exists, for every Σ-constraint Γ and for every set $\underline{a}$ of constants, by using König lemma. Unfortunately, such a basis is not always computable; this motivates the following notion.

Definition 3.5. *The theory T is an* effectively Noetherian extension of T_0 *if and only if T_0 is Noetherian and there exists a T-residue enumerator for T_0 w.r.t. every finite set $\underline{a}$ of free constants.*

For example, the theory of commutative K-algebras is an effectively Noetherian extension of the theory of K-vector spaces, where K is a field (see [16,28] for details). Locally finite theories and Linear Real Arithmetic are further examples taken from the literature about automated theorem proving.

A Σ_0-theory T_0 is *locally finite* iff Σ_0 is finite and, for every finite set of free constants $\underline{a}$, there are finitely many ground $\Sigma_0^{\underline{a}}$-terms $t_1, \ldots, t_{k_{\underline{a}}}$ such that for every ground $\Sigma_0^{\underline{a}}$-term u, $T_0 \models u = t_i$ (for some $i \in \{1, \ldots, k_{\underline{a}}\}$). If such $t_1, \ldots, t_{k_{\underline{a}}}$ are effectively computable from $\underline{a}$, then T_0 is *effectively locally finite* and there are finitely many (*representative*) $\Sigma_0^{\underline{a}}$-atoms $\psi_1(\underline{a}), \ldots, \psi_m(\underline{a})$ such that for any $\Sigma_0^{\underline{a}}$-atom $\psi(\underline{a})$, there is some i such that $T_0 \models \psi_i(\underline{a}) \leftrightarrow \psi(\underline{a})$. Examples of effectively locally finite theories are Boolean algebras, Linear Integer Arithmetic modulo a given integer, and any theory over a finite purely relational signature. Also, theories consisting of sentences which are true in a fixed finite Σ_0-structure $\mathcal{M} = (M, \mathcal{I})$ are locally finite. Enumerated datatypes can be formalized by theories in this class. The class of locally finite theories is (strictly) contained in that of Noetherian theories: to see this, it is sufficient to notice that, once fixed a finite set of free constants $\underline{a}$, there are only finitely many $\Sigma_0^{\underline{a}}$-atoms which are not equivalent to each other modulo the locally finite theory. From this, it is obvious that any infinite ascending chain of sets of such atoms must be eventually constant. Under the hypotheses that T_0 is effectively locally finite and its extension T has decidable constraint satisfiability problem, it is straightforward to build a T-residue enumerator for T_0.

Example. Let us consider the signature $\Sigma = \{0, +, -, \{f_r\}_{r\in\mathbb{R}}, \leq\}$ where 0 is a constant, $-$ and f_r are unary function symbols, $+$ is a binary function symbol, $\leq$ is a binary predicate symbol, and $\Sigma_0 = \Sigma \setminus \{\leq\}$. We consider the theory $T_{\mathbb{R}}^{\leq} = Th_{\Sigma}(\mathbb{R})$, i.e. the set of all Σ-sentences true in $\mathbb{R}$, which is seen as an $\mathbb{R}$-vector space equipped with a linear ordering, where the f_r's represent the external product so that terms are all equivalent to homogeneous linear polynomials. Finally, let $T_{\mathbb{R}}$ be $Th_{\Sigma_0}(\mathbb{R})$, i.e. the set of all Σ_0-sentences true in $\mathbb{R}$, which is seen as an $\mathbb{R}$-vector space without the ordering (so $T_{\mathbb{R}}$ is the theory of the $\mathbb{R}$-vector spaces, not reduced to $\{0\}$). The Noetherianity of $T_{\mathbb{R}}$ follows from general algebraic properties (see, e.g., [22]). A $T_{\mathbb{R}}^{\leq}$-residue enumerator for $T_{\mathbb{R}}$ can be obtained as follows. Let $\Gamma = \{C_1, \ldots, C_m\}$ be a set of inequalities, i.e. Σ-atoms whose main predicate symbol is $\leq$. By Definition 3.2, a Σ_0-basis for Γ is the set of all the disjunctions of equalities implied by Γ. Actually, to compute a basis, it is sufficient to identify the set of *implicit* equalities in Γ, i.e. the equalities $C_i^{=}$ such that $T_{\mathbb{R}}^{\leq} \models \Gamma \rightarrow C_i^{=}$ (here $C_i^{=}$ is obtained from C_i by substituting $\leq$ with $=$). This is so because (i) $T_{\mathbb{R}}^{\leq}$ is Σ_0-*convex* (i.e. if $T_{\mathbb{R}}^{\leq} \models \Gamma \rightarrow (e_1 \vee \cdots \vee e_n)$, then there exists $i \in \{1, \ldots, n\}$ such that $T_{\mathbb{R}}^{\leq} \models \Gamma \rightarrow e_i$, for $n \geq 1$ and equalities $e_1, \ldots, e_n$) and (ii) given a system of inequalities Γ, if Δ is the collection of all the implicit equalities of Γ and e is an equality such that $T_{\mathbb{R}}^{\leq} \models \Gamma \rightarrow e$, then $T_{\mathbb{R}} \models \Delta \rightarrow e$ (see [21] for full details, [28] for the adaptation to our context). The interest of implicit equalities is that they can be easily identified by using the Fourier-Motzkin variable elimination method (see [20] for details on how to do this).

3.1 Combination over Noetherian Theories

Preliminarily, we recall the notion of T_0-compatibility [13] which is crucial for the completeness of our combination technique.

Definition 3.6 (T_0-compatibility [13]). *Let T be a theory in the signature Σ and let T_0 be a universal theory in a subsignature $\Sigma_0 \subseteq \Sigma$. We say that T is* T_0-compatible *iff $T_0 \subseteq T$ and there is a Σ_0-theory $T_0^\star$ such that (i) $T_0 \subseteq T_0^\star$; (ii) $T_0^\star$ has quantifier elimination; (iii) every model of T_0 can be embedded into a model of $T_0^\star$; and (iv) every model of T can be embedded into a model of $T \cup T_0^\star$.*

The requirements (i)-(iii) guarantee the uniqueness of the theory $T_0^\star$, provided it exists ($T_0^\star$ is the *model completion* of T_0, see e.g. [7]). Notice that if T_0 is the empty theory over the empty signature, then $T_0^\star$ is the theory axiomatizing an infinite domain, (i)-(iii) hold trivially, and (iv) can be shown equivalent to the stably infinite requirement of the Nelson-Oppen schema [27,31]. Examples of theories satisfying the compatibility condition are the following: (a) the theory of K-algebras is compatible with the theory of K-vector spaces, where K is a field (see [16,28]), (b) $T_{\mathbb{R}}^{\leq}$ is compatible with the universal fragment of $T_{\mathbb{R}}$ (this is so for $T_{\mathbb{R}}^{\leq} \supseteq T_{\mathbb{R}}$ and $T_{\mathbb{R}}$ eliminates quantifiers), (c) any equational extension over a larger signature of the theory BA of Boolean algebras is BA-compatible [13],

and (d) any extension of T_0 whatsoever is T_0-compatible whenever T_0 eliminates quantifiers.

The following lemma is *our main technical tool* allowing us to reduce satisfiability in a "temporalized" extension of a (Noetherian) theory to satisfiability in first-order logic.

Lemma 3.7 (Amalgamation). *Let I be a (possibly infinite) set of indexes; $\Sigma_i^{\underline{c},\underline{a}_i}$ (for $i \in I$) be signatures (expanded with free constants $\underline{c}, \underline{a}_i$), whose pairwise intersections are all equal to a certain signature $\Sigma_r^{\underline{c}}$ (i.e. $\Sigma_i^{\underline{c},\underline{a}_i} \cap \Sigma_j^{\underline{c},\underline{a}_j} = \Sigma_r^{\underline{c}}$, for all distinct $i, j \in I$); T_i be Σ_i-theories (for $i \in I$) which are all T_r-compatible, where $T_r \subseteq \bigcap_i T_i$ is a universal Σ_r-theory; $\{\Gamma_i\}_{i\in I}$ be sets of ground $\Sigma_i^{\underline{c},\underline{a}_i}$-clauses; and $\mathcal{B}^\star$ be a set of positive ground $\Sigma_r^{\underline{c}}$-clauses not containing the empty clause and satisfying the following condition:*

$$\text{if } T_i \cup \Gamma_i \cup \mathcal{B}^\star \models C, \text{ then } C \in \mathcal{B}^\star,$$

for $i \in I$ and every positive ground $\Sigma_r^{\underline{c}}$-clause C. Then, there exists a $\bigcup_i(\Sigma_i^{\underline{c},\underline{a}_i})$-structure $\mathcal{M}$ such that $\mathcal{M} \models \bigcup_i(T_i \cup \Gamma_i)$ or, equivalently, there exist $\Sigma_i^{\underline{c},\underline{a}_i}$-structures $\mathcal{M}_i$ ($i \in I$) satisfying $T_i \cup \Gamma_i$, whose $\Sigma_r^{\underline{c}}$-reducts coincide.

This lemma can also be used to prove the "first-order version" of the combination result in [16], where residue enumerators permit the exchange of positive clauses between theories.

3.2 The Theory of a Free Unary Function Symbol

By collecting the observations above, it is easy to identify pairs of theories (T, T_0) such that T satisfies our relevant requirements to be 'combined over T_0' (i.e. T is such that $T_0 \subseteq T$ and T is a T_0-compatible effectively Noetherian extension of T_0). Here, we consider an entirely new (and somewhat remarkable) class of examples of such pairs (T, T_0) of theories.

Let f be a unary function symbol. If T is a theory, then T_f is the theory obtained from T by adding f to its signature (as a new free function symbol). So, e.g., if E the empty theory over the empty signature, E_f denotes the empty theory over the signature $\{f\}$.

Proposition 3.8. *E_f is Noetherian.*

A theory T is stably infinite (see, e.g., [27,31]) iff it is E-compatible, or, equivalently, iff any T-satisfiable constraint is satisfiable in a model of T whose domain is infinite.

Proposition 3.9. *If T is stably infinite and has decidable constraint satisfiability problem, then T_f is an effectively Noetherian extension of E_f.*

Proposition 3.10. *If T is stably infinite, then T_f is E_f-compatible.*

We are now ready to characterize our new class of theories.

Theorem 3.11. *Let T be a theory with decidable constraint satisfiability problem. If T is stably infinite, then T_f is an effectively Noetherian extension of E_f, which is also E_f-compatible.*

This result is a first step towards the integration in our framework of some theories that are useful for verification. For example, the theory of integer offsets can be seen as an extension of the theory of a loop-free unary function symbol (see, e.g., [1]). Properties of hardware systems can be expressed in a mixture of temporal logic – e.g., LTL or Computation Tree Logic (CTL) – and the theory of integer offsets [6]. Our decidability results on "temporalized" first-order theories below (see Theorems 4.11 and 5.4) can then be used to augment the degree of automation of tools attempting to solve this kind of verification problems.

4 Temporalizing a First-Order Theory

We introduce "temporalized" first-order theories, by using LTL to describe the temporal dimension. We use the formal framework introduced in [14] where formulae are obtained by applying temporal and Boolean operators (but no quantifiers) to first-order formulae over a given signature.

Definition 4.1 (LTL($\Sigma^{\underline{a}}$)-Sentences [14]). *Given a signature Σ and a (finite or infinite) set of free constants $\underline{a}$, the set of LTL($\Sigma^{\underline{a}}$)-sentences is inductively defined as follows: (a) if φ is a first-order $\Sigma^{\underline{a}}$-sentence, then φ is an LTL($\Sigma^{\underline{a}}$)-sentence and (b) if ψ_1, ψ_2 are LTL($\Sigma^{\underline{a}}$)-sentence, so are $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, $\neg\psi_1$, $X\psi_1$, $\Box\psi_1$, $\Diamond\psi_1$, $\psi_1 U \psi_2$.*

The free constants $\underline{a}$ allowed in LTL($\Sigma^{\underline{a}}$)-sentences will be used to model the variables and the parameters of (reactive) systems.

Definition 4.2 ([14]). *Given a signature Σ and a set $\underline{a}$ of free constants, an* LTL($\Sigma^{\underline{a}}$)-structure *(or simply a* structure*) is a sequence $\mathcal{M} = \{\mathcal{M}_n = (M, \mathcal{I}_n)\}_{n\in\mathbb{N}}$ of $\Sigma^{\underline{a}}$-structures. The set M is called the* domain *(or the* universe*) and $\mathcal{I}_n$ is called the n-*th level interpretation function *of the LTL($\Sigma^{\underline{a}}$)-structure.*[1]

When considering a background Σ-theory T, the structures $\mathcal{M}_n = (M_n, \mathcal{I}_n)$ will be taken to be models of T (further requirements will be analyzed later on).

Definition 4.3 ([14]). *Given an LTL($\Sigma^{\underline{a}}$)-sentence φ and $t \in \mathbb{N}$, the notion of "φ being true in the LTL($\Sigma^{\underline{a}}$)-structure $\mathcal{M} = \{\mathcal{M}_n = (M, \mathcal{I}_n)\}_{n\in\mathbb{N}}$ at the instant t" (in symbols $\mathcal{M} \models_t \varphi$) is inductively defined as follows:*

- *if φ is an first-order sentence, $\mathcal{M} \models_t \varphi$ iff $\mathcal{M}_t \models \varphi$;*
- *$\mathcal{M} \models_t \neg\varphi$ iff $\mathcal{M} \not\models_t \varphi$;*
- *$\mathcal{M} \models_t \varphi \wedge \psi$ iff $\mathcal{M} \models_t \varphi$ and $\mathcal{M} \models_t \psi$;*
- *$\mathcal{M} \models_t \varphi \vee \psi$ iff $\mathcal{M} \models_t \varphi$ or $\mathcal{M} \models_t \psi$;*
- *$\mathcal{M} \models_t X\varphi$ iff $\mathcal{M} \models_{t+1} \varphi$;*

[1] In more detail, $\mathcal{I}_n$ is such that $\mathcal{I}_n(P) \subseteq M^k$ for every predicate symbols $P \in \Sigma$ of arity k, and $\mathcal{I}_n(f) : M^k \longrightarrow M$ for each function symbol $f \in \Sigma$ of arity k.

- $\mathcal{M} \models_t \Box\varphi$ *iff for each* $t' \geq t$, $\mathcal{M} \models_{t'} \varphi$;
- $\mathcal{M} \models_t \Diamond\varphi$ *iff for some* $t' \geq t$, $\mathcal{M} \models_{t'} \varphi$;
- $\mathcal{M} \models_t \varphi U \psi$ *iff there exists* $t' \geq t$ *such that* $\mathcal{M} \models_{t'} \psi$ *and for each* t'', $t \leq t'' < t' \Rightarrow \mathcal{M} \models_{t''} \varphi$.

Let φ be an LTL($\Sigma^{\underline{a}}$)-sentence; we say that φ *is true in* $\mathcal{M}$ or, equivalently, that $\mathcal{M}$ *satisfies* φ (in symbols $\mathcal{M} \models \varphi$) iff $\mathcal{M} \models_0 \varphi$.

Since we distinguish between rigid (i.e. time-independent) and flexible (i.e. time-dependent) symbols of the signature, we need to introduce a notion of first-order theory that fixes a sub-signature and distinguish between two kinds of free constants.

Definition 4.4. *A* data-flow theory *is a 5-tuple* $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ *where* Σ *is a signature,* T *is a* Σ*-theory (called the underlying theory of* $\mathcal{T}$*),* Σ_r *is the* rigid subsignature *of* Σ*,* $\underline{a}$ *is a set of free constants (called* system variables*), and* $\underline{c}$ *is a set of free constants (called* system parameters*).*

A data-flow theory $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ is *totally flexible* iff Σ_r is empty and is *totally rigid* iff $\Sigma_r = \Sigma$. In [14], data-flow theories are called LTL-theories. Here, we prefer to use the more abstract term of data-flow theory in order to prepare for the generalization of the decidability result in the next section.

Definition 4.5 ([14]). *An LTL($\Sigma^{\underline{a},\underline{c}}$)-structure* $\mathcal{M} = \{\mathcal{M}_n = (M, \mathcal{I}_n)\}_{n \in \mathbb{N}}$ *is* appropriate *for a data-flow theory* $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ *iff for all* $m, n \in \mathbb{N}$*, for all function symbol* $f \in \Sigma_r$*, for all relational symbol* $P \in \Sigma_r$*, and for all constant* $c \in \underline{c}$*, we have*

$$\mathcal{M}_n \models T, \quad \mathcal{I}_n(f) = \mathcal{I}_m(f), \quad \mathcal{I}_n(P) = \mathcal{I}_m(P), \quad \mathcal{I}_n(c) = \mathcal{I}_m(c).$$

The satisfiability problem for $\mathcal{T}$ *is the following: given an LTL($\Sigma^{\underline{a},\underline{c}}$)-sentence* φ*, decide whether there is an LTL($\Sigma^{\underline{a},\underline{c}}$)-structure* $\mathcal{M}$ *appropriate for* $\mathcal{T}$ *such that* $\mathcal{M} \models \varphi$*. The* ground satisfiability problem for $\mathcal{T}$ *is similarly introduced, but* φ *is assumed to be ground.*

Notice that appropriate structures are such that the equality symbol is always interpreted as the identity relation, since the equality is included in every signature (hence also in the rigid signature Σ_r).

In the sequel, we shall concentrate on the ground satisfiability problem for data-flow theories; for this reason, we shall assume from now on that **the underlying theory** T **of any data-flow theory** $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ **has decidable constraint satisfiability problem**. Unfortunately, this assumption is insufficient to guarantee decidability.

Theorem 4.6 ([14]). *There exists a totally flexible data-flow theory* $\mathcal{T}$ *whose ground satisfiability problem is undecidable.*

Notwithstanding the undecidability of the ground satisfiability problem, the following compatibility requirement can be used to re-gain decidability.

Definition 4.7. *A data-flow theory $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ is said to be* Noetherian compatible *iff there is a Σ_r-universal theory T_r such that T is an effectively Noetherian and T_r-compatible extension of T_r.*

The definition above refers to a Σ_r-theory T_r such that T is T_r-compatible. Although not relevant for the results in this paper, we notice that if such a theory T_r exists, then one can always take T_r to be the theory axiomatized by the universal Σ_r-sentences which are logical consequences of T.

4.1 A Decision Procedure for the Noetherian Compatible Case

Preliminarily, we recall that it is possible to define the notion of ground model-checking problem in our framework [14] and to show its undecidability when the underlying theory is Noetherian. The argument of the proof is a simple reduction to the (undecidable) reachability problem of Minsky machines [26,10] by using the reduct of Presburger Arithmetic obtained by forgetting addition and ordering, which is capable of encoding counters (see [15] for details). This is why here we focus on the ground satisfiability problem in the Noetherian compatible case.

Before developing our decision procedure, some preliminary notions are required.

Definition 4.8 (PLTL-Abstraction [14]). *Given a signature $\Sigma^{\underline{a}}$ and a set of propositional letters $\mathcal{L}$ of the appropriate cardinality, let $[\![\cdot]\!]$ be a bijection from the set of ground $\Sigma^{\underline{a}}$-atoms into $\mathcal{L}$. By translating identically Boolean and temporal connectives, the map is inductively extended to a bijective map (also called $[\![\cdot]\!]$) from the set of ground LTL($\Sigma^{\underline{a}}$)-sentences onto the set of propositional $\mathcal{L}$-formulae.*

Given a ground LTL($\Sigma^{\underline{a}}$)-sentence φ, we call $[\![\varphi]\!]$ the *PLTL-abstraction* of φ; moreover, if Θ is a set of ground LTL($\Sigma^{\underline{a}}$)-sentences, the PLTL-abstraction $[\![\Theta]\!]$ of Θ denotes the set $\{[\![\varphi]\!] \mid \varphi \in \Theta\}$.

Definition 4.9 (φ-Guessing). *Let φ be a ground LTL($\Sigma^{\underline{a},\underline{c}}$)-sentence. A* φ-guessing *is a Boolean assignment to literals of φ (we view a guessing as the set $\{\ell \mid \ell$ is an atom occurring in φ and ℓ is assigned to true$\} \cup \{\neg\ell \mid \ell$ is an atom occurring in φ and ℓ is assigned to false$\}$).*

We say that a (non-empty) set of φ-guessings $\mathcal{G}_{(\varphi)} := \{G_1, \ldots, G_k\}$ is *φ-compatible* if and only if $[\![\varphi \wedge \Box \bigvee_{i=1}^{k} G_i]\!]$ is PLTL-satisfiable.

Let $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ be a Noetherian compatible data-flow theory. The procedure NSAT (see Algorithm 1) takes a ground LTL($\Sigma^{\underline{a},\underline{c}}$)-sentence φ as input and returns *"satisfiable"* if there is an appropriate LTL($\Sigma^{\underline{a},\underline{c}}$)-structure $\mathcal{M}$ for $\mathcal{T}$ such that $\mathcal{M} \models \varphi$; otherwise, it returns *"unsatisfiable"*. The procedure relies on a decision procedure for the PLTL-satisfiability problem in order to recognize the φ-compatible sets of φ-guessings (cf. the outer loop of NSAT). Moreover, DP-T is a decision procedure for the satisfiability problem of arbitrary Boolean

Algorithm 1 The satisfiability procedure for the Noetherian compatible case

Require: φ ground LTL($\Sigma^{\underline{a},\underline{c}}$)-sentence
1: **procedure** NSAT(φ)
2: **for all** φ-compatible set of φ-guessing $\mathcal{G}_{(\varphi)}$ **do**
3: $\mathcal{B} \leftarrow \emptyset$
4: **repeat**
5: $\mathcal{B}' \leftarrow \mathcal{B}$
6: **for all** $G_i \in \mathcal{G}_{(\varphi)}$ **do**
7: $\mathcal{B}_i \leftarrow Res_T^{\underline{c}}(G_i \cup \mathcal{B})$
8: **end for**
9: $\mathcal{B} \leftarrow \bigcup_i \mathcal{B}_i$
10: **until** DP-T($\mathcal{B}' \wedge \neg\mathcal{B}$) = *"unsatisfiable"*
11: **if** DP-T($\mathcal{B}$) = *"satisfiable"* **then**
12: **return** *"satisfiable"*
13: **end if**
14: **end for**
15: **return** *"unsatisfiable"*
16: **end procedure**

combinations of atoms of the theory T (i.e., it is capable of checking the T-satisfiability of sets of ground $\Sigma^{\underline{a},\underline{c}}$-clauses and not only of ground $\Sigma^{\underline{a},\underline{c}}$-literals). Notice that DP-T can be implemented by Satisfiability Modulo Theories solvers (see, e.g., [30]). Finally, $Res_T^{\underline{c}}$ is the T-residue enumerator for T_r w.r.t. $\underline{c}$.

In the outer loop of NSAT, all possible φ-compatible sets of φ-guessings are enumerated. Let $\mathcal{G}_{(\varphi)} := \{G_1, \ldots, G_n\}$ be the current φ-guessing. The local variable $\mathcal{B}$ is initialized to the empty set (line 3) and then updated in the inner loop (lines 4-10) as follows: the T_r-bases $\mathcal{B}_i$ for $G_i \cup \mathcal{B}$ w.r.t. $\underline{c}$ are computed (for $i = 1, \ldots, n$), and the new value of $\mathcal{B}$ is set to $\bigcup_i \mathcal{B}_i$ (line 5 saves the old value of $\mathcal{B}$ in $\mathcal{B}'$). The inner loop is iterated until $\mathcal{B}$ is logically equivalent to $\mathcal{B}'$ modulo T. At this point, if $\mathcal{B}$ is T-consistent, the procedure stops and returns *"satisfiable"*; otherwise it tries another φ-compatible set of φ-guessings. If for all φ-compatible sets of φ-guessings the $\mathcal{B}$'s returned after the execution of the inner loop are T-inconsistent, the procedure returns *"unsatisfiable"*.

Proposition 4.10 (Correctness of NSat). *Let $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ be a Noetherian compatible data-flow theory and φ be a ground LTL($\Sigma^{\underline{a},\underline{c}}$)-sentence. Then,* NSAT*(φ) returns "satisfiable" iff there exists an LTL($\Sigma^{\underline{a},\underline{c}}$)-structure $\mathcal{M}$ appropriate for $\mathcal{T}$ such that $\mathcal{M} \models \varphi$.*

Indeed, the termination of NSAT is a consequence of the Noetherianity of the underlying theory of $\mathcal{T}$ by using the fact that every infinite ascending chain of sets of positive ground $\Sigma_r^{\underline{c}}$-clauses is eventually constant for logical consequence. The correctness and termination of NSAT yield our main decidability result.

Theorem 4.11. *The ground satisfiability problem for Noetherian compatible data-flow theories is decidable.*

The theories considered in the previous section (especially, those in Section 3.2) satisfy the hypothesis of the theorem above.

5 Extensions to Abstract Temporal Logics

By considering the proof of the correctness of NSAT, it becomes evident that only very few of the characteristic properties of LTL are used. It turns out that a simple generalization of NSAT can be used to decide satisfiability problems of "temporalized" extensions of Noetherian theories whose flow of time is not linear, e.g., branching as in CTL.

In order to formalize the observation above, we regard modal/temporal operators as functions operating on powerset Boolean algebras. In this way, logics for various flows of time, as well as CTL, Propositional Dynamic Logic (PDL), and the μ-calculus fall within the scope of our result (see [2] for a similar approach).

Definition 5.1. *An* abstract temporal signature[2] *I is a purely functional signature extending the signature BA of Boolean algebras.*[3] *An* abstract temporal logic *L is a class of I-structures, whose Boolean reducts are powerset Boolean algebras. Given an I-term t, deciding whether $t \neq 0$ is satisfied in some member of L is the* satisfiability *problem for L. Given I-terms t, u, deciding whether $u = 1$ & $t \neq 0$ is satisfied in some member of L is the* relativized satisfiability *problem for L.*

In many cases (e.g., LTL, CTL, PDL, and the μ-calculus), it is possible to reduce the relativized satisfiability problem to that of satisfiability (by using the so-called "master modality"); however, there are logics for which the latter is decidable whereas the former is undecidable (see [12]).

Definition 5.2 ($I(\Sigma^{\underline{a}})$-sentence). *Given a signature Σ, a (finite or infinite) set of free constants $\underline{a}$, and an abstract temporal signature I, the set of $I(\Sigma^{\underline{a}})$-sentences is inductively defined as follows: (a) if φ is a first-order $\Sigma^{\underline{a}}$-sentence, then φ is an $I(\Sigma^{\underline{a}})$-sentence, (b) if φ_1, φ_2 are $I(\Sigma^{\underline{a}})$-sentences, so are $\varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2, \neg\varphi_1$, and (c) if $\psi_1, \ldots, \psi_n$ are $I(\Sigma^{\underline{a}})$-sentences and $O \in I \setminus BA$ has arity n, then $O(\psi_1, \ldots, \psi_n)$ is a $I(\Sigma^{\underline{a}})$-sentence.*

When I is LTL, $I(\Sigma^{\underline{a},\underline{c}})$-sentences coincide with LTL$(\Sigma^{\underline{a},\underline{c}})$-sentences (cf. Definition 4.1). We defined an abstract temporal logic L (based on I) as a class of I-structures based on powerset Boolean algebras: such structures (also called I-frames) will be denoted with $\mathcal{F} = (\wp(F), \{O^{\mathcal{F}}\}_{O \in I \setminus BA})$.

[2] From the modal/temporal literature viewpoint, the adjective "intensional" might be preferable to "abstract temporal". We have chosen the latter, in order to emphasize that our results are deemed as significant for a class of logics whose modalities concern flows of time.

[3] This signature contains two binary function symbols for meet and join, a unary function symbol for complement, and two constants for zero and one (the latter are denoted with 0 and 1, respectively).

Definition 5.3. *Let a signature Σ, a set $\underline{a}$ of free constants, and an abstract temporal signature I be given; an $I(\Sigma^{\underline{a}})$-*structure *(or simply a* structure*) is a pair formed by an I-frame $\mathcal{F} = (\wp(F), \{O^{\mathcal{F}}\}_{O \in I \setminus BA})$ and a collection $\mathcal{M} = \{\mathcal{M}_n = (M, \mathcal{I}_n)\}_{n \in F}$ of $\Sigma^{\underline{a}}$-structures (all based on the same domain).*

An $I(\Sigma^{\underline{a}})$-sentence φ is true in the $I(\Sigma^{\underline{a}})$-structure $(\mathcal{F}, \mathcal{M})$ at $t \in F$ (noted $\mathcal{F}, \mathcal{M} \models_t \varphi$) iff the following holds: (a) if φ is a first-order sentence, then $\mathcal{F}, \mathcal{M} \models_t \varphi$ holds iff $\mathcal{M}_t \models \varphi$ and (b) if the main operator of φ is a Boolean connective, truth of φ is defined in a truth-table manner; (c) if φ is of the kind $O(\psi_1, \ldots, \psi_n)$, then $\mathcal{F}, \mathcal{M} \models_t \varphi$ holds iff $t \in O^{\mathcal{F}}(\{u \mid \mathcal{F}, \mathcal{M} \models_u \psi_1\}, \ldots, \{u \mid \mathcal{F}, \mathcal{M} \models_u \psi_n\})$.

If a data-flow theory $\mathcal{T}$ is given, we say that an $I(\Sigma^{\underline{a}})$-structure is appropriate for $\mathcal{T}$ iff it satisfies the requirements of Definition 4.5. The (ground) satisfiability problem for an abstract temporal logic L (based on I) and for a data-flow theory $\mathcal{T}$ is now the following: given a (ground) $I(\Sigma^{\underline{a}})$-sentence φ, decide whether there is a $I(\Sigma^{\underline{a}})$-structure $(\mathcal{F}, \mathcal{M})$ appropriate for $\mathcal{T}$, such that $\mathcal{F} \in L$ and such that $\mathcal{F}, \mathcal{M} \models_t \varphi$ holds for some t.

Theorem 5.4. *The ground satisfiability problem for $\mathcal{T}$ and L is decidable if (i) $\mathcal{T}$ is Noetherian compatible and (ii) the relativized satisfiability problem for L is decidable.*

When I is LTL, this result simplifies to Theorem 4.11. To prove Theorem 5.4, it is possible to re-use NSAT (cf. Algorithm 1) almost 'off-the-shelf', by preliminarily adapting the definition of PLTL-abstraction function $[\![\cdot]\!]$ (cf. Definition 4.8) to L in the obvious way. It turns out that only the compatibility of guessings should be changed: a finite set of φ-guessings $\mathcal{G}_{(\varphi)} := \{G_1, \ldots, G_k\}$ is φ-compatible if and only if the relativized satisfiability problem

$$[\![\varphi]\!] \neq 0 \quad \& \quad [\![\bigvee_{i=1}^{k} G_i]\!] = 1$$

is satisfiable in L (this is the only modification required to the definitions and proofs from Section 4.1).

While Theorem 4.11 is relevant to augment the degree of mechanization of deductive approaches for the verification of reactive systems based on LTL (e.g., the one put-forward by Manna and Pnueli [23]), one may wonder about the relevance of its generalization, i.e. Theorem 5.4. To see its usefulness, consider TLA [19]. For such a specification formalism, it is difficult to reuse techniques and tools for (classic) temporal/modal logic since TLA features some non-standard characteristics which are quite useful for practitioners (see [25] for an extensive discussion on this and related issues). On the other hand, deductive verification of TLA specifications can be supported by proof assistants (e.g., [24]). While applying the inference rules of TLA [19], it has been observed [25] that some of the resulting sub-goals may belong to a fragment of TLA which is equivalent to the modal logic S4.2 [4]. Now, the relativized satisfiability problem for this logic is decidable (see again [4]) so that NSAT can be used to automatically discharge some of the sub-goals, whenever the data-flow theory formalizing

the data structure manipulated by the system modelled in TLA is Noetherian compatible.

6 Conclusions

We have investigated the role of Noetherianity for the decidability of the satisfiability problem for "temporalized" first-order theories (cf. Sections 4 and 5). The key technical contribution is Lemma 3.7, which allows us to obtain amalgamations of (possibly infinite) sequences of first-order structures corresponding to temporal structures. This lemma is the basis of a method for combinations of first-order theories over Noetherian theories. An important class of stably infinite theories extending the empty theory over a single unary function symbol has been shown to satisfy the hypotheses for the decidability of both the combination schema and the satisfiability of "temporalized" first-order theories (cf. Section 3.2).

The results in this paper extends those of [14] in two ways. First, the requirement of local finiteness of the (rigid) sub-theory is weakened to that of Noetherianity. Second, decidability is parametric w.r.t. a modal/temporal logic, provided that relativized satisfiability problem is decidable in the latter.

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A Proofs

The appendix is organized as follows: Section A.1 recalls some basic results coming from model theory and some theorems about structure amalgamations; Section A.2 presents the proofs regarding correctness and termination of the procedure NSAT, and finally Section A.3 contains the details about the new examples presented in Section 3.2.

A.1 Model-Theoretic Background

We first recall some further standard background. Given a Σ-structure $\mathcal{M} = (M, \mathcal{I})$ and a subset $C \subseteq M$, the substructure of $\mathcal{M}$ generated by C is the substructure obtained from $\mathcal{M}$ by restricting $\mathcal{I}$ to the subset $\{t^{\mathcal{M}}(\underline{c}) \mid \underline{c} \subseteq C$ and $t(\underline{x})$ is a Σ-term$\}$ (here $t^{\mathcal{M}}$ is the function interpreting the term t in $\mathcal{M}$). In case this substructure coincides with $\mathcal{M}$, we say that C is a set of generators for $\mathcal{M}$.

If C is a set of generators for $\mathcal{M}$, the *diagram* $\Delta(\mathcal{M})$ of $\mathcal{M}$ (w.r.t. Σ, C) consists of all ground Σ^C-literals that hold in $\mathcal{M}$; analogously, the *elementary diagram* $\Delta^e(\mathcal{M})$ of $\mathcal{M}$ (w.r.t. Σ, C) consists of all ground Σ^C-sentences that hold in $\mathcal{M}$ (often C is not specified at all, in these cases it is assumed to coincide with the whole carrier set of $\mathcal{M}$).

Diagrams (in combination with the compactness of the logical consequence relation) will be repeatedly used. A typical standard use is the following: suppose that we want to embed $\mathcal{M}$ into a model of a theory T, then it is sufficient to check that $T \cup \Delta(\mathcal{M})$ is consistent. This argument is justified by Robinson's Diagram Lemma [33], which relates embeddings and diagrams as follows.

Lemma A.1 (Robinson's Diagram Lemma). *Let $\mathcal{M}$ be a Σ-structure generated by a set C, and let $\mathcal{N}$ be another Σ-structure; then $\mathcal{M}$ can be embedded (resp. elementarily embedded) into $\mathcal{N}$ iff $\mathcal{N}$ can be expanded to Σ^C-model of the diagram $\Delta(\mathcal{M})$ (resp. of the elementary diagram $\Delta^e(\mathcal{M})$) of $\mathcal{M}$ w.r.t. Σ, C.*

The technique used for proving Lemma A.1 is simple, we sketch it. If we have an expansion of $\mathcal{N}$ to a Σ^C-structure (to be called $\mathcal{N}$ again for simplicity), then, since every element of the support of $\mathcal{M}$ is of the kind $t^{\mathcal{M}}(\underline{c})$ for some $c \subseteq C$, we can define the embedding μ by putting $\mu(t^{\mathcal{M}}(\underline{c})) := t^{\mathcal{N}}(\underline{c}^{\mathcal{N}})$: this is well-defined and it is an embedding precisely because $\mathcal{N} \models \Delta(\mathcal{M})$. Conversely, if we have the embedding μ, then we can get the desired expansion by taking $c^{\mathcal{N}} := \mu(c)$ for all $c \in C$.

Since a surjective embedding is just an isomorphism, the argument just sketched shows also the following fact:

Lemma A.2. *If two Σ-structures $\mathcal{M}$, $\mathcal{N}$ are both generated by a set C and if one of them, say $\mathcal{N}$, satisfies the other's diagram (w.r.t. Σ, C), then the two structures are Σ^C-isomorphic.*

Ground formulae are invariant under embeddings in the following sense.

Lemma A.3. *Let $\mathcal{M} = (M, \mathcal{I})$ be a Σ-structure that can be embedded into another Σ-structure $\mathcal{N}$. For all ground Σ^M-sentences φ, we have that*

$$\mathcal{M} \models \varphi \qquad \Leftrightarrow \qquad \mathcal{N} \models \varphi,$$

where $\mathcal{N}$ is extended to a Σ^M-structure by interpreting each $a \in M$ by its image under the embedding.

Next lemma states the well-known property (called submodel-completeness) of theories enjoying quantifier-elimination:

Lemma A.4. *Suppose that $T^\star$ is a Σ_r-theory enjoying quantifier elimination and that Δ is a diagram of a substructure $\mathcal{R} = (R, \mathcal{J})$ of a model $\mathcal{M}$ of $T^\star$; then the Σ^R-theory $T^\star \cup \Delta$ is complete.*

Proof. By Robinson Diagram Lemma A.1, the models of $T^\star \cup \Delta$ are the models of $T^\star$ endowed with a Σ_r-embedding from $\mathcal{R}$. One such model is $\mathcal{M}$; we show that any other model $\mathcal{M}'$ satisfies the same Σ^R-sentences as $\mathcal{M}$ (we assume without loss of generality the Σ_r-embedding from $\mathcal{R}$ into $\mathcal{M}'$ to be an inclusion). Pick an arbitrary Σ^R-sentence $\varphi(\underline{c})$ (where the $\underline{c}$ are parameters from the set of generators of $\mathcal{R}$ used in order to build Δ): this sentence is equivalent, modulo $T^\star$, to a ground Σ^R-sentence $\varphi^\star(\underline{c})$. Since truth of ground sentences is preserved by substructures (see Lemma A.3), we have the following chain of equivalences

$$\mathcal{M}' \models \varphi(\underline{c}) \Leftrightarrow \mathcal{M}' \models \varphi^\star(\underline{c}) \Leftrightarrow \mathcal{R} \models \varphi^\star(\underline{c}) \Leftrightarrow \mathcal{M} \models \varphi^\star(\underline{c}) \Leftrightarrow \mathcal{M} \models \varphi(\underline{c}),$$

showing our claim.

Next result is also part of basic classical model theory: a proof of it can be easily deduced from Craig's Interpolation Theorem (alternatively, a direct proof using a double chain argument is possible, see [33], pp. 141-142):

Theorem A.5 (Robinson's Joint Consistency Theorem). *Let H_1, H_2 be, respectively, consistent Θ_1, Θ_2-theories and let Θ_0 be the signature $\Theta_1 \cap \Theta_2$. Suppose that there is a complete Θ_0-theory H_0 such that $H_0 \subseteq H_1$ and $H_0 \subseteq H_2$; then $H_1 \cup H_2$ is a consistent $\Theta_1 \cup \Theta_2$-theory.*

Structure Amalgamations The statement of next Lemma extends the statement of Lemma 9.3 from [13] (and is proved in the same way):

Lemma A.6. *Let T_i be Σ_i-theories (for $i \in I$) and let Σ_r be a subsignature of all the Σ_i's. Let*

$$\Gamma_1, \ldots, \Gamma_i, \ldots \quad (i \in I)$$

be sets of ground $\Sigma_i^{\underline{a}_i, \underline{c}}$-clauses (here $\underline{a}_i, \underline{c}$ are free constants); a set $\mathcal{B}^\star$ of positive ground $\Sigma_r^{\underline{c}}$-clauses is said to be saturated iff for every $i \in I$ and for every positive ground $\Sigma_r^{\underline{c}}$-clause C it happens that:

$$T_i \cup \Gamma_i \cup \mathcal{B}^\star \models C \quad \Rightarrow \quad C \in \mathcal{B}^\star.$$

Suppose now that $\mathcal{B}^\star$ is saturated and does not contain the empty clause. Then there are $\Sigma_i^{\underline{a}_i,\underline{c}}$-structures $\mathcal{M}_i$ such that $\mathcal{M}_i \models T_i \cup \Gamma_i \cup \mathcal{B}^\star$; moreover, the $\Sigma_r^{\underline{c}}$-substructures generated by the elements (denoted by) $\underline{c}$ coincide for all the $\mathcal{M}_i$'s.

Proof. A set of ground $\Sigma_r^{\underline{c}}$-literals is said to be exhaustive iff it contains, for every ground $\Sigma_r^{\underline{c}}$-literal A, either A itself or its negation. The statement of the lemma is proved if we are able to find an exhaustive set Δ of ground $\Sigma_r^{\underline{c}}$-literals which is consistent with $T_i \cup \Gamma_i \cup \mathcal{B}^\star$ for each $i \in I$. In this case, in fact, given models $\mathcal{M}_i \models T_i \cup \Gamma_i \cup \mathcal{B}^\star \cup \Delta$, we have that the $\Sigma_r^{\underline{c}}$-substructures generated by $\underline{c}$ in all the $\mathcal{M}_i$'s all have diagram Δ, consequently they are $\Sigma_r^{\underline{c}}$-isomorphic (and can be made coincident by suitable renaming).

We shall adapt the notion of productive clause used in nowadays refutational completeness proofs for e.g. resolution or paramodulation based calculi. Consider any strict total terminating order on ground $\Sigma_r^{\underline{c}}$-atoms and extend it to a strict total terminating order $>$ for positive ground $\Sigma_r^{\underline{c}}$-clauses by taking standard multiset extension. We shall define increasing sets Δ_C^+ (varying $C \in \mathcal{B}^\star$) of ground $\Sigma_r^{\underline{c}}$-atoms as follows. Recall that, as the empty clause is not in $\mathcal{B}^\star$, all positive clauses in $\mathcal{B}^\star$ are of the kind $A \vee A_1 \vee \cdots \vee A_n$ ($n \geq 0$).

The definition is by transfinite induction on $>$. Say that the clause $C \equiv A \vee A_1 \vee \cdots \vee A_n$ from $\mathcal{B}^\star$ is *productive* iff (i) $\{A\} > \{A_1, \ldots, A_n\}$ and (ii) $A_1, \ldots, A_n \notin \Delta_{<C}^+$ (where $\Delta_{<C}^+$ is $\bigcup_{D<C} \Delta_D^+$). Now, if C is productive, we let Δ_C^+ to be $\Delta_{<C}^+ \cup \{A\}$, otherwise Δ_C^+ is simply $\Delta_{<C}^+$.

Let Δ^+ be $\bigcup_{C \in \mathcal{B}^\star} \Delta_C^+$ and Δ be $\Delta^+ \cup \{\neg A \mid A$ is a ground $\Sigma_r^{\underline{c}}$-atom not belonging to $\Delta^+\}$. By construction, $\Delta \models \mathcal{B}^\star$, so we simply need to show that $T_i \cup \Gamma_i \cup \Delta$ is consistent for each $i \in I$. We need a preliminary claim.

Claim: if the clause $A \vee A_1 \vee \cdots \vee A_n$ is productive and A is the maximum atom in it, then $A_1, \ldots, A_n \notin \Delta^+$: this is evident, as the A_i's could only be produced by clauses smaller than $A \vee A_1 \vee \cdots \vee A_n$.

Suppose now that $T_i \cup \Gamma_i \cup \Delta$ is not consistent. Then there are ground atoms $B_1, \ldots, B_m \notin \Delta^+$ and productive clauses

$$\begin{aligned} C_1 &\equiv A_1 \vee A_{11} \vee \cdots \vee A_{1k_1} \\ &\cdots \\ C_n &\equiv A_n \vee A_{n1} \vee \cdots \vee A_{nk_n} \end{aligned}$$

(with maximum atoms $A_1, \ldots, A_n$, respectively), such that

$$T_i \cup \Gamma_i \cup \{A_1, \ldots, A_n\} \models B_1 \vee \cdots \vee B_m.$$

By trivial logical manipulations, it follows that

$$T_i \cup \Gamma_i \cup \{C_1, \ldots, C_n\} \models \bigvee_{i,j} A_{ij} \vee B_1 \vee \cdots \vee B_m.$$

As $C_1, \ldots, C_n$ are clauses in $\mathcal{B}^\star$ and as $\mathcal{B}^\star$ is saturated, the clause

$$D \equiv \bigvee_{i,j} A_{ij} \vee B_1 \vee \cdots \vee B_m$$

is also in $\mathcal{B}^\star$. By construction (anyway, either D is productive or not) some of the atoms $\{A_{11}, \ldots, A_{nk_n}, B_1, \ldots, B_m\}$ are in Δ^+. By the claim, $A_{11}, \ldots, A_{nk_n}$ cannot be there, so one of the B_j's is in Δ^+, contradiction.

Next Lemma also extends a fact (namely Lemma 9.4) established in [13]:

Lemma A.7. *Let I be a (possibly infinite) set of indexes, $\Sigma_i^{\underline{c},\underline{a}_i}$ be signatures expanded with free constants $\underline{c}, \underline{a}_i$ (for $i \in I$), whose pairwise intersections are all equal to a certain signature $\Sigma_r^{\underline{c}}$ (i.e., we have $\Sigma_i^{\underline{c},\underline{a}_i} \cap \Sigma_j^{\underline{c},\underline{a}_j} = \Sigma_r^{\underline{c}}$ for all distinct $i, j \in I$). Suppose we are also given Σ_i-theories T_i which are all T_r-compatible, where $T_r \subseteq \bigcap_i T_i$ is a universal Σ_r-theory; let finally $\{\mathcal{M}_i = (M_i, \mathcal{I}_i)\}_{i \in I}$ be a sequence of $\Sigma_i^{\underline{c},\underline{a}_i}$-structures which are models of T_i and satisfy the same $\Sigma_r^{\underline{c}}$-atoms. Under these hypotheses, there exist a $\bigcup_i(\Sigma_i^{\underline{c},\underline{a}_i})$-structure $\mathcal{M} \models \bigcup_i T_i$ such that for each i, $\mathcal{M}_i$ has a $\Sigma_i^{\underline{c},\underline{a}_i}$-embedding into $\mathcal{M}$.*

Proof. By Robinson Diagram Lemma A.1 and by Lemma A.2 (and up to a partial renaming of the support sets), the fact that the $\mathcal{M}_i$ satisfy the same $\Sigma_r^{\underline{c}}$-atoms is another way of saying that they share the same $\Sigma_r^{\underline{c}}$-substructure generated by the $\underline{c}$ (let us call $\mathcal{R} = (R, \mathcal{J})$ this substructure); by T_r-compatibility, we may also freely assume that $\mathcal{M}_i \models T_i \cup T_r^\star$. Notice also that, by Lemma A.4 above, the theory $T_r^\star \cup \Delta$ is complete, where Δ is the diagram of $\mathcal{R}$ as a Σ_r-structure.

Again by Robinson Diagram Lemma, we only need to show that the union of the elementary diagrams $\Delta_i^e(\mathcal{M}_i)$ is consistent:[4] here $\Delta_i^e(\mathcal{M}_i)$ is the elementary diagram of $\mathcal{M}_i$ as a $\Sigma_i^{\underline{c},\underline{a}_i}$-structure.

By compactness, we can freely assume that the index set I is finite, let it be $\{1, \ldots, k\}$ and let us argue by induction on k. For $k = 1$, there is nothing to prove and for $k > 1$, we use Robinson's Joint Consistency Theorem as follows.

By renaming some elements in the supports if needed, we can freely suppose that the sets $M_1 \setminus R$ and $(M_2 \cup \cdots \cup M_k) \setminus R$ are disjoint. Given the hypotheses of the Lemma on the signatures $\Sigma_i^{\underline{c},\underline{a}_i}$, we can apply the Joint Consistency Theorem to the theories $\Delta^e(\mathcal{M}_1)$ and $\Delta^e(\mathcal{M}_2) \cup \cdots \cup \Delta^e(\mathcal{M}_k)$: in fact, they are both consistent (the latter by induction) and their both contain the complete subtheory $T_r^\star \cup \Delta$ in the shared subsignature. This proves that $\Delta^e(\mathcal{M}_1) \cup \cdots \cup \Delta^e(\mathcal{M}_k)$ is consistent, as desired.

If we put together the two previous lemmas, we get the following fact:

Lemma 3.7. *Suppose we are given the following data:*

(i) *I is a (possibly infinite) set of indexes;*
(ii) *$\Sigma_i^{\underline{c},\underline{a}_i}$ (for $i \in I$) are signatures (expanded with free constants $\underline{c}, \underline{a}_i$), whose pairwise intersections are all equal to a certain signature $\Sigma_r^{\underline{c}}$ (that is, we have $\Sigma_i^{\underline{c},\underline{a}_i} \cap \Sigma_j^{\underline{c},\underline{a}_j} = \Sigma_r^{\underline{c}}$ for all distinct $i, j \in I$);*
(iii) *T_i are Σ_i-theories (for $i \in I$) which are all T_r-compatible, where $T_r \subseteq \bigcap_i T_i$ is a universal Σ_r-theory;*

[4] We need the elementary diagrams here, and not just diagrams, because we want the model to be built to be a model of $\bigcup_i T_i$.

(iv) $\{\Gamma_i\}_{i\in I}$ *are sets of ground* $\Sigma_i^{\underline{a}_i,\underline{c}}$*-clauses;*
(v) $\mathcal{B}^\star$ *is a set of positive ground* $\Sigma_r^{\underline{c}}$*-clauses not containing the empty clause and satisfying the following condition for every* $i \in I$ *and for every positive ground* $\Sigma_r^{\underline{c}}$*-clause* C*:*

$$T_i \cup \Gamma_i \cup \mathcal{B}^\star \models C \quad \Rightarrow \quad C \in \mathcal{B}^\star.$$

If the above data are given, then there exists a $\bigcup_i(\Sigma_i^{\underline{c},\underline{a}_i})$*-structure* $\mathcal{M} \models \bigcup_i(T_i \cup \Gamma_i)$*. Equivalently: there exist* $\Sigma_i^{\underline{c},\underline{a}_i}$*-structures* $\mathcal{M}_i$ *(*$i \in I$*) satisfying* $T_i \cup \Gamma_i$*, whose* $\Sigma_r^{\underline{c}}$*-reducts coincide.*

A.2 Proofs of the Decidability Results

In order to introduce the reader to our decidability result, we need to prove two technical lemmas. The following generalizes the notion of residue enumerator from constraints to set of clauses:

Lemma 3.4. *Given a finite set* $\underline{a}$ *of free constants and a* T*-residue enumerator for* T_0 *w.r.t.* $\underline{a}$*, there is a computable function* $Res_T^{\underline{a}}(\Theta)$ *mapping a finite set of ground clauses* Θ *to a finite* T_0*-basis of* Θ *w.r.t.* $\underline{a}$*.*

Proof. We proceed as follows. First of all, let us convert Θ into its disjunctive normal form $\bigvee_i \Gamma_i$. Let $\Delta_i := Res_T^{\underline{a}}(\Gamma_i)$. We claim that Δ, namely the conversion into conjunctive normal form of $\bigvee_i \Delta_i$, is a T_0-basis for Θ w.r.t. $\underline{a}$. Indeed, 3.2-(i) is verified since, for each i, $T \cup \Gamma_i \models \Delta_i$ (because Δ_i is a T_0-basis for Γ_i), so it follows $T \cup \bigvee_i \Gamma_i \models \bigvee_i \Delta_i$, hence $T \cup \Theta \models \Delta$ (recall that Δ is logically equivalent to $\bigvee_i \Delta_i$). Moreover, 3.2-(ii) is verified because $T \cup \Theta \models C$ iff $T \cup \bigvee_i \Gamma_i \models C$ if and only if, for each i, $T \cup \Gamma_i \models C$, hence, for each i, $T_0 \cup \Delta_i \models C$ (again because Δ_i is a T_0-basis for Γ_i), and finally $T_0 \cup \Delta \models C$.

The next lemma transfers the termination property from sets of atoms to sets of positive clauses:

Lemma A.8. *Every infinite ascending chain of sets of positive ground* $\Sigma_r^{\underline{c}}$*-clauses is eventually constant for logical consequence modulo a Noetherian* Σ*-theory* T_r*.*

Proof. By contradiction, suppose not; in this case it is immediate to see that there are infinitely many positive ground T_r-clauses $C_1, C_2, \ldots$ such that for all i the clause C_i is not a logical consequence of $T_r \cup \{C_1, \ldots, C_{i-1}\}$.

Let us build a chain of trees $\mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \cdots$, whose nodes are labeled by positive ground $\Sigma_r^{\underline{c}}$-atoms as follows. $\mathcal{T}_0$ consists of the root only, which is labeled $\top$. Suppose $\mathcal{T}_{i-1}$ is already built and consider the clause $C_i \equiv B_1 \vee \cdots \vee B_m$. To build $\mathcal{T}_i$, do the following for every leaf K of $\mathcal{T}_{i-1}$ (let the branch leading to K be labeled by $A_1, \ldots, A_k$): append new sons to K labeled $B_1, \ldots, B_m$, respectively, if C_i is such that $T_r \cup \{A_1, \ldots, A_k\} \not\models C_i$ (if this is not the case, do nothing for the leaf K).

Consider now the union tree $\mathcal{T} = \bigcup \mathcal{T}_i$: since, whenever a node labeled A_{k+1} is added, A_{k+1} is not a logical consequence w.r.t. T_r of the formulae labeling the

predecessor nodes, by the Noetherianity of T_r all branches are then finite and by König lemma the whole tree is itself finite. This means that for some index j, the examination of clauses C_i (for $i > j$) did not yield any modification of the already built tree. Now, C_{j+1} is not a logical consequence of $T_r \cup \{C_1, \ldots, C_j\}$: this means that there is a Σ_r^c-structure $\mathcal{M}$ which is a model of T_r and in which all atoms of C_{j+1} are false and the $C_1, \ldots, C_j$ are all true. By induction on $i = 0, \ldots, j$, it is easily seen that there is a branch in $\mathcal{T}_i$ whose labeling atoms are true in $\mathcal{M}$: this contradicts the fact that the tree $\mathcal{T}_j$ has not been modified in step $j + 1$.

The termination of the procedure NSAT is stated by the following

Lemma A.9. *The procedure* NSAT *always terminates.*

Proof. Since the number of literals occurring in φ is finite, there is only a finite number of φ-guessings, and thus there is a finite number of sets of φ-guessings $\mathcal{G}_{(\varphi)}$. So, it remains to prove that the inner loop of lines 4-10 of Algorithm 1 terminates; to this aim we recall the fact (proved in Lemma A.8) that every infinite ascending chain of sets of positive ground Σ_r^c-clauses is eventually constant for logical consequence w.r.t. a Noetherian theory T_r. The test on line 10 eventually have to succeed by the following reason: if we let $\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2, \ldots$ be the values of the local variable $\mathcal{B}$ after each execution of the loop, we have that $T_r \cup \mathcal{B}^{i+1} \models \mathcal{B}^i$, for each i, by Definition 3.2(ii). Thus, if we let $\mathcal{D}_i := \bigcup_{j \leq i} \mathcal{B}_j$, then the succession

$$\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots$$

is increasing and hence eventually constant modulo $T_r \subseteq T$, which means that also the above mentioned test eventually succeeds.

The following straightforward lemma explains why PLTL-abstractions are relevant for satisfiability checking of LTL($\Sigma^{\underline{a}}$)-sentences.

Lemma A.10. *Let $\mathcal{L}$ be a set of propositional letters, Σ be a signature, $\underline{a}$ be a set of free constants, and $[\![\cdot]\!]$ be a PLTL-abstraction function mapping ground LTL($\Sigma^{\underline{a}}$)-sentences into propositional $\mathcal{L}$-formulae. Suppose we are given a ground LTL($\Sigma^{\underline{a}}$)-sentence φ, a Kripke model V for $\mathcal{L}$ (based on $\mathbb{N}$ as a temporal flow) and an LTL($\Sigma^{\underline{a}}$)-structure $\mathcal{M} = \{\mathcal{M}_n\}_{n \in \mathbb{N}}$ such that for every $t \in \mathbb{N}$ and for every $\Sigma^{\underline{a}}$-ground atom ℓ occurring in φ we have*

$$\mathcal{M}_t \models \ell \qquad \textit{iff} \qquad V_t([\![\, \ell \,]\!]) = 1.$$

Then we have also

$$\mathcal{M} \models_t \varphi \qquad \textit{iff} \qquad V \models_t [\![\, \varphi \,]\!],$$

for every $t \in \mathbb{N}$.

Proof. The proof is by an easy induction on the complexity of the subformulae ψ occurring in φ.

The key to define a reduction to the satisfiability problem in PLTL is guessing. The following two lemmas state the correctness of the procedure NSAT.

Lemma A.11 (Soundness). *Let $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ be a Noetherian compatible data-flow theory and φ be a ground LTL($\Sigma^{\underline{a},\underline{c}}$)-sentence. If* NSAT*($\varphi$) returns "satisfiable", then there is an LTL($\Sigma^{\underline{a},\underline{c}}$)-structure $\mathcal{M}$ appropriate for $\mathcal{T}$ such that $\mathcal{M} \models \varphi$.*

Proof. If NSAT(φ) returns *"satisfiable"*, then there exists a (non-empty) set of φ-guessings $\mathcal{G}_{(\varphi)} := \{G_1, \dots G_n\}$ that are φ-compatible, i.e. such that $[\![\varphi \wedge \Box(\bigvee_i G_i)]\!]$ is satisfiable (as usual, with a little abuse of notation, we confuse the set G_i with the conjunction of the literals occurring in it). NSAT(φ) will produce the list of sets of positive ground $\Sigma_r^{\underline{c}}$-clauses

$$\mathcal{B}_1^0, \dots, \mathcal{B}_n^0, \mathcal{B}_1^1, \dots, \mathcal{B}_n^1, \dots, \mathcal{B}_1^h, \dots, \mathcal{B}_n^h,$$

such that:

- $\mathcal{B}^0, \dots, \mathcal{B}^h, \mathcal{B}^{h+1}$ are the values of the local variable $\mathcal{B}$ in the iterations of the inner loop (we have $\mathcal{B}^0 = \emptyset, \mathcal{B}^1 = \bigcup_i \mathcal{B}_i^0, \dots, \mathcal{B}^{h+1} = \bigcup_i \mathcal{B}_i^h$);
- for $j = 0, \dots, h$ and for $i = 1, \dots, n$, the set $\mathcal{B}_i^j$ is a T_r-basis for $G_i \cup \mathcal{B}^j$ w.r.t. $\underline{c}$,
- $\mathcal{B}^{h+1}$ is T-consistent and logically equivalent to $\mathcal{B}^h$ modulo T.

Let $\mathcal{B}^\star := \{C \mid T \cup \mathcal{B}^h \models C$ and C is a positive ground $\Sigma_r^{\underline{c}}$-clause$\}$; notice that $\mathcal{B}^\star$ does not contain the empty clause, moreover we claim that for every positive ground $\Sigma_r^{\underline{c}}$-clause C and for each $i \in \{1, \dots, n\}$, we have

$$T \cup G_i \cup \mathcal{B}^\star \models C \quad \Rightarrow \quad C \in \mathcal{B}^\star. \tag{2}$$

In fact, if $T \cup G_i \cup \mathcal{B}^\star \models C$, then $T \cup G_i \cup \mathcal{B}^h \models C$ and so, by Definition 3.2(ii) $T_r \cup \mathcal{B}_i^h \models C$; but then $T_r \cup \mathcal{B}^{h+1} \models C$, meaning that $T \cup \mathcal{B}^h \models C$ (because $\mathcal{B}^{h+1}$ is logically equivalent to $\mathcal{B}^h$) and finally $C \in \mathcal{B}^\star$ by the definition of the latter. Let $V := V_0 \to V_1 \to \dots \to V_n \to \dots$ be the infinite succession of Boolean assignments that is a PLTL model for $[\![\varphi \wedge \Box(\bigvee_i G_i)]\!]$. Let us consider the infinite sequence $\{G'_n\}_{n \in \mathbb{N}}$ of guessings such that $G'_n := G_i$ and $V \models_n [\![G_i]\!]$ (this is well-set since for every $n \geq 0$ there exists only one G_i such that $V \models_n [\![G_i]\!]$). By (2) and by Lemma 3.7,[5] we obtain an infinite sequence $\mathcal{M}_0, \dots, \mathcal{M}_i, \dots$ of $\Sigma^{\underline{a},\underline{c}}$-structures such that (i) they all have the same support M and $\mathcal{M}_{i|\Sigma_r^{\underline{c}}} = \mathcal{M}_{j|\Sigma_r^{\underline{c}}}$ $(i, j \in \mathbb{N})$; (ii) $\mathcal{M}_i \models T \cup G'_i$. These $\mathcal{M}_i$ consequently form an LTL($\Sigma^{\underline{a},\underline{c}}$)-structure $\mathcal{M} := \{\mathcal{M}_i\}_{i \in \mathbb{N}}$ that, by construction, for every atom ℓ occurring in φ satisfies the condition: $\mathcal{M} \models_i \ell$ iff $V \models_i [\![\ell]\!]$. Applying Lemma A.10 we have that $\mathcal{M} \models_0 \varphi$, because $V \models_0 [\![\varphi]\!]$, thus $\mathcal{M} \models \varphi$ obtains.

[5] Lemma 3.7 is used with $I := \mathbb{N}$, and $T_i := T$, but symbols from $\Sigma \setminus \Sigma_r$ are disjointly renamed when building the signature Σ_i for the i-th copy of T (the same observation applies also to the flexible constants $\underline{a}$). In this way, a model of $\bigcup_i T_i$ is the same thing as a sequence of models $\{\mathcal{M}'_n\}_{n \in \mathbb{N}}$ of T whose $\Sigma_r^{\underline{c}}$-reducts coincide.

Lemma A.12 (Completeness). *Let $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ be a Noetherian compatible data-flow theory and φ be a ground LTL($\Sigma^{\underline{a},\underline{c}}$)-sentence. If there is an LTL($\Sigma^{\underline{a},\underline{c}}$)-structure $\mathcal{M}$ appropriate for $\mathcal{T}$ such that $\mathcal{M} \models \varphi$, then* NSAT($\varphi$) *returns "satisfiable".*

Proof. Let $\mathcal{M} = \{\mathcal{M}_n = (M, \mathcal{I}_n)\}_{n \in \mathbb{N}}$ be an LTL($\Sigma^{\underline{a},\underline{c}}$)-structure appropriate for $\mathcal{T}$ such that $\mathcal{M} \models \varphi$. Let us consider the set of φ-guessings $\mathcal{G}_{(\varphi)} := \{G_1, \ldots G_k\}$ defined as follows: $G_i \in \mathcal{G}_{(\varphi)}$ iff there exists an n such that $\mathcal{M} \models_n G_i$. It is easy to verify that $\mathcal{G}_{(\varphi)}$ is φ-compatible, i.e. that $[\![\varphi \wedge \Box(\bigvee_i G_i)]\!]$ is satisfiable (here $G_i \in \mathcal{G}_{(\varphi)}$). Infact, the PLTL structure V that satisfies the condition $V \models_n [\![\ell]\!]$ iff $\mathcal{M} \models_n \ell$ for every atom ℓ occurring in φ is a model for $[\![\varphi \wedge \Box(\bigvee_i G_i)]\!]$ by Lemma A.10.

When examining the φ-guessing $\mathcal{G}_{(\varphi)}$, the procedure DP-LTL produces a (finite, by lemma A.9) list of sets of positive ground $\Sigma_r^{\underline{c}}$-clauses

$$\mathcal{B}_1^0, \ldots, \mathcal{B}_k^0, \mathcal{B}_1^1, \ldots, \mathcal{B}_k^1, \ldots, \mathcal{B}_1^h, \ldots, \mathcal{B}_k^h,$$

such that:

- $\mathcal{B}^0, \ldots, \mathcal{B}^h, \mathcal{B}^{h+1}$ are the values of the local variable $\mathcal{B}$ in the iterations of the inner loop (we have $\mathcal{B}^0 = \emptyset, \mathcal{B}^1 = \bigcup_i \mathcal{B}_i^0, \ldots, \mathcal{B}^{h+1} = \bigcup_i \mathcal{B}_i^h$);
- for $j = 0, \ldots, h$ and for $i = 1, \ldots, k$, the set $\mathcal{B}_i^j$ is a T_r-basis for $G_i \cup \mathcal{B}^j$ w.r.t. $\underline{c}$;
- $\mathcal{B}^{h+1}$ is logically equivalent to $\mathcal{B}^h$ modulo T.

We need to show that $\mathcal{B}^h$ is T-consistent. To this aim it is sufficient to observe (by induction on $j \leq h$) that the a $\Sigma_r^{\underline{c}}$-clause belonging to $\mathcal{B}^j$ is true in $\mathcal{M}_0$ (in fact in all the $\mathcal{M}_n$, because the symbols of $\Sigma_r^{\underline{c}}$ are rigidly interpreted): this is obvious for $j = 0$ and for $j > 0$ it is a direct consequence of the fact that every G_i is true in some $\mathcal{M}_n$, by induction hypothesis and Definition 3.2(i).

The above Lemmas A.11 and A.12 lead to the following

Proposition 4.10. *Let $\mathcal{T} = \langle \Sigma, T, \Sigma_r, \underline{a}, \underline{c} \rangle$ be a Noetherian compatible data-flow theory and φ be a ground LTL($\Sigma^{\underline{a},\underline{c}}$)-sentence. Then* NSAT($\varphi$) *returns "satisfiable" iff there exists an LTL($\Sigma^{\underline{a},\underline{c}}$)-structure $\mathcal{M}$ appropriate for $\mathcal{T}$ such that $\mathcal{M} \models \varphi$.*

As an immediate corollary of Proposition 4.10, together with Lemma A.9, we obtain

Theorem 4.11. *The ground satisfiability problem for a Noetherian compatible data-flow theory is decidable.*

A.3 Effectively Noetherian Extensions and Unary Functions Symbols

This Section is devoted to the proofs of the results from Section 3.2 concerning free unary function symbols. We recall that if f is a unary function symbol and

T is a theory, we denote by T_f the theory obtained from T by adding it f as a new free function symbol (thus, if we denote by E the empty theory in the empty signature, E_f denotes the free theory over the signature $\{f\}$).

Proposition 3.8. *E_f is Noetherian.*

Proof. By contradiction, suppose that there is a chain $\Theta_1 \subseteq \Theta_2 \subseteq \cdots \subseteq \Theta_n \subseteq \cdots$ of sets of ground $\Sigma^{\underline{a}}$-atoms which is not eventually constant for logical consequence w.r.t. T. Without loss of generality, we can assume that $\Theta_1 \subseteq \Theta_2 \subseteq \cdots \subseteq \Theta_n \subseteq \cdots$ is such that for each i there exists a $\Sigma^{\underline{a}}$-atom $\ell_i \in \Theta_i$ such that $T \cup \Theta_{i-1} \not\models \ell_i$.

Notice that, since f is a unary function symbol, each element of the infinite sequence $\{\ell_i\}_{i\in\mathbb{N}}$ of $\Sigma^{\underline{a}}$-atoms is a Σ^{a_i,a_j}-atom (for some $a_i, a_j \in \underline{a}$). Thus, since $\underline{a}$ is finite, we can extract an infinite subsequence of ground $\Sigma^{a,b}$-atoms (for some fixed elements $a, b \in \underline{a}$) inducing an infinite ascending chain $\Theta_{1|\Sigma^{a,b}} \subseteq \Theta_{2|\Sigma^{a,b}} \subseteq \cdots \subseteq \Theta_{n|\Sigma^{a,b}} \subseteq \cdots$ which is not eventually constant for logical consequence w.r.t. T (here $\Theta_{i|\Sigma^{a,b}}$ is the collection of all the ground $\Sigma^{a,b}$-atoms occurring in Θ_i).

Suppose that a $\Sigma^{a,b}$-atom of the kind $\ell := f^m(a) = f^n(a)$ occurs in such an infinite subsequence (here $m \neq n$ otherwise $T \models \ell$, contrary to our choice of these atoms). Notice that $T \cup \ell$ is such that there are only finitely many Σ^a-terms that are not logically equivalent w.r.t. $T \cup \ell$, which implies that every infinite ascending chain of sets of ground Σ^a-atoms is eventually constant for logical consequence w.r.t. $T \cup \ell$ (the same argument apply to atoms of the kind $\ell := f^m(b) = f^n(b)$).

Suppose now that a $\Sigma^{a,b}$-atom of the kind $\ell := f^m(a) = f^n(b)$ belongs to such an infinite chain of $\Sigma^{a,b}$-atoms. The only $\Sigma^{a,b}$-atoms of the form $f^{m'}(a) = f^{n'}(b)$ not implied by $T \cup \ell$ are such that either (i) $m - n \neq m' - n'$ or (ii) $m' < m$ and $n' < n$. It is clear that there are only finitely many atoms of the kind (ii); for (i), notice that $f^m(a) = f^n(b) \wedge f^{m'}(a) = f^{n'}(b)$ implies that $f^{m+n'}(a) = f^{n+n'}(b) = f^{m'+n}(a)$ and that $f^{n+m'}(b) = f^{m+m'}(a) = f^{n'+m}(b)$ (where $m + n' \neq m' + n$ by (i)), so we are reduced to the first case.

The arguments above imply that the chain $\Theta_{1|\Sigma^{a,b}} \subseteq \Theta_{2|\Sigma^{a,b}} \subseteq \cdots \subseteq \Theta_{n|\Sigma^{a,b}} \subseteq \cdots$ is eventually constant for logical consequence w.r.t. T. Contradiction.

We assume the reader is familiar with the fundamentals of Superposition Calculus $\mathcal{SP}$, as explained for instance in [35]. We shall be especially interested in the saturation (modulo redundancy) $\mathcal{SP}(\Gamma)$ of a finite set of ground literals Γ: we recall that this can be achieved by $\mathcal{SP}$ in finitely many steps with respect to any reduction ordering. In fact, on this kind of inputs, $\mathcal{SP}$ behaves like standard Knuth-Bendix completion (with simplification). We just fix the relevant facts for future reference:

Lemma A.13. *[35] Let Γ be a consistent[6] ground constraint; given any reduction ordering total on ground terms, the saturation $\mathcal{SP}(\Gamma)$ of a Γ consists of a finite set R of equations and a finite set I of inequations such that:*

(0) *Γ is logically equivalent to $I \cup R$;*
(i) *the equation in R (once oriented from left to right) form a convergent ground rewriting system;*
(ii) *every equation $l = r \in R$ is in normal form with respect to $R \setminus \{l \to r\}$;*
(iii) *the inequations in I are in R-normal form;*
(iv) *every positive clause C is a logical consequence of Γ iff there is a disjunct $s = t$ in C such that s and t have the same R-normal form.*

For the last claim, notice that free theories are convex,[7] hence we have that $\Gamma \models C$ holds iff there is an equation $s = t$ in C such that $\Gamma \models s = t$ and the latter holds iff s and t have the same R-normal form.

Proposition 3.9. *If T is stably infinite and has decidable constraint satisfiability problem, then T_f is an effectively Noetherian extension of E_f.*

Proof. Let Γ be a T_f-constraint (we write $\Gamma(\underline{a}, \underline{b})$ to emphasize that the free constants occurring in Γ are in the tuple $\underline{a}, \underline{b}$): we want to compute an E_f-basis of Γ w.r.t. $\underline{a}$. Notice that $T_f = T \cup E_f$: since both theories are stably infinite and their intersection is E, Nelson-Oppen results apply. In particular, the following is a decision procedure for T_f-consistency of Γ [27,31,13]:

(a) produce a T-constraint $H(\underline{a}, \underline{b}, \underline{c})$ and an E_f-constraint $L(\underline{a}, \underline{b}, \underline{c})$ such that $\Gamma(\underline{a}, \underline{b})$ is logically equivalent to $\exists \underline{x}(H(\underline{a}, \underline{b}, \underline{x}) \wedge L(\underline{a}, \underline{b}, \underline{x}))$ (this is a standard purification step);
(b) guess an $(\underline{a}, \underline{b}, \underline{c})$-arrangement $G(\underline{a}, \underline{b}, \underline{c})$ (an $(\underline{a}, \underline{b}, \underline{c})$-arrangement is a set of literals containing for each $c_1, c_2 \in \underline{a} \cup \underline{b} \cup \underline{c}$ either $c_1 = c_2$ or $c_1 \neq c_2$);
(c) check $H(\underline{a}, \underline{b}, \underline{c}) \wedge G(\underline{a}, \underline{b}, \underline{c})$ for T-satisfiability and $L(\underline{a}, \underline{b}, \underline{c}) \wedge G(\underline{a}, \underline{b}, \underline{c})$ for E_f-satisfiability;
(d) output *satisfiable* iff both tests are successful and *unsatisfiable* iff they fail for all arrangements.

The correctness of the procedure is obvious, its completeness is due to the fact that, given a T-model $\mathcal{M}$ for $H(\underline{a}, \underline{b}, \underline{c}) \wedge G(\underline{a}, \underline{b}, \underline{c})$ and an E_f-model $\mathcal{N}$ for $L(\underline{a}, \underline{b}, \underline{c}) \wedge G(\underline{a}, \underline{b}, \underline{c})$, one can produce out of them a T_f-model $\mathcal{G}$ whose reducts to the signatures of T and of E_f are such that $\mathcal{M}$ and $\mathcal{N}$ respectively embed into them.[8]

[6] If Γ is not consistent, $\mathcal{SP}(\Gamma)$ just consists of the empty clause.
[7] A theory T is said to be convex iff whenever for a constraint Γ we have $T \cup \Gamma \models A_1 \vee \cdots \vee A_n$ (here the A_i are atoms and $n \geq 1$), then there is i such that $T \cup \Gamma \models A_i$. Among examples of convex theories, we have all Horn theories.
[8] The argument is the following: one can suppose that $\mathcal{M}, \mathcal{N}$ to be both infinite and of the same cardinality (by stable infiniteness and Löwenheim-Skolem theorem). Then, one can simply glue them because (up to renaming) they agree on the interpretation of the shared constants $\underline{a}, \underline{b}, \underline{c}$. Notice that stable infiniteness of a theory T can

Notice that the E_f-satisfiability test for $L(\underline{a},\underline{b},\underline{c}) \wedge G(\underline{a},\underline{b},\underline{c})$ can be obtained through Superposition: when doing that, we use a lexicographic path ordering [32] induced by a precedence giving the $\underline{b},\underline{c}$'s higher precedence with respect to both f and the $\underline{a}$'s. As a consequence, Lemma A.13(iv) immediately implies the following:

Claim: let $B_G(\underline{a})$ be the set of equations from $\mathcal{SP}(L(\underline{a},\underline{b},\underline{c}) \wedge G(\underline{a},\underline{b},\underline{c}))$ not involving the $\underline{b},\underline{c}$. We have that a positive clause $C(\underline{a})$ is a logical consequence of $L(\underline{a},\underline{b},\underline{c}) \wedge G(\underline{a},\underline{b},\underline{c})$ iff $B_G(\underline{a}) \models C(\underline{a})$.

We now show that $\bigvee_G B_G(\underline{a})$ is an E_f-basis for $\Gamma(\underline{a},\underline{b})$ with respect to $\underline{a}$ (the index G ranges over all arrangements for which the consistency tests in (c) are both positive).[9]

That $T_f \cup \{\Gamma(\underline{a},\underline{b})\} \models \bigvee_G B_G(\underline{a})$ is clear: by (a), $\Gamma(\underline{a},\underline{b})$ is logically equivalent to $\exists \underline{x}(H(\underline{a},\underline{b},\underline{x}) \wedge L(\underline{a},\underline{b},\underline{x}))$, the latter is equivalent to $\exists \underline{x}(H(\underline{a},\underline{b},\underline{x}) \wedge L(\underline{a},\underline{b},\underline{x}) \wedge \bigvee_G G(\underline{a},\underline{b},\underline{x}))$ and finally $L(\underline{a},\underline{b},\underline{c}) \wedge \bigvee_G G(\underline{a},\underline{b},\underline{c})$ entails $\bigvee_G B_G(\underline{a})$.

Conversely, suppose that $C(\underline{a})$ is a positive E_f-clause such that $T_f \cup \{\Gamma(\underline{a},\underline{b})\} \models C(\underline{a})$; we need to show that $B_G(\underline{a}) \models C(\underline{a})$ for any given arrangement $G(\underline{a},\underline{b},\underline{c})$ (such that both consistency tests in (c) are positive). We first show that $L(\underline{a},\underline{b},\underline{c}) \wedge G(\underline{a},\underline{b},\underline{c}) \models C(\underline{a})$: to see this, let $\mathcal{N}$ be an arbitrary model of $L(\underline{a},\underline{b},\underline{c}) \wedge G(\underline{a},\underline{b},\underline{c})$. Since the first consistency test in (c) is positive, there is a T-model $\mathcal{M}$ of $H(\underline{a},\underline{b},\underline{c}) \wedge G(\underline{a},\underline{b},\underline{c})$: by the above Nelson-Oppen combination argument, there is a model $\mathcal{G}$ of T_f whose reducts to the signatures of T and of E_f are such that $\mathcal{M}$ and $\mathcal{N}$ respectively embed into them. Since $\mathcal{G}$ is a model of T_f and of $\Gamma(\underline{a},\underline{b})$, $\mathcal{G} \models C(\underline{a})$, hence also $\mathcal{N} \models C(\underline{a})$ (because $\mathcal{N}$ embeds into the E_f-reduct of $\mathcal{G}$); being $\mathcal{N}$ arbitrary, this means that $L(\underline{a},\underline{b},\underline{c}) \wedge G(\underline{a},\underline{b},\underline{c}) \models C(\underline{a})$. But now the above Claim shows that $B_G(\underline{a}) \models C(\underline{a})$.

The Theory $\mathbf{E_f^\star}$. To the aim of proving Theorem 3.11, we need to introduce the theory $E_f^\star$ and to prove that it admits quantifier elimination. The theory $E_f^\star$ in the signature consisting of a unary function symbol f says the following:

(i) for each positive integer n there exist infinite elements x such that $f^n(x) = x$ and $f^m(x) \neq x$ (for all $0 < m < n$);
(ii) every element x is of the form $f(y)$ for infinitely many y.

$E_f^\star$ is a consistent theory: this is shown by producing a chain of E_f-models whose union is a model of $E_f^\star$ (the first model of the chain consists of infinitely many loops of any finite size, the $i+1$-model is obtained by adding an f-predecessor to any element of the i-th model).

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be formulated either by saying that every constraint is satisfiable in an infinite model of T or by saying that every model of T embeds into an infinite model of T (the equivalence of the two statements follows from the diagram theorem and compactness).

[9] Of course, if there are none of them, the index set is empty and $\bigvee_G B_G(\underline{a})$ is the empty disjunction, namely $\bot$. Formally, the notion of an E_f-basis requires a set of clauses, hence $\bigvee_G B_G(\underline{a})$ should be brought in conjunctive normal form.

Lemma A.14. *The theory $E_f^\star$ admits quantifier elimination; moreover, every model of E_f embeds into a model of $E_f^\star$.*[10]

Proof. We first show how to reduce the whole statement of the Lemma to the following:

Claim. Suppose that $\Gamma(a, b_1, \ldots, b_k)$ is a constraint satisfying the following conditions: (i) the free constant a occurs in all literals from Γ; (ii) Γ is saturated (i.e. $\mathcal{SP}(\Gamma) = \Gamma$) with respect to the lexicographic path ordering induced by the precedence

$$a > b_1 > \cdots > b_k > f. \tag{3}$$

Then $E_f^\star \models \forall y_1 \cdots \forall y_k \exists x\, \Gamma(x, y_1, \ldots, y_k)$.

If the Claim holds, we can eliminate quantifiers from any simply primitive formula $\exists x G(x, y_1, \ldots, y_k)$ as follows: first, saturate $G(a, b_1, \ldots, b_k)$ and then, keep only the literals not involving a (or output $\bot$ if the saturation produces the empty clause). The Claim shows also that every model $\mathcal{M}$ of E_f embeds into a model of $E_f^\star$: in fact, $E_f^\star$ is consistent and hence (by the above argument) consistent with the diagram of $\mathcal{M}$.

Thus, it only remains to prove the Claim: let $\Gamma(a, b_1, \ldots, b_k)$ be a constraint satisfying the two conditions of the Claim. By our choice of the reduction ordering, it is straightforward to see that (a) $f^n(a) > f^m(b_i)$ for each b_i and $n, m \geq 0$ and (b) $f^n(c) > f^m(c)$ iff $n > m$ for each constant c. Now, since Γ is saturated and all literals from Γ contains an occurrence of a, we see that Γ is either of the kind

$$\{f^m(a) = u, f^{m-k_1}(a) \neq u_1, \ldots, f^{m-k_n}(a) \neq u_n\}$$

or of the kind

$$\{f^{m_1}(a) \neq u_1, \ldots, f^{m_n}(a) \neq u_n\}$$

(here $n, m, m_i \geq 0$ and $0 < k_i \leq m$). Indeed, by contradiction, suppose that two equalities involving a occur in Γ or that the equality $f^m(a) = u$ and an inequality of the kind $f^{m+k}(a) \neq t$ occur in Γ; in both cases, by our hypothesis of the ordering, a occurs in the maximum term of the equations, thus a reduction rewriting rule would apply, contradicting the fact that Γ is saturated. To simplify the matter further, notice that we can get rid of the case in which the equation $f^m(a) = u$ does not appear, because we can add it freely, taking as u the constant b_{k+1} which is not among the original $b_1, \ldots, b_k$ (proving the claim for this case would in fact be stronger).

We now distinguish two cases, depending on the form of the term u occurring in the only equation $f^m(a) = u$ of Γ:

(i) *a does not occur in u* (that is, u is of the form $f^l(b_j)$): the constraint Γ is

$$\{f^m(a) = u, f^{m-k_1}(a) \neq u_1, \ldots, f^{m-k_n}(a) \neq u_n\}.$$

[10] The reader interested in a purely model-theoretic proof of the model-completability of the 'loop-free extension' of E_f can consult [34].

Pick a model $\mathcal{M}$ of $E_f^\star$ and for simplicity let us indicate directly with $b_1, \ldots, b_k$ a given k-tuple of elements of the support of $\mathcal{M}$: we must show that we can find a so that $\mathcal{M} \models \Gamma(a, b_1, \ldots, b_k)$. Notice that any term t not involving a is of the kind $f^j(b_i)$ and hence gets interpreted as a specific element of $\mathcal{M}$ (that we still call t), because $b_1, \ldots, b_k$ have been assigned an interpretation. We let X be the set of such terms among the $u, u_1, \ldots, u_n$ (notice that the complement set $\{u, u_1, \ldots, u_n\} \setminus X$ is formed by terms of the kind $f^j(a)$, where $j < m$).[11]
By induction, we define elements $a_m, a_{m-1}, \ldots, a_1, a_0$ in the following way: we let a_m to be u and, when defining a_{i-1} we choose it in such a way that $f^{\mathcal{M}}(a_{i-1}) = a_i$ and a_{i-1} is different from all interpretations of elements from X and also from $a_m, \ldots, a_i$: this is possible by the second group of axioms for $E_f^\star$. If we let a to be a_0, it is clear that $\mathcal{M} \models \Gamma(a, b_1, \ldots, b_k)$ holds (saturation prevents the constraint from containing inconsistent inequations like $t \neq t$).

(ii) *a occurs in u* (that is, u is of the form $f^{m-l}(a)$, for $0 < l \leq m$): the constraint Γ is

$$\{f^m(a) = f^{m-l}(a), f^{m-k_1}(a) \neq u_1, \ldots, f^{m-k_n}(a) \neq u_n\}.$$

Again we pick a model $\mathcal{M}$ of $E_f^\star$, a k-tuple $b_1, \ldots, b_k$ of elements from the support of $\mathcal{M}$, and we still follow the convention of indicating with t the resulting interpretation of terms t of the kind $f^j(b_i)$ (we also collect in a set called X these terms). We have to find a in such a way that $\mathcal{M} \models \Gamma(a, b_1, \ldots, b_k)$ holds.
By the first group of axioms for $E_f^\star$, it is possible to pick a loop of length l formed by elements $a_{m-1}, \ldots, a_{m-l}$ which are pairwise distinct from each other and also distinct from the interpretations of the terms in X. We then define, by induction, elements $a_{m-l}, a_{m-l-1}, \ldots, a_1, a_0$ as in the previous case, starting from the already defined element a_{m-l}. If we finally take a to be a_0, we can ensure the condition $\mathcal{M} \models \Gamma(a, b_1, \ldots, b_k)$.

Proposition 3.10. *If T is stably infinite, then T_f is E_f-compatible.*

Proof. We need to show that

(i) $E_f \subseteq E_f^\star$;
(ii) $E_f^\star$ has quantifier elimination;
(iii) every model of E_f can be embedded into a model of $E_f^\star$;
(iv) every model of T_f can be embedded into a model of $T_f \cup E_f^\star$.

We already know that (i)-(ii) and (iii) hold from Lemma A.14.

To show (iv), let $\mathcal{M}_0 = (M_0, \mathcal{I}_0)$ be a model of $T_f = T \cup E_f$. Take models $\mathcal{M}_1, \mathcal{M}_2$ such that: (1) $\mathcal{M}_1$ is an infinite model of T such that the reduct of $\mathcal{M}_0$

[11] We cannot have $j \geq m$, otherwise the constraint would not be saturated (a rewriting demodulation applies).

to the signature Σ of T embeds into $\mathcal{M}_1$ (it exists because T is stably infinite); (2) $\mathcal{M}_2$ is a model of $E_f^\star$ such that the reduct of $\mathcal{M}_0$ to the signature $\{f\}$ of E_f embeds into $\mathcal{M}_2$ (it exists by (iii) above).

We are now in the position of applying Lemma A.7: we take $I := \{1, 2\}$, $\underline{c} := M_0$, $\Sigma_1 := \Sigma$, $\Sigma_2 := \{f\}$, $\Sigma_r := \emptyset$, $\underline{a}_1 := \underline{a}_2 := \emptyset$, $T_1 := T$, $T_2 := E_f^\star$, $T_r := E$. The hypotheses of Lemma A.7 are satisfied because T_1, T_2 are both stably infinite (alias E-compatible), hence there exists $\mathcal{M} \models T \cup E_f^\star$ such that $\mathcal{M}_0$ has a $\Sigma \cup \{f\}$-embedding into $\mathcal{M}$: in fact, for $i = 1, 2$, $\mathcal{M}_0$ has a Σ_i-embedding into $\mathcal{M}_i$ and the latter $\Sigma_i^{M_0}$-embeds into $\mathcal{M}$.

Theorem 3.11 is immediate: it is just the conjunction of the statements of Propositions 3.9 and 3.10.

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