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Appendix to POPLMark Reloaded: Mechanizing Proofs by Logical Relations

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A Appendix

A.1 Basic Properties of Typed Reductions

Lemma A.1 (Reductions preserve Typing). *If* $\Gamma \vdash M \longrightarrow N : A$ *then* $\Gamma \vdash M : A$ *and* $\Gamma \vdash N : A$.

Proof. By induction on the given derivation. □

Lemma A.2 (Weakening and Exchange for Typing and Typed Substitutions).

- *If* $\Gamma \vdash M : B$ *then* $\Gamma, x{:}A \vdash M : B$.
- *If* $\Gamma, y{:}A, x{:}A' \vdash M : B$ *then* $\Gamma, x{:}A', y{:}A \vdash M : B$.
- *If* $\Gamma' \vdash \sigma : \Gamma$ *then* $\Gamma', x{:}A \vdash \sigma : \Gamma$.

Proof. By induction on the given derivation. □

Corollary A.1 (Weakening of Renamings). *If* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma', x{:}A \leq_\rho \Gamma$.

Lemma A.3 (Anti-renaming of Typing). *If* $\Gamma' \vdash [\rho]M : A$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma \vdash M : A$.

Proof. By induction on the given typing derivation taking into account equational properties of substitutions. □

Lemma A.4 (Weakening and Exchange of Typed Reductions).

- *If* $\Gamma \vdash M \longrightarrow N : B$ *then* $\Gamma, x{:}A \vdash M \longrightarrow N : B$.
- *If* $\Gamma, y{:}A, x{:}A' \vdash M \longrightarrow N : B$ *then* $\Gamma, x{:}A', y{:}A \vdash M \longrightarrow N : B$.

Proof. By mutual induction on the first derivation. □

Lemma A.5 (Substitution Property of Typed Reductions)**.** *If* $\Gamma, x{:}A \vdash M \longrightarrow M' : B$ *and* $\Gamma \vdash N : A$ *then* $\Gamma \vdash [N/x]M \longrightarrow [N/x]M' : B$.

Proof. By induction on the first derivation, using the usual properties of composition of substitutions as well as weakening and exchange. □

Lemma A.6 (Properties of Multi-Step Reductions)**.**

1. *If* $\Gamma \vdash M_1 \longrightarrow^* M_2 : B$ *and* $\Gamma \vdash M_2 \longrightarrow^* M_3 : B$ *then* $\Gamma \vdash M_1 \longrightarrow^* M_3 : B$.
2. *If* $\Gamma \vdash M \longrightarrow^* M' : A \Rightarrow B$ *and* $\Gamma \vdash N : A$ *then* $\Gamma \vdash M\ N \longrightarrow^* M'\ N : B$.
3. *If* $\Gamma \vdash M : A \Rightarrow B$ *and* $\Gamma \vdash N \longrightarrow^* N' : A$ *then* $\Gamma \vdash M\ N \longrightarrow^* M\ N' : B$.
4. *If* $\Gamma, x{:}A \vdash M \longrightarrow^* M' : B$ *then* $\Gamma \vdash \lambda x{:}A.M \longrightarrow^* \lambda x{:}A.M' : A \Rightarrow B$.
5. *If* $\Gamma, x{:}A \vdash M : B$ *and* $\Gamma \vdash N \longrightarrow N' : A$ *then* $\Gamma \vdash [N/x]M \longrightarrow^* [N'/x]M : B$.

Proof. Properties 1, 2, 3, and 4 are proven by induction on the given multi-step relation. Property 5 is proven by induction on $\Gamma, x{:}A \vdash M : B$ using weakening and exchange (Lemma A.4). □

Lemma A.7 (Simultaneous Substitution and Renaming)**.**

1. *If* $\Gamma' \vdash \sigma : \Gamma$ *and* $\Gamma \vdash M \longrightarrow N : A$ *then* $\Gamma' \vdash [\sigma]M \longrightarrow [\sigma]N : A$.
2. *If* $\Gamma \vdash M \longrightarrow N : B$ *and* $\Gamma' \leq_\rho \Gamma$*, then* $\Gamma' \vdash [\rho]M \longrightarrow [\rho]N : B$.

A.2 Challenge 1a: Properties of sn

Lemma A.8 (Multi-step Strong Normalization)**.** *If* $\Gamma \vdash M \longrightarrow^* M' : A$ *and* $\Gamma \vdash M : A \in \mathsf{sn}$ *then* $\Gamma \vdash M' : A \in \mathsf{sn}$.

Proof. Induction on $\Gamma \vdash M \longrightarrow^* M' : A$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow M' : A}{\Gamma \vdash M \longrightarrow^* M' : A}$ M-Refl

$\Gamma \vdash M' : A \in \mathsf{sn}$ by using $\Gamma \vdash M : A \in \mathsf{sn}$

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow N : A \qquad \Gamma \vdash N \longrightarrow^* M' : A}{\Gamma \vdash M \longrightarrow^* M' : A}$ M-Trans

$\Gamma \vdash M : A \in \mathsf{sn}$ by assumption
$\Gamma \vdash N : A \in \mathsf{sn}$ by using $\Gamma \vdash M : A \in \mathsf{sn}$
$\Gamma \vdash M' : A \in \mathsf{sn}$ by IH
□

Lemma A.9 (Properties of strongly normalizing terms)**.**

1. *For all variables* $x : A \in \Gamma$, $\Gamma \vdash x : A \in \mathsf{sn}$.
2. *If* $\Gamma \vdash [N/x]M : B \in \mathsf{sn}$ *and* $\Gamma \vdash N : A$ *then* $\Gamma, x{:}A \vdash M : B \in \mathsf{sn}$.
3. *If* $\Gamma, x{:}A \vdash M : B \in \mathsf{sn}$ *then* $\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \in \mathsf{sn}$.
4. *If* $\Gamma \vdash M\ N : B \in \mathsf{sn}$ *then* $\Gamma \vdash M : A \Rightarrow B \in \mathsf{sn}$ *and* $\Gamma \vdash N : A \in \mathsf{sn}$.

Proof. In all the proofs below we silently exploit type uniqueness and do not track explicitly the reasoning about well-typed terms.

1. For all variables $x : A \in \Gamma$, $\Gamma \vdash x : A \in \mathsf{sn}$.

$\forall M'.\ \Gamma \vdash x \longrightarrow M' : A \Longrightarrow \Gamma \vdash M' : A \in \mathsf{sn}$ — since $\Gamma \vdash x \longrightarrow M'$ is impossible
$\Gamma \vdash x : A$ — since $x : A \in \Gamma$
$\Gamma \vdash x : A \in \mathsf{sn}$

2. If $\Gamma \vdash [N/x]M : B \in \mathsf{sn}$ and $\Gamma \vdash N : A$ then $\Gamma, x{:}A \vdash M : B \in \mathsf{sn}$.

Induction on $\Gamma \vdash [N/x]M : B \in \mathsf{sn}$.

Assume $\Gamma, x{:}A \vdash M \longrightarrow M' : B$
$\Gamma \vdash [N/x]M \longrightarrow [N/x]M' : B$ — by Lemma A.5
$\Gamma \vdash [N/x]M' : B \in \mathsf{sn}$ — by using $\Gamma \vdash [N/x]M : B \in \mathsf{sn}$
$\Gamma, x{:}A \vdash M' : B \in \mathsf{sn}$ — by IH
$\Gamma, x{:}A \vdash M : B \in \mathsf{sn}$ — since $\Gamma, x{:}A \vdash M \longrightarrow M' : B$ was arbitrary.

3. If $\Gamma, x{:}A \vdash M : B \in \mathsf{sn}$ then $\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \in \mathsf{sn}$.

Induction on $\Gamma, x{:}A \vdash M : B \in \mathsf{sn}$.

Assume $\Gamma \vdash \lambda x{:}A.M \longrightarrow Q : A \Rightarrow B$
$\Gamma, x{:}A \vdash M \longrightarrow M' : B$ and $Q = \lambda x{:}A.M'$ — by reduction rule for λ.
$\Gamma, x{:}A \vdash M' : B \in \mathsf{sn}$ — by assumption $\Gamma, x{:}A \vdash M : B \in \mathsf{sn}$
$\Gamma \vdash \lambda x{:}A.M' : A \Rightarrow B \in \mathsf{sn}$ — by IH
$\Gamma \vdash Q : A \Rightarrow B \in \mathsf{sn}$ — since $Q = \lambda x{:}A.M'$
$\Gamma \vdash \lambda x.M : A \Rightarrow B \in \mathsf{sn}$ — since $\Gamma \vdash \lambda x.M \longrightarrow Q : A \Rightarrow B$ was arbitrary

4. If $\Gamma \vdash M\ N : B \in \mathsf{sn}$ then $\Gamma \vdash M : A \Rightarrow B \in \mathsf{sn}$ and $\Gamma \vdash N : A \in \mathsf{sn}$.

We prove first: If $\Gamma \vdash M\ N : B \in \mathsf{sn}$ then $\Gamma \vdash M : A \Rightarrow B \in \mathsf{sn}$. Proving $\Gamma \vdash M\ N : B \in \mathsf{sn}$ implies also $\Gamma \vdash N : A \in \mathsf{sn}$ is similar.
By induction on $\Gamma \vdash M\ N : B \in \mathsf{sn}$.

Assume $\Gamma \vdash M \longrightarrow M' : A \Rightarrow B$
$\Gamma \vdash M\ N \longrightarrow M'\ N : B$ — by reduction rule for application
$\Gamma \vdash M'\ N : B \in \mathsf{sn}$ — by assumption $\Gamma \vdash M\ N : B \in \mathsf{sn}$
$\Gamma \vdash M' : A \Rightarrow B \in \mathsf{sn}$ — by IH
$\Gamma \vdash M : A \Rightarrow B \in \mathsf{sn}$ — since $\Gamma \vdash M \longrightarrow M' : A \Rightarrow B$ was arbitrary
□

Lemma A.10 (Weak head expansion)**.** *If* $\Gamma \vdash N : A \in \mathsf{sn}$ *and* $\Gamma \vdash [N/x]M : B \in \mathsf{sn}$ *then* $\Gamma \vdash (\lambda x{:}A.M)\ N : B \in \mathsf{sn}$.

Proof. Proof by induction — either $\Gamma \vdash N : A \in \mathsf{sn}$ is getting smaller or $\Gamma \vdash [N/x]M : B \in \mathsf{sn}$ is getting smaller.

Assume $\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow P : B$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow [N/x]M : B}$ and $Q = [N/x]M$

$\Gamma \vdash [N/x]M : B \in \mathsf{sn}$ by assumption

Case $\mathscr{D} = \dfrac{\dfrac{\Gamma, x{:}A \vdash M \longrightarrow M' : B}{\Gamma \vdash \lambda x{:}A.M \longrightarrow \lambda x{:}A.M' : A \Rightarrow B} \quad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow (\lambda x{:}A.M')\ N : B}$ and $Q = (\lambda x{:}A.M')\ N$

$\Gamma \vdash [N/x]M \longrightarrow [N/x]M' : B$ by Lemma A.5
$\Gamma \vdash [N/x]M' : B \in \mathsf{sn}$ using $\Gamma \vdash [N/x]M : B \in \mathsf{sn}$
$\Gamma \vdash N : A \in \mathsf{sn}$ by assumption
$\Gamma \vdash (\lambda x{:}A.M')\ N : B \in \mathsf{sn}$ by IH (since $\Gamma \vdash [N/x]M' : B \in \mathsf{sn}$ is smaller)

Case $\mathscr{D} = \dfrac{\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \quad \Gamma \vdash N \longrightarrow N' : A}{\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow (\lambda x{:}A.M)\ N' : B}$

$\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B$ by assumption
$\Gamma, x{:}A \vdash M : B$ by inversion on typing
$\Gamma \vdash [N/x]M \longrightarrow^* [N'/x]M : B$ by Lemma A.6 (5) using $\Gamma \vdash N \longrightarrow N' : A$
$\Gamma \vdash [N'/x]M : B \in \mathsf{sn}$ Lemma A.8 using $\Gamma \vdash [N/x]M : B \in \mathsf{sn}$
$\Gamma \vdash N' : A \in \mathsf{sn}$ using $\Gamma \vdash N : A \in \mathsf{sn}$
$\Gamma \vdash (\lambda x{:}A.M)\ N' : B \in \mathsf{sn}$ by IH (since $\Gamma \vdash N' : A \in \mathsf{sn}$ is smaller)

□

Lemma A.11 (Closure properties of neutral terms)**.**

1. *If* $\Gamma \vdash R : A\ \mathsf{ne}$ *and* $\Gamma \vdash R \longrightarrow R' : A$*, then* $\Gamma \vdash R' : A\ \mathsf{ne}$.
2. *If* $\Gamma \vdash R : A \Rightarrow B\ \mathsf{ne}$, $\Gamma \vdash R : A \Rightarrow B \in \mathsf{sn}$*, and* $\Gamma \vdash N : A \in \mathsf{sn}$ *then* $\Gamma \vdash R\ N : B \in \mathsf{sn}$.

Proof.

1. If $\Gamma \vdash R : A\ \mathsf{ne}$ and $\Gamma \vdash R \longrightarrow R' : A$, then $\Gamma \vdash R' : A\ \mathsf{ne}$.

By induction on $\Gamma \vdash R : A\ \mathsf{ne}$.

Case $\mathscr{D} = \dfrac{x : A \in \Gamma}{\Gamma \vdash x : A\ \mathsf{ne}}$

Contradiction with the assumption $\Gamma \vdash R \longrightarrow R' : A$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash R'' : A \Rightarrow B\ \mathsf{ne} \qquad \Gamma \vdash N : A}{\Gamma \vdash R''\ N : B\ \mathsf{ne}}$
$\Gamma \vdash R'' : A \Rightarrow B\ \mathsf{ne}$ by assumption

We proceed by cases on $\Gamma \vdash R \longrightarrow R' : A$.

Sub-case $\mathscr{D} = \dfrac{\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow [N/x]M : B}$

Contradiction with the assumption $\Gamma \vdash R : A$ ne.

Sub-case $\mathscr{D} = \dfrac{\Gamma \vdash R'' \longrightarrow P : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash R''\ N \longrightarrow P\ N : B}$

$R'' \longrightarrow P : A \Rightarrow B$ by assumption
$\Gamma \vdash P : A \Rightarrow B$ ne by IH
$\Gamma \vdash P\ N : B$ ne by definition of neutral terms

Sub-case $\mathscr{D} = \dfrac{\Gamma \vdash R'' : A \Rightarrow B \qquad \Gamma \vdash N \longrightarrow N' : B}{\Gamma \vdash R'' N \longrightarrow R'' N' : B}$

$\Gamma \vdash R'' : A \Rightarrow B$ ne by assumption
$\Gamma \vdash R''\ N' : B$ ne by definition of neutral terms

2. If $\Gamma \vdash R : A \Rightarrow B$ ne, $\Gamma \vdash R : A \Rightarrow B \in$ sn, and $\Gamma \vdash N : A \in$ sn then $\Gamma \vdash R\ N : B \in$ sn.

By simultaneous induction on $\Gamma \vdash R : A \Rightarrow B \in$ sn, $\Gamma \vdash N : A \in$ sn.

Assume $\Gamma \vdash R\ N \longrightarrow Q : B$

Case $\mathscr{D} = \dfrac{\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow [N/x]M : B}$

Contradiction with the assumption $\Gamma \vdash R : A \Rightarrow B$ ne.

Case $\mathscr{D} = \dfrac{\Gamma \vdash R \longrightarrow R' : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash R\ N \longrightarrow R'\ N : B}$

$\Gamma \vdash R' : A \Rightarrow B \in$ sn by using $\Gamma \vdash R : A \Rightarrow B \in$ sn
$\Gamma \vdash R : A \Rightarrow B$ ne by assumption
$\Gamma \vdash R \longrightarrow R' : A \Rightarrow B$ by assumption
$\Gamma \vdash R' : A \Rightarrow B$ ne by Property (1)
$\Gamma \vdash R'\ N : B \in$ sn by IH (since $\Gamma \vdash R' : A \Rightarrow B \in$ sn is smaller)

Case $\mathscr{D} = \dfrac{\Gamma \vdash R : A \Rightarrow B \qquad \Gamma \vdash N \longrightarrow N' : A}{\Gamma \vdash R\ N \longrightarrow R\ N' : B}$

$\Gamma \vdash N' : A \in$ sn by using $\Gamma \vdash N : A \in$ sn
$\Gamma \vdash R\ N' : B \in$ sn by IH (since $\Gamma \vdash N' : A \in$ sn is smaller)

□

Lemma A.12 (Confluence of sn). *If* $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} N : A$ *and* $\Gamma \vdash M \longrightarrow N' : A$ *then either* $N = N'$ *or there* $\exists Q$ *s.t.* $\Gamma \vdash N' \longrightarrow_{\mathsf{sn}} Q : A$ *and* $\Gamma \vdash N \longrightarrow^* Q : A$.

Proof. By induction on $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} N : A$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash N : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M)\; N \longrightarrow_{\mathsf{sn}} [N/x]M : B}$ $\quad \dfrac{\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x{:}A.M)\; N \longrightarrow [N/x]M : B}$

$[N/x]M : B = [N/x]M : B$ by reflexivity

Case $\mathscr{D} = \dfrac{\Gamma \vdash N : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M)\; N \longrightarrow_{\mathsf{sn}} [N/x]M : B}$ $\quad \dfrac{\dfrac{\Gamma, x{:}A \vdash M \longrightarrow M' : B}{\Gamma \vdash \lambda x{:}A.M \longrightarrow \lambda x{:}A.M' : A \Rightarrow B} \quad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x{:}A.M)\; N \longrightarrow (\lambda x{:}A.M')\; N : B}$

WE SHOW: $\exists Q$ s.t. $\Gamma \vdash (\lambda x{:}A.M')\; N \longrightarrow_{\mathsf{sn}} Q : B$ and $\Gamma \vdash [N/x]M \longrightarrow^* Q : B$
Let $Q = [N/x]M'$.
$\Gamma \vdash (\lambda x{:}A.M')\; N \longrightarrow_{\mathsf{sn}} [N/x]M' : B$ by def. of $\longrightarrow_{\mathsf{sn}}$
$\Gamma \vdash [N/x]M \longrightarrow [N/x]M' : B$ by Lemma A.5
$\Gamma \vdash [N/x]M \longrightarrow^* [N/x]M' : B$ by M-TRANS

Case $\mathscr{D} = \dfrac{\Gamma \vdash N : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M)\; N \longrightarrow_{\mathsf{sn}} [N/x]M : B}$ $\quad \dfrac{\Gamma \vdash N \longrightarrow N' : A \quad \Gamma \vdash \lambda x{:}A.M : A \Rightarrow B}{\Gamma \vdash (\lambda x{:}A.M)\; N \longrightarrow (\lambda x{:}A.M)\; N' : B}$

WE SHOW: $\exists Q$ s.t. $\Gamma \vdash (\lambda x{:}A.M)\; N' \longrightarrow_{\mathsf{sn}} Q : B$ and $\Gamma \vdash [N/x]M \longrightarrow^* Q : B$

Let $Q = [N'/x]M$.
$\Gamma \vdash (\lambda x{:}A.M)\; N' \longrightarrow_{\mathsf{sn}} [N'/x]M : B$ by def. of $\longrightarrow_{\mathsf{sn}}$
$\Gamma \vdash [N/x]M \longrightarrow^* [N'/x]M : B$ by Lemma A.6 (5)

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M_1 : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M\; N \longrightarrow_{\mathsf{sn}} M_1\; N : B}$ $\quad \dfrac{\Gamma \vdash M \longrightarrow M_2 : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M\; N \longrightarrow M_2\; N : B}$

Either $M_2 = M_1$ or $\exists P$ s.t. $\Gamma \vdash M_2 \longrightarrow_{\mathsf{sn}} P : A \Rightarrow B$ and $\Gamma \vdash M_1 \longrightarrow^* P : A \Rightarrow B$ by IH

Sub-case $M_2 = M_1$

$M_1\; N = M_2\; N$ trivial

Sub-case $\exists P$ s.t. $\Gamma \vdash M_2 \longrightarrow_{\mathsf{sn}} P : A \Rightarrow B$ and $\Gamma \vdash M_1 \longrightarrow^* P : A \Rightarrow B$

WE SHOW: $\exists Q$ s.t. $\Gamma \vdash M_2\; N \longrightarrow_{\mathsf{sn}} Q : B$ and $\Gamma \vdash M_1\; N \longrightarrow^* Q : B$

Let $Q = P\; N$
$\Gamma \vdash M_2\; N \longrightarrow_{\mathsf{sn}} P\; N : B$ using def. of $\longrightarrow_{\mathsf{sn}}$ and $\Gamma \vdash M_2 \longrightarrow_{\mathsf{sn}} P : A \Rightarrow B$
$\Gamma \vdash M_1\; N \longrightarrow^* P\; N : B$ by Lemma A.6 (2)

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M\; N \longrightarrow_{\mathsf{sn}} M'\; N : B}$ $\quad \dfrac{\Gamma \vdash N \longrightarrow N' : A \quad \Gamma \vdash M : A \Rightarrow B}{\Gamma \vdash M\; N \longrightarrow M\; N' : B}$

WE SHOW: $\exists Q$ s.t. $\Gamma \vdash M\; N' \longrightarrow_{\mathsf{sn}} Q : B$ and $\Gamma \vdash M'\; N \longrightarrow^* Q : B$

Let $Q = M'\; N'$
$\Gamma \vdash M\; N' \longrightarrow_{\mathsf{sn}} M'\; N' : B$ by $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A \Rightarrow B$

$\Gamma \vdash N \longrightarrow^* N' : A$ by M-Trans
$\Gamma \vdash M'\ N \longrightarrow^* M'\ N' : B$ by Lemma A.6 (3)

□

Lemma A.13 (Backward closure of sn)**.**

1. *If* $\Gamma \vdash N : A \in \mathsf{sn}$, $\Gamma \vdash M : A \Rightarrow B \in \mathsf{sn}$, $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A \Rightarrow B$ *and* $\Gamma \vdash M'\ N : B \in \mathsf{sn}$, *then* $\Gamma \vdash M\ N : B \in \mathsf{sn}$.
2. *If* $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$ *and* $\Gamma \vdash M' : A \in \mathsf{sn}$ *then* $\Gamma \vdash M : A \in \mathsf{sn}$.

Proof.

1. If $\Gamma \vdash N : A \in \mathsf{sn}$, $\Gamma \vdash M : A \Rightarrow B \in \mathsf{sn}$, $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A \Rightarrow B$ and $\Gamma \vdash M'\ N : B \in \mathsf{sn}$, then $\Gamma \vdash M\ N : B \in \mathsf{sn}$.

By induction on $\Gamma \vdash N : A \in \mathsf{sn}$ and $\Gamma \vdash M : A \Rightarrow B \in \mathsf{sn}$.

Assume $\Gamma \vdash M\ N \longrightarrow Q : B$.

Case $\mathscr{D} = \dfrac{}{\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow [N/x]M : B}$

Contradiction with $\Gamma \vdash (\lambda x{:}A.M) \longrightarrow_{\mathsf{sn}} M' : A \Rightarrow B$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow M'' : A \Rightarrow B}{\Gamma \vdash M N \longrightarrow M'' N : B}$

$\Gamma \vdash M \longrightarrow M'' : A \Rightarrow B$ by assumption
$\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A \Rightarrow B$ by assumption
$\Gamma \vdash M' = M''$ or $\exists P$ s.t. $\Gamma \vdash M'' \longrightarrow_{\mathsf{sn}} P : A \Rightarrow B$ and $\Gamma \vdash M' \longrightarrow^* P : A \Rightarrow B$ by Conf. Lemma A.12

Sub-case $\Gamma \vdash M' = M''$

$\Gamma \vdash M'\ N : B \in \mathsf{sn}$ by assumption
$\Gamma \vdash M''\ N : B \in \mathsf{sn}$ since $M' = M''$

Sub-case $\exists P$ s.t. $\Gamma \vdash M'' \longrightarrow_{\mathsf{sn}} P : A \Rightarrow B$ and $\Gamma \vdash M' \longrightarrow^* P : A \Rightarrow B$

$\Gamma \vdash M'\ N \longrightarrow^* P\ N : A \Rightarrow B$ by Lemma A.6 (2)
$\Gamma \vdash M'\ N : B \in \mathsf{sn}$ by assumption
$\Gamma \vdash P\ N : B \in \mathsf{sn}$ by Lemma A.8
$\Gamma \vdash M'' \longrightarrow_{\mathsf{sn}} P : A \Rightarrow B$ by assumption
$\Gamma \vdash M \longrightarrow M'' : A \Rightarrow B$ by assumption
$\Gamma \vdash M'' : A \Rightarrow B \in \mathsf{sn}$ using $\Gamma \vdash M : A \Rightarrow B \in \mathsf{sn}$ and $\Gamma \vdash M \longrightarrow M'' : A \Rightarrow B$
$\Gamma \vdash M''\ N : B \in \mathsf{sn}$ by IH (since $\Gamma \vdash M'' : A \Rightarrow B \in \mathsf{sn}$ is smaller)

Case $\mathscr{D} = \dfrac{\Gamma \vdash N \longrightarrow N' : A}{\Gamma \vdash M\,N \longrightarrow M\,N' : B}$

$\Gamma \vdash N \longrightarrow N' : A$	by assumption
$\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A \Rightarrow B$	by assumption
$\Gamma \vdash M : A \in \mathsf{sn}$	by assumption
$\Gamma \vdash M'\ N : B \in \mathsf{sn}$	by assumption
$\Gamma \vdash M'\ N' : B \in \mathsf{sn}$	as $M'\,N \longrightarrow M'\,N'$
$\Gamma \vdash N' : A \in \mathsf{sn}$	using $\Gamma \vdash N : A \in \mathsf{sn}$ and $\Gamma \vdash N \longrightarrow N' : A$
$\Gamma \vdash M\ N' : B \in \mathsf{sn}$	by IH (since $\Gamma \vdash N' : A \in \mathsf{sn}$ is smaller)

2. If $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$ and $\Gamma \vdash M' : A \in \mathsf{sn}$ then $\Gamma \vdash M : A \in \mathsf{sn}$.

By induction on $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash N : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M)\ N \longrightarrow_{\mathsf{sn}} [N/x]M : B}$

$\Gamma \vdash [N/x]M : B \in \mathsf{sn}$	by assumption
$\Gamma \vdash N : A \in \mathsf{sn}$	by assumption
$\Gamma \vdash (\lambda x{:}A.M)\ N : B \in \mathsf{sn}$	by Lemma A.9 (A.10)

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N \longrightarrow_{\mathsf{sn}} M'\,N : B}$

$\Gamma \vdash M'\ N : B \in \mathsf{sn}$	by assumption
$\Gamma \vdash M' : A \Rightarrow B \in \mathsf{sn}$	by Lemma A.9 (4)
$\Gamma \vdash M : A \Rightarrow B \in \mathsf{sn}$	by IH
$\Gamma \vdash N : A \in \mathsf{sn}$	by Lemma A.9 (4)
$\Gamma \vdash M\ N : B \in \mathsf{sn}$	by Property (1)

□

A.3 Soundness

Lemma A.14. *If* $\Gamma \vdash M : A \in \mathit{SNe}$ *then* $\Gamma \vdash M : A\ \mathsf{ne}$.

Proof. By induction on $\Gamma \vdash M : A \in \mathsf{SNe}$.

Case $\mathscr{D} = \dfrac{x{:}A \in \Gamma}{\Gamma \vdash x : A \in \mathsf{SNe}}$

$\Gamma \vdash x : A\ \mathsf{ne}$	by definition

Case $\mathscr{D} = \dfrac{\Gamma \vdash R : A \Rightarrow B \in \mathsf{SNe} \qquad \Gamma \vdash M : A \in \mathsf{SN}}{\Gamma \vdash R\ M : B \in \mathsf{SNe}}$

$\Gamma \vdash R : A \Rightarrow B \in \mathsf{SNe}$	by assumption
$\Gamma \vdash R : A \Rightarrow B\ \mathsf{ne}$	by IH
$\Gamma \vdash R\ M : B\ \mathsf{ne}$	by definition of neutral terms

□

Theorem A.1 (Soundness of SN)**.**

1. *If* $\Gamma \vdash M : A \in \mathsf{SN}$ *then* $\Gamma \vdash M : A \in \mathsf{sn}$.
2. *If* $\Gamma \vdash M : A \in \mathsf{SNe}$ *then* $\Gamma \vdash M : A \in \mathsf{sn}$.
3. *If* $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A$ *then* $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$.

Proof. By mutual structural induction on the given derivations using the closure properties.

1. If $\Gamma \vdash M : A \in \mathsf{SN}$ then $\Gamma \vdash M : A \in \mathsf{sn}$.

Induction on $\Gamma \vdash M : A \in \mathsf{SN}$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash R : A \in \mathsf{SNe}}{\Gamma \vdash R : A \in \mathsf{SN}}$

$\Gamma \vdash R : A \in \mathsf{sn}$ by IH (2)

Case $\mathscr{D} = \dfrac{\Gamma, x{:}A \vdash M : B \in \mathsf{SN}}{\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \in \mathsf{SN}}$

$\Gamma, x{:}A \vdash M : B \in \mathsf{sn}$ by IH (1)
$\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \in \mathsf{sn}$ by Lemma A.9 (3)

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A \qquad \Gamma \vdash M' : A \in \mathsf{SN}}{\Gamma \vdash M : A \in \mathsf{SN}}$

$\Gamma \vdash M' : A \in \mathsf{sn}$ by IH (1)
$\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$ by IH (3)
$\Gamma \vdash M : A \in \mathsf{sn}$ by Backwards Closure (Lemma A.13 (2))

2. If $\Gamma \vdash M : A \in \mathsf{SNe}$ then $\Gamma \vdash M : A \in \mathsf{sn}$.

Induction on $\Gamma \vdash M : A \in \mathsf{SNe}$.

Case $\mathscr{D} = \dfrac{x : A \in \Gamma}{\Gamma \vdash x : A \in \mathsf{SNe}}$
$\Gamma \vdash x : A \in \mathsf{sn}$ by Lemma A.9 (1)

Case $\mathscr{D} = \dfrac{\Gamma \vdash R : A \Rightarrow B \in \mathsf{SNe} \qquad \Gamma \vdash M : A \in \mathsf{SN}}{\Gamma \vdash R\, M : B \in \mathsf{SNe}}$
$\Gamma \vdash R : A \Rightarrow B \in \mathsf{sn}$ by IH (2)
$\Gamma \vdash M : A \in \mathsf{sn}$ by IH (1)
$\Gamma \vdash R : A \Rightarrow B\ \mathsf{ne}$ by Lemma A.14
$\Gamma \vdash R\, M : B \in \mathsf{sn}$ by Lemma A.11 (2)

3. If $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A$ then $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$.

Induction on $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A$

Case $\mathscr{D} = \dfrac{\Gamma \vdash N : A \in \mathsf{SN} \quad \Gamma, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M)\; N \longrightarrow_{\mathsf{SN}} [N/x]M : B}$

$\Gamma \vdash N : A \in \mathsf{sn}$ by IH (1)
$\Gamma \vdash (\lambda x.M)\; N \longrightarrow_{\mathsf{sn}} [N/x]M : B$ by def. of $\longrightarrow_{\mathsf{sn}}$

Case $\mathscr{D} = \dfrac{\Gamma \vdash R \longrightarrow_{\mathsf{SN}} R' : A \Rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash RM \longrightarrow_{\mathsf{SN}} R'M : B}$

$\Gamma \vdash R \longrightarrow_{\mathsf{sn}} R' : A \Rightarrow B$ by IH(3)
$\Gamma \vdash RM \longrightarrow_{\mathsf{sn}} R'M : B$ by def. of $\longrightarrow_{\mathsf{sn}}$ ☐

A.3.1 Properties of the inductive definition of SN

Lemma A.15 (SN and SNe characterize well-typed terms)**.**

1. *If* $\Gamma \vdash M : A \in \mathit{SN}$ *then* $\Gamma \vdash M : A$.
2. *If* $\Gamma \vdash M : A \in \mathit{SNe}$ *then* $\Gamma \vdash M : A$.
3. *If* $\Gamma \vdash M \longrightarrow_{\mathit{SN}} M' : A$ *then* $\Gamma \vdash M : A$ *and* $\Gamma \vdash M' : A$.

Proof. By induction on the definition of SN, SNe, and $\longrightarrow_{\mathsf{SN}}$. ☐

Lemma A.16 (Renaming)**.**

1. *If* $\Gamma \vdash M : A \in \mathit{SN}$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma' \vdash [\rho]M : A \in \mathit{SN}$
2. *If* $\Gamma \vdash M : A \in \mathit{SNe}$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma' \vdash [\rho]M : A \in \mathit{SNe}$
3. *If* $\Gamma \vdash M \longrightarrow_{\mathit{SN}} N : A$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma' \vdash [\rho]M \longrightarrow_{\mathit{SN}} [\rho]N : A$.

Proof. By induction on the first derivation.

Case: $\mathscr{D} = \dfrac{\Gamma \vdash R : A \in \mathsf{SNe}}{\Gamma \vdash R : A \in \mathsf{SN}}$

$\Gamma' \vdash [\rho]R : A \in \mathsf{SNe}$ by IH (2)
$\Gamma' \vdash [\rho]R : A \in \mathsf{SN}$ by def. of SN

Case: $\mathscr{D} = \dfrac{\Gamma, x{:}A \vdash M : B \in \mathsf{SN}}{\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \in \mathsf{SN}}$

$\Gamma', x{:}A \leq_{\rho, x/x} \Gamma, x{:}A$ by def. of $\leq_\rho$
$\Gamma', x{:}A \vdash [\rho, x/x]M : B \in \mathsf{SN}$ by IH (1)
$\Gamma' \vdash \lambda x{:}A.[\rho, x/x]M : A \Rightarrow B \in \mathsf{SN}$ by def. of SN
$\Gamma' \vdash [\rho](\lambda x{:}A.M) : A \Rightarrow B \in \mathsf{SN}$ by subst. def.

Case: $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A \qquad \Gamma \vdash M' : A \in \mathsf{SN}}{\Gamma \vdash M : A \in \mathsf{SN}}$

$\Gamma' \vdash [\rho]M \longrightarrow_{\mathsf{SN}} [\rho]M' : A$ by IH (3)
$\Gamma' \vdash [\rho]M' : A \in \mathsf{SN}$ by IH (1)
$\Gamma' \vdash [\rho]M : A \in \mathsf{SN}$ by def. of SN

Case: $\mathcal{D} = \dfrac{x{:}A \in \Gamma}{\Gamma \vdash x : A \in \mathsf{SNe}}$

$\Gamma' \leq_\rho \Gamma$ by assumption
$\Gamma' \vdash [\rho]x : A$ by Renaming of Typing (Lemma A.1)
$\Gamma' \vdash [\rho]x : A \in \mathsf{SNe}$ by def. of SNe

Case: $\mathcal{D} = \dfrac{\Gamma \vdash R : A \Rightarrow B \in \mathsf{SNe} \qquad \Gamma \vdash M : A \in \mathsf{SN}}{\Gamma \vdash RM : A \Rightarrow B \in \mathsf{SNe}}$

$\Gamma' \vdash [\rho]R : A \Rightarrow B \in \mathsf{SNe}$ by IH (2)
$\Gamma' \vdash [\rho]M : A \in \mathsf{SN}$ by IH (1)
$\Gamma' \vdash [\rho]R\ [\rho]M : A \Rightarrow B \in \mathsf{SNe}$ by def. of SNe
$\Gamma' \vdash [\rho](R\ M) : B \in \mathsf{SNe}$ by subst. def.

Case: $\mathcal{D} = \dfrac{\Gamma, x{:}A \vdash M : B \qquad \Gamma \vdash N : A \in \mathsf{SN}}{\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow_{\mathsf{SN}} [N/x]M : B}$

$\Gamma' \vdash [\rho]N : A \in \mathsf{SN}$ by IH (1)
$\Gamma' \leq_\rho \Gamma$ by assumption
$\Gamma', x{:}A \leq_\rho \Gamma$ by Weakening of Renaming (Lemma A.1)
$\Gamma', x{:}A \leq_{\rho, x/x} \Gamma, x{:}A$ by def. of well-typed subst.
$\Gamma', x{:}A \vdash [\rho, x/x]M : B$ by Weakening Lemma A.2
$\Gamma' \vdash (\lambda x{:}A.[\rho, x/x]M)\ [\rho]N \longrightarrow_{\mathsf{SN}} [\rho, [\rho]N/x]M : B$ by def. of $\longrightarrow_{\mathsf{SN}}$
$\Gamma' \vdash [\rho]((\lambda x{:}AM)\ N) \longrightarrow_{\mathsf{SN}} [\rho]([N/x]M) : B$ by def. of subst

Case: $\mathcal{D} = \dfrac{\Gamma \vdash R \longrightarrow_{\mathsf{SN}} R' : A \Rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash RM \longrightarrow_{\mathsf{SN}} R'M : B}$

$\Gamma' \vdash [\rho]R \longrightarrow_{\mathsf{SN}} [\rho]R' : A \Rightarrow B$ by IH(3)
$\Gamma' \vdash [\rho]M : A$ by Weakening of Typing (Lemma A.2)
$\Gamma \vdash [\rho]R\ [\rho]M \longrightarrow_{\mathsf{SN}} [\rho]R'\ [\rho]M : B$ by def. of $\longrightarrow_{\mathsf{SN}}$
$\Gamma \vdash [\rho](R\ M) \longrightarrow_{\mathsf{SN}} [\rho](R'\ M) : B$ by def. of subst.

□

Lemma A.17 (Anti-Renaming)**.**

1. *If* $\Gamma' \vdash [\rho]M : A \in SN$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma \vdash M : A \in SN$
2. *If* $\Gamma' \vdash [\rho]M : A \in SNe$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma \vdash M : A \in SNe$
3. *If* $\Gamma' \vdash [\rho]M \longrightarrow_{SN} N' : A$ *and* $\Gamma' \leq_\rho \Gamma$ *then there exists* N *s.t.* $\Gamma \vdash M \longrightarrow_{SN} N : A$ *and* $[\rho]N = N'$.

Proof. By induction on the first derivation. We exploit the fact that ρ is a renaming substitution and take into account equational properties of substitutions when considering different cases. We only show a few cases.

Case $\mathscr{D} = \dfrac{\Gamma', x{:}A \vdash [\rho, x/x]M : B \in \mathsf{SN}}{\Gamma' \vdash \lambda x{:}A.[\rho, x/x]M : A \Rightarrow B \in \mathsf{SN}}$ using $[\rho](\lambda x{:}A.M) = \lambda x{:}A.[\rho, x/x]M$.

$\Gamma', x{:}A \leq_{\rho, x/x} \Gamma, x{:}A$ by Weakening (Lemma A.1) and well-typed substitution rule
$\Gamma, x{:}A \vdash M : B \in \mathsf{SN}$ by IH (1)
$\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \in \mathsf{SN}$ by def. of SN

Case $\mathscr{D} = \dfrac{y_i{:}A_1 \in \Gamma'}{\Gamma' \vdash [\rho]x_i : A_i \in \mathsf{SNe}}$ using $[\rho]x_i = y_i$

where $\rho = y_1/x_1, \dots, y_n/x_n$ and $\Gamma = x_1{:}A_1, \dots, x_n{:}A_n$ and $\Gamma' = y_1{:}A_1, \dots, y_n{:}A_n$

$\Gamma \vdash x_i : A_i$ since $x_i{:}A_i \in \Gamma$

Case $\mathscr{D} = \dfrac{\Gamma' \vdash [\rho]M \longrightarrow_{\mathsf{SN}} N' : A \qquad \Gamma' \vdash N' : A \in \mathsf{SN}}{\Gamma' \vdash [\rho]M : A \in \mathsf{SN}}$ using $[\rho]M = M'$

$\Gamma \vdash M \longrightarrow_{\mathsf{SN}} N : A$ and $[\rho]N = N'$ by IH (3)
$\Gamma' \vdash [\rho]N : A \in \mathsf{SN}$ using assumption $\Gamma' \vdash N' : A \in \mathsf{SN}$ and $[\rho]N = N'$
$\Gamma \vdash N : A \in \mathsf{SN}$ by IH (1)
$\Gamma \vdash M : A \in \mathsf{SN}$ by def. of $\longrightarrow_{\mathsf{SN}}$

Case $\mathscr{D} = \dfrac{\Gamma' \vdash [\rho]R : A \Rightarrow B \in \mathsf{SNe} \quad \Gamma' \vdash [\rho]M : A \in \mathsf{SN}}{\Gamma' \vdash [\rho](RM) : B \in \mathsf{SNe}}$ using $[\rho](R\ M) = [\rho]R\ [\rho]M$

$\Gamma \vdash R : A \Rightarrow B \in \mathsf{SNe}$ by IH(2)
$\Gamma \vdash M : A \in \mathsf{SN}$ by IH(1)
$\Gamma \vdash R\ M : B \in \mathsf{SNe}$ by def. of SNe

Case $\mathscr{D} = \dfrac{\Gamma' \vdash [\rho]N : A \in \mathsf{SN} \quad \Gamma', x{:}A \vdash [\rho, x/x]M : B}{\Gamma' \vdash [\rho]((\lambda x{:}A.M)\ N) \longrightarrow_{\mathsf{SN}} [\rho, [\rho]N/x]M : B}$

using $[\rho]((\lambda x{:}A.M)\ N) = (\lambda x{:}A.[\rho, x/x]M)\ [\rho]N$
and $[[\rho]N/x]([\rho, x/x]M) = [\rho, [\rho]N/x]M = [\rho]([N/x]M)$

$\Gamma \vdash N : A \in \mathsf{SN}$ by IH(1)
$\Gamma, x{:}A \vdash M : B$ by Anti-Weakening for Typing (Lemma A.3)
$\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow_{\mathsf{SN}} [N/x]M : B$ by def. of $\longrightarrow_{\mathsf{SN}}$

Case $\mathscr{D} = \dfrac{\Gamma' \vdash [\rho]R \longrightarrow_{\mathsf{SN}} R' : A \Rightarrow B \quad \Gamma' \vdash [\rho]M : A}{\Gamma' \vdash [\rho](RM) \longrightarrow_{\mathsf{SN}} R'\ [\rho]M}$ using $[\rho](R\ M) = [\rho]R\ [\rho]M$

$\Gamma \vdash M : A$ by Anti-weakening for Typing (Lemma A.3)
$\Gamma \vdash R \longrightarrow_{\mathsf{SN}} R_0 : A \Rightarrow B$ and $[\rho]R_0 = R'$ by IH(3)
$\Gamma \vdash R\ M \longrightarrow_{\mathsf{SN}} R_0\ M : B$ by def. of $\longrightarrow_{\mathsf{SN}}$
$[\rho](R_0\ M) = [\rho]R_0\ [\rho]M = R'\ [\rho]M$ by previous lines and subst. properties

□

Lemma A.18 (Extensionality of SN)**.** *If $x{:}A \in \Gamma$ and $\Gamma \vdash M\ x : B \in \mathsf{SN}$ then $\Gamma \vdash M : A \Rightarrow B \in \mathsf{SN}$.*

Proof. By induction on SN.

Case: $\mathscr{D} = \dfrac{\Gamma \vdash M\ x : B \in \mathsf{SNe}}{\Gamma \vdash M\ x : B \in \mathsf{SN}}$

$\Gamma \vdash M : A \Rightarrow B \in \mathsf{SNe}$ by def. of SNe
$\Gamma \vdash M : A \Rightarrow B \in \mathsf{SN}$ by def. of SN

Case: $\mathscr{D} = \dfrac{\Gamma \vdash M\ x \longrightarrow_{\mathsf{SN}} Q : B \qquad \Gamma \vdash Q : B \in \mathsf{SN}}{\Gamma \vdash M\ x : B \in \mathsf{SN}}$

Sub-case: $\Gamma \vdash (\lambda y{:}A.M')\ x \longrightarrow_{\mathsf{SN}} [x/y]M' : B$

$\Gamma \vdash [x/y]M' : B \in \mathsf{SN}$ by assumption
$\Gamma, y{:}A \vdash M' : B \in \mathsf{SN}$ by Anti-Renaming Property (Lemma A.17)
$\Gamma \vdash \lambda y{:}A.M' : A \Rightarrow B \in \mathsf{SN}$ by def. of SN

Sub-case: $\Gamma \vdash M\ x \longrightarrow_{\mathsf{SN}} M'\ x : B$ and $Q = M'\ x$

$\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A \Rightarrow B$ by def. of $\longrightarrow_{\mathsf{SN}}$
$\Gamma \vdash M' : A \Rightarrow B \in \mathsf{SN}$ by IH
$\Gamma \vdash M : A \Rightarrow B \in \mathsf{SN}$ by def. of SN □

A.3.2 Reducibility Candidates

Theorem A.2.

1. *CR1: If $\Gamma \vdash M \in \mathscr{R}_C$ then $\Gamma \vdash M : C \in \mathsf{SN}$.*
2. *CR2: If $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : C$ and $\Gamma \vdash M' \in \mathscr{R}_C$ then $\Gamma \vdash M \in \mathscr{R}_C$.*
3. *CR3: If $\Gamma \vdash M : C \in \mathsf{SNe}$ then $\Gamma \vdash M \in \mathscr{R}_C$.*

Proof. We prove these three properties simultaneously, each by induction on the structure of C.

CR 1. If $\Gamma \vdash M \in \mathscr{R}_C$ then $\Gamma \vdash M : C \in \mathsf{SN}$.

By induction on the structure of C.

Case $C = \mathsf{i}$

$\Gamma \vdash M \in \mathscr{R}_{\mathsf{i}}$	by assumption
$\Gamma \vdash M : \mathsf{i} \in \mathsf{SN}$	by def. of sem. interpretation for i

Case $C = A \Rightarrow B$

$\Gamma, x{:}A \vdash x : A \in \mathsf{SNe}$	by def. of SNe
$\Gamma.x{:}A \vdash x \in \mathscr{R}_A$	by IH (3)
$\Gamma, x{:}A \leq_{\mathsf{wk}} \Gamma$	by def. of context extensions
$\Gamma, x{:}A \vdash [\mathsf{wk}]M\ x \in \mathscr{R}_B$	by def. of $\Gamma, x{:}A \vdash M \in \mathscr{R}_{A \Rightarrow B}$
$\Gamma, x{:}A \vdash [\mathsf{wk}]M\ x : B \in \mathsf{SN}$	by IH (CR 1)
$\Gamma, x{:}A \vdash [\mathsf{wk}]M : A \Rightarrow B \in \mathsf{SN}$	by Extensionality Lemma A.18
$\Gamma \vdash M : A \Rightarrow B \in \mathsf{SN}$	by Anti-renaming Lemma A.17

CR 2. If $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : C$ and $\Gamma \vdash M' \in \mathscr{R}_C$ then $\Gamma \vdash M \in \mathscr{R}_C$.

By induction on the structure of C.

Case: $C = \mathsf{i}$.

$\Gamma \vdash M' : \mathsf{i} \in \mathsf{SN}$	since $\Gamma \vdash M' \in \mathscr{R}_{\mathsf{i}}$
$\Gamma \vdash M : \mathsf{i} \in \mathsf{SN}$	by closure rule for SN
$\Gamma \vdash M \in \mathscr{R}_{\mathsf{i}}$	by definition of semantic typing

Case: $C = A \Rightarrow B$.

Assume $\Gamma' \leq_\rho \Gamma$, $\Gamma' \vdash N \in \mathscr{R}_A$

$\Gamma' \vdash M'[\rho]\ N \in \mathscr{R}_B$	by assumption $\Gamma \vdash M' \in \mathscr{R}_{A \Rightarrow B}$
$\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A \Rightarrow B$	by assumption
$\Gamma' \vdash [\rho]M \longrightarrow_{\mathsf{SN}} [\rho]M' : A \Rightarrow B$	by Renaming Lemma A.16
$\Gamma' \vdash [\rho]M\ N \longrightarrow_{\mathsf{SN}} [\rho]M'\ N : B$	by $\longrightarrow_{\mathsf{SN}}$
$\Gamma \vdash [\rho]M\ N \in \mathscr{R}_B$	by IH (CR2)
$\Gamma \vdash M \in \mathscr{R}_{A \Rightarrow B}$	since $\Gamma' \vdash N \in \mathscr{R}_A$ was arbitrary

CR 3. If $\Gamma \vdash M : C \in \mathsf{SNe}$ then $\Gamma \vdash M \in \mathscr{R}_C$.

By induction on the structure of C.

Case: $C = \mathsf{i}$.

$\Gamma \vdash M : C \in \mathsf{SNe}$	by assumption
$\Gamma \vdash M : C \in \mathsf{SN}$	by def. of SN
$\Gamma \vdash M \in \mathscr{R}_{\mathsf{i}}$	by def. of semantic typing

Case: $C = A \Rightarrow B$.

Assume $\Gamma' \leq_\rho \Gamma$ and $\Gamma' \vdash N \in \mathscr{R}_A$

$\Gamma' \vdash N : A \in \mathsf{SN}$	by IH (CR 1)
$\Gamma \vdash M : A \Rightarrow B \in \mathsf{SNe}$	by assumption
$\Gamma' \vdash [\rho]M : A \Rightarrow B \in \mathsf{SNe}$	by Renaming Lemma A.16
$\Gamma' \vdash [\rho]M\ N : B \in \mathsf{SNe}$	by def. of SNe

$\Gamma' \vdash [\rho]M\,N \in \mathscr{R}_B$ by IH (CR 3)
$\Gamma \vdash M \in \mathscr{R}_{A \Rightarrow B}$ since $\Gamma' \vdash N \in \mathscr{R}_A$ was arbitrary
□

A.4 Proving strong normalization

Lemma A.19 (Fundamental lemma)**.** *If* $\Gamma \vdash M : A$ *and* $\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$ *then* $\Gamma' \vdash [\sigma]M \in \mathscr{R}_A$.

Proof. By induction on $\Gamma \vdash M : A$.

Case $\mathscr{D} = \dfrac{\Gamma(x) = A}{\Gamma \vdash x : A}$

$\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$ by assumption
$\Gamma' \vdash [\sigma]x \in \mathscr{R}_A$ by definition of $[\sigma]x$ and $\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B}$

$\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$ by assumption
$\Gamma' \vdash [\sigma]M \in \mathscr{R}_{A\to B}$ by IH
$\Gamma' \vdash [\sigma]N \in \mathscr{R}_A$ by IH
$\Gamma' \vdash [\sigma]M\ [\sigma]N \in \mathscr{R}_B$ by $\Gamma' \vdash [\sigma]M \in \mathscr{R}_{A\to B}$
$\Gamma' \vdash [\sigma](M\,N) \in \mathscr{R}_B$ by subst. definition

Case $\mathscr{D} = \dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$

$\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$ by assumption
Assume $\Gamma'' \leq_\rho \Gamma'$ and $\Gamma'' \vdash N : A$
$\Gamma'' \vdash [\rho]\sigma \in \mathscr{R}_\Gamma$ by weakening
$\Gamma'' \vdash ([\rho]\sigma, N/x) \in \mathscr{R}_{\Gamma,x:A}$ by definition of semantic substitutions
$\Gamma'' \vdash [[\rho]\sigma, N/x]M \in \mathscr{R}_B$ by IH
$\Gamma'' \vdash (\lambda x.[[\rho]\sigma, x/x]M)\ N \longrightarrow_{\mathsf{SN}} [[\rho]\sigma, N/x]M$ by reduction $\longrightarrow_{\mathsf{SN}}$
$(\lambda x.[[\rho]\sigma, x/x]M) = [[\rho]\sigma](\lambda x.M)$ by subst. def
$\Gamma'' \vdash ([[\rho]\sigma]\lambda x.M)\ N \in \mathscr{R}_B$ by CR 2
$\Gamma' \vdash [\sigma](\lambda x.M) \in \mathscr{R}_{A\Rightarrow B}$ since $\Gamma'' \leq_\rho \Gamma'$ and $\Gamma'' \vdash N : A$ was arbitrary
□

Corollary A.2. *If* $\Gamma \vdash M : A$ *then* $\Gamma \vdash M : A \in \mathit{SN}$.

Proof. Using the fundamental lemma with the identity substitution $\Gamma \vdash \mathsf{id} \in \mathscr{R}_\Gamma$, we obtain $\Gamma \vdash M \in \mathscr{R}_A$. By CR1, we know $\Gamma \vdash M \in \mathsf{SN}$. □

A.5 Extension with disjoint sums

A.5.1 Soundness of the inductive definition

Lemma A.20 (Properties of Multi-Step Reductions)**.**

Type-directed reduction : $\boxed{\Gamma \vdash M \longrightarrow N : A}$

$$\frac{\Gamma \vdash M \longrightarrow N : A}{\Gamma \vdash \mathsf{inl}\, M \longrightarrow \mathsf{inl}\, N : A+B}\ \text{E-INL} \qquad \frac{\Gamma \vdash M \longrightarrow N : B}{\Gamma \vdash \mathsf{inr}\, M \longrightarrow \mathsf{inr}\, N : A+B}\ \text{E-INR}$$

$$\frac{\Gamma \vdash M \longrightarrow M' : A+B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}b \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C}\ \text{E-CASE}$$

$$\frac{\Gamma \vdash M : A+B \quad \Gamma, x{:}A \vdash N_1 \longrightarrow N_1' : C \quad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 : C}\ \text{E-CASE-L}$$

$$\frac{\Gamma \vdash M : A+B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}B \vdash N_2 \longrightarrow N_2' : C}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2' : C}\ \text{E-CASE-R}$$

$$\frac{\Gamma \vdash M : A \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}b \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow [M/x]N_1 : C}\ \text{E-CASE-INL}$$

$$\frac{\Gamma \vdash M : B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}b \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\,(\mathsf{inr}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow [M/y]N_2 : C}\ \text{E-CASE-INR}$$

Fig. 1. Type-Directed Reduction, Extended with Disjoint Sums

Head reduction : $\boxed{\Gamma \vdash M \longrightarrow_{\mathsf{sn}} N : A}$

$$\frac{\Gamma \vdash M : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_1 : C}$$

$$\frac{\Gamma \vdash M : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inr}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/y]N_2 : C}$$

$$\frac{\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A+B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C}$$

Fig. 2. Head Reduction, Extended with Disjoint Sums

1. *If* $\Gamma, x{:}A \vdash M : B$ *and* $\Gamma \vdash N \longrightarrow N' : A$ *then* $\Gamma \vdash [N/x]M \longrightarrow^* [N'/x]M : B$.
2. *If* $\Gamma \vdash M \longrightarrow^* M' : A$ *then* $\Gamma \vdash \mathit{inl}\, M \longrightarrow^* \mathit{inl}\, M' : A+B$.
3. *If* $\Gamma \vdash M \longrightarrow^* M' : B$ *then* $\Gamma \vdash \mathit{inr}\, M \longrightarrow^* \mathit{inr}\, M' : A+B$.
4. *If* $\Gamma \vdash M \longrightarrow^* M' : A+B$ *then* $\Gamma \vdash \mathit{case} M\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2 \longrightarrow^* \mathit{case} M'\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2 : C$.
5. *If* $\Gamma, x{:}A \vdash N_1 \longrightarrow^* N_1' : C$ *then* $\Gamma \vdash \mathit{case} M\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2 \longrightarrow^* \mathit{case} M\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1' \mid \mathit{inr} y \Rightarrow N_2 : C$.
6. *If* $\Gamma, y{:}B \vdash N_2 \longrightarrow^* N_2' : C$ *then* $\Gamma \vdash \mathit{case} M\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2 \longrightarrow^* \mathit{case} M\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2' : C$.

Proof. (1) adds new cases to Lemma A.6 (5). The rest of the properties are proven by induction on the multi-step relation. □

Lemma A.21 (Properties of strongly normalizing terms)**.**

1. *If* $\Gamma \vdash M : A \in \mathsf{sn}$ *then* $\Gamma \vdash \mathit{inl}\, M : A+B \in \mathsf{sn}$.

2. *If* $\Gamma \vdash M : B \in \mathsf{sn}$ *then* $\Gamma \vdash \mathit{inr}\, M : A + B \in \mathsf{sn}$.
3. *If* $\Gamma \vdash \mathit{case} M\, \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2 : C \in \mathsf{sn}$*, then* $\Gamma \vdash M : A + B \in \mathsf{sn}$ *and* $\Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn}$ *and* $\Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}$.

Proof.

1. If $\Gamma \vdash M : A \in \mathsf{sn}$ then $\Gamma \vdash \mathsf{inl}\, M : A + B \in \mathsf{sn}$.

Induction on $\Gamma \vdash M : A \in \mathsf{sn}$.

Assume $\Gamma \vdash \mathsf{inl}\, M \longrightarrow Q : A + B$.
$\Gamma \vdash M \longrightarrow M' : A$ and $Q = \mathsf{inl}\, M'$ — by inversion on the only applicable red. rule
$\Gamma \vdash M' : A \in \mathsf{sn}$ — by assumption $\Gamma \vdash M : A \in \mathsf{sn}$
$\Gamma \vdash \mathsf{inl}\, M' : A + B \in \mathsf{sn}$ — by IH
$\Gamma \vdash \mathsf{inl}\, M : A + B \in \mathsf{sn}$ — since $\Gamma \vdash \mathsf{inl}\, M \longrightarrow Q : A + B$ was arbitrary

2. If $\Gamma \vdash M : B \in \mathsf{sn}$ then $\Gamma \vdash \mathsf{inr}\, M : A + B \in \mathsf{sn}$.

Similar to above.

3. If $\Gamma \vdash \mathsf{case} M\, \mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 : C \in \mathsf{sn}$, then $\Gamma \vdash M : A + B \in \mathsf{sn}$ and $\Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn}$ and $\Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}$.

Induction on $\Gamma \vdash \mathsf{case} M\, \mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 : C \in \mathsf{sn}$. We show that if $\Gamma \vdash \mathsf{case} M\, \mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 : C \in \mathsf{sn}$ then $\Gamma \vdash M : A + B \in \mathsf{sn}$; the other two proofs are similar.

Assume $\Gamma \vdash M \longrightarrow M' : A + B$.
$\Gamma \vdash \mathsf{case} M\, \mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow \mathsf{case} M'\, \mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 : C$ — by rule E-CASE
$\Gamma \vdash \mathsf{case} M\, \mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 : C \in \mathsf{sn}$ — by assumption
$\Gamma \vdash \mathsf{case} M'\, \mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 : C \in \mathsf{sn}$ — by definition of sn
$\Gamma \vdash M' : A + B \in \mathsf{sn}$ — by IH
$\Gamma \vdash M : A + B \in \mathsf{sn}$ — since $\Gamma \vdash M \longrightarrow M' : A + B$ was arbitrary

□

Lemma A.22 (Weak head expansion)**.**

1. *If* $\Gamma \vdash M : A \in \mathsf{sn}$ *and* $\Gamma \vdash [M/x]N_1 : C \in \mathsf{sn}$ *and* $\Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}$ *then* $\Gamma \vdash \mathit{case}\,(\mathit{inl}\, M)\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2 \in \mathsf{sn}$.
2. *If* $\Gamma \vdash M : B \in \mathsf{sn}$ *and* $\Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn}$ *and* $\Gamma \vdash [M/y]N_2 : C \in \mathsf{sn}$ *then* $\Gamma \vdash \mathit{case}\,(\mathit{inr}\, M)\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2 \in \mathsf{sn}$.

Proof.

1. If $\Gamma \vdash M : A \in \mathsf{sn}$ and $\Gamma \vdash [M/x]N_1 : C \in \mathsf{sn}$ and $\Gamma, y : B \vdash N_2 : C \in \mathsf{sn}$, then $\Gamma \vdash \mathsf{case}\,\mathsf{inl}\, M\, \mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 : C \in \mathsf{sn}$.

Proof by induction — either $\Gamma \vdash M : A \in \mathsf{sn}$ is getting smaller or $\Gamma, x : A \vdash N_1 : C \in \mathsf{sn}$ is getting smaller or $\Gamma, y : B \vdash N_2 : C \in \mathsf{sn}$ is getting smaller.

Assume $\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow P : B$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A \quad \Gamma, x : A \vdash N_1 : C \quad \Gamma, y : B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, \mathsf{inl}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow [M/x]N_1 : C}$ and $P = [M/x]N_1$

$\Gamma \vdash [M/x]N_1 : C \in \mathsf{sn}$ by assumption

Case $\mathscr{D} = \dfrac{\dfrac{\Gamma \vdash M \longrightarrow M' : A}{\Gamma \vdash \mathsf{inl}\, M \longrightarrow \mathsf{inl}\, M' : A + B} \quad \Gamma, x : A \vdash N_1 : C \quad \Gamma, y : B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, \mathsf{inl}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, \mathsf{inl}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C}$

and $Q = \mathsf{case}\, \mathsf{inl}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2$

$\Gamma \vdash M' : A \in \mathsf{sn}$ using $\Gamma \vdash M : A \in \mathsf{sn}$
$\Gamma \vdash [M/x]N_1 \longrightarrow^* [M'/x]N_1 : C$ by Lemma A.20 (1) using $\Gamma \vdash M \longrightarrow M' : A$
$\Gamma \vdash [M'/x]N_1 : C \in \mathsf{sn}$ by Lemma A.8 using $\Gamma \vdash [M/x]N_1 : C \in \mathsf{sn}$
$\Gamma \vdash \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$ by IH (since $\Gamma \vdash M' : A \in \mathsf{sn}$ is smaller)

Case $\mathscr{D} = \dfrac{\Gamma, x{:}A \vdash N_1 \longrightarrow N_1' : C}{\Gamma \vdash \mathsf{case}\, \mathsf{inl}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, \mathsf{inl}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 : C}$

and $Q = \mathsf{case}\, \mathsf{inl}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2$

$\Gamma \vdash [M/x]N_1 \longrightarrow [M/x]N_1' : C$ by Lemma A.5
$\Gamma \vdash [M/x]N_1' : C \in \mathsf{sn}$ using $\Gamma \vdash [M/x]N_1 : C \in \mathsf{sn}$
$\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$ by IH (since $\Gamma \vdash [M/x]N_1' : C \in \mathsf{sn}$ is smaller)

Case $\mathscr{D} = \dfrac{\Gamma, y{:}B \vdash N_2 \longrightarrow N_2' : C}{\Gamma \vdash \mathsf{case}\, \mathsf{inl}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, \mathsf{inl}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2' : C}$

and $Q = \mathsf{case}\, \mathsf{inl}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2'$

$\Gamma, y : B \vdash N_2' : C \in \mathsf{sn}$ using $\Gamma, y : B \vdash N_2 : C \in \mathsf{sn}$
$\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2' : C \in \mathsf{sn}$ by IH (since $\Gamma \vdash N_2' : C \in \mathsf{sn}$ is smaller)

2. If $\Gamma \vdash M : B \in \mathsf{sn}$ and $\Gamma, x : A \vdash N_1 : C \in \mathsf{sn}$ and $\Gamma \vdash [M/y]N_2 : C \in \mathsf{sn}$, then $\Gamma \vdash \mathsf{case}\, \mathsf{inr}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$.

Similar to above. □

Lemma A.23 (Closure properties of neutral terms)**.**

1. *If* $\Gamma \vdash R : A$ ne *and* $\Gamma \vdash R \longrightarrow R' : A$*, then* $\Gamma \vdash R' : A$ ne.
2. *If* $\Gamma \vdash M : A + B \in \mathsf{sn}$, $\Gamma \vdash M : A + B$ ne, $\Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn}$, *and* $\Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}$, *then* $\Gamma \vdash \mathit{case}\, M\ \mathit{of\ inl}\, x \Rightarrow N_1 \mid \mathit{inr}\, y \Rightarrow N_2 \in \mathsf{sn}$.

Proof.

1. If $\Gamma \vdash R : A$ ne and $\Gamma \vdash R \longrightarrow R' : A$, then $\Gamma \vdash R' : A$ ne.

By induction on $\Gamma \vdash R : A$ ne. We highlight the case for disjoint sums.

Case $$\frac{\Gamma \vdash R'' : A + B \text{ ne} \qquad \Gamma, x{:}A \vdash N_1 : C \qquad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, R''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \text{ ne}}$$

$\Gamma \vdash R'' : A + B$ ne by assumption

We proceed by cases on $\Gamma \vdash R \longrightarrow R' : A$.

Sub-case $$\frac{\Gamma \vdash R'' \longrightarrow P : A + B \qquad \Gamma, x{:}A \vdash N_1 : C \qquad \Gamma, y{:}b \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, R''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, P\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C}\ \text{E-CASE}$$

$\Gamma \vdash R'' \longrightarrow P : A + B$ by assumption
$\Gamma \vdash P : A + B$ ne by IH
$\Gamma \vdash \mathsf{case}\, P\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C$ ne by definition of neutral terms

Sub-case $$\frac{\Gamma \vdash R'' : A + B \qquad \Gamma, x{:}A \vdash N_1 \longrightarrow N_1' : C \qquad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, R''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, R''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 : C}\ \text{E-CASE-L}$$

$\Gamma \vdash \mathsf{case}\, R''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 : C$ ne by definition of neutral terms

Sub-case $$\frac{\Gamma \vdash R'' : A + B \qquad \Gamma, x{:}A \vdash N_1 : C \qquad \Gamma, y{:}B \vdash N_2 \longrightarrow N_2' : C}{\Gamma \vdash \mathsf{case}\, R''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, R''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2' : C}\ \text{E-CASE-R}$$

$\Gamma \vdash \mathsf{case}\, R''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2' : C$ ne by definition of neutral terms

Sub-case $$\frac{\Gamma \vdash M : A \qquad \Gamma, x{:}A \vdash N_1 : C \qquad \Gamma, y{:}b \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, (\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow [M/x]N_1 : C}\ \text{E-CASE-INL}$$

Contradiction with the assumption $\Gamma \vdash R'' : A + B$ ne.

Sub-case $$\frac{\Gamma \vdash M : B \qquad \Gamma, x{:}A \vdash N_1 : C \qquad \Gamma, y{:}b \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, (\mathsf{inr}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow [M/y]N_2 : C}\ \text{E-CASE-INR}$$

Contradiction with the assumption $\Gamma \vdash R'' : A + B$ ne.

2. If $\Gamma \vdash R : A + B \in$ sn, $\Gamma \vdash R : A + B$ ne, $\Gamma, x{:}A \vdash N_1 : C \in$ sn, and $\Gamma, y{:}B \vdash N_2 : C \in$ sn, then $\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \in$ sn.

By simultaneous induction on $\Gamma \vdash R : A + B \in$ sn, $\Gamma, x{:}A \vdash N_1 : C \in$ sn, and $\Gamma, y{:}B \vdash N_2 : C \in$ sn.

Assume $\Gamma \vdash \mathsf{case}\, R\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow Q : C$.

Case $$\frac{\Gamma \vdash R \longrightarrow R' : A+B \qquad \Gamma, x{:}A \vdash N_1 : C \qquad \Gamma, y{:}b \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, R\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, R'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C}\ \text{E-CASE}$$

$\Gamma \vdash R : A+B \in \mathsf{sn}$ by assumption
$\Gamma \vdash R' : A+B \in \mathsf{sn}$ by definition of sn
$\Gamma \vdash R : A+B$ ne by assumption
$\Gamma \vdash R' : A+B$ ne by (1)
$\Gamma \vdash \mathsf{case}\, R'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \in \mathsf{sn}$ by IH (since $\Gamma \vdash R' : A+B \in \mathsf{sn}$ is smaller)

Case $$\frac{\Gamma \vdash R : A+B \qquad \Gamma, x{:}A \vdash N_1 \longrightarrow N_1' : C \qquad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, R\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, R\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 : C}\ \text{E-CASE-L}$$

$\Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn}$ by assumption
$\Gamma, x{:}A \vdash N_1' : C \in \mathsf{sn}$ by definition of sn
$\Gamma \vdash \mathsf{case}\, R\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 \in \mathsf{sn}$ by IH (since $\Gamma, x{:}A \vdash N_1' : C \in \mathsf{sn}$ is smaller)

Case $$\frac{\Gamma \vdash R : A+B \qquad \Gamma, x{:}A \vdash N_1 : C \qquad \Gamma, y{:}B \vdash N_2 \longrightarrow N_2' : C}{\Gamma \vdash \mathsf{case}\, R\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, R\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2' : C}\ \text{E-CASE-R}$$

Similar to above.

Case $$\frac{\Gamma \vdash M : A \qquad \Gamma, x{:}A \vdash N_1 : C \qquad \Gamma, y{:}b \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, (\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow [M/x]N_1 : C}\ \text{E-CASE-INL}$$

Contradiction with the assumption $\Gamma \vdash R : A+B$ ne.

Case $$\frac{\Gamma \vdash M : B \qquad \Gamma, x{:}A \vdash N_1 : C \qquad \Gamma, y{:}b \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, (\mathsf{inr}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow [M/y]N_2 : C}\ \text{E-CASE-INR}$$

Contradiction with the assumption $\Gamma \vdash R : A+B$ ne.

□

Lemma A.24 (Confluence of sn). *If* $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} N : A$ *and* $\Gamma \vdash M \longrightarrow N' : A$ *then either* $N = N'$ *or there* $\exists Q$ *s.t.* $\Gamma \vdash N' \longrightarrow_{\mathsf{sn}} Q : A$ *and* $\Gamma \vdash N \longrightarrow^* Q : A$.

Proof. By induction on $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} N : A$. We highlight the cases for disjoint sums.

Case $$\mathscr{D} = \frac{\Gamma \vdash M : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\, (\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_1 : C}$$

$$\frac{\Gamma \vdash M : A \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\, (\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow [M/x]N_1 : C}$$

$[M/x]N_1 : C = [M/x]N_1 : C$ by reflexivity

Case $$\mathscr{D} = \frac{\Gamma \vdash M : B \in \mathsf{sn} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\, (\mathsf{inr}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_2 : C}$$

$$\frac{\Gamma \vdash M : B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\,(\mathsf{inr}\,M)\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow [M/x]N_2 : C}$$

$[M/x]N_2 : C = [M/x]N_2 : C$ by reflexivity

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A+B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C}$

$$\frac{\Gamma \vdash M \longrightarrow M'' : A+B}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow \mathsf{case}\,M''\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C}$$

$\Gamma \vdash M' = M'' : A+B$ or $\exists M'''.\Gamma \vdash M' \longrightarrow_{\mathsf{sn}} M''' : A+B$ and $\Gamma \vdash M'' \longrightarrow^{*} M''' : A+B$ by IH

Subcase $M' = M''$.

$\mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 = \mathsf{case}\,M''\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2$ by reflexivity

Subcase $\Gamma \vdash M' \longrightarrow_{\mathsf{sn}} M''' : A+B$ and $\Gamma \vdash M'' \longrightarrow^{*} M''' : A+B$.

$\Gamma \vdash \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} \mathsf{case}\,M'''\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C$ by definition
$\Gamma \vdash \mathsf{case}\,M''\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow^{*} \mathsf{case}\,M'''\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C$ by Lemma A.20 (4)

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A+B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C}$

$$\frac{\Gamma, x : A \vdash N_1 \longrightarrow N_1' : C}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1' \mid \mathsf{inr}\,y \Rightarrow N_2 : C}$$

$\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1' \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1' \mid \mathsf{inr}\,y \Rightarrow N_2 : C$ by definition
$\Gamma \vdash \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow^{*} \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1' \mid \mathsf{inr}\,y \Rightarrow N_2 : C$ by E-CASE-L

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A+B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C}$

$$\frac{\Gamma, y : B \vdash N_2 \longrightarrow N_2' : C}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1' \mid \mathsf{inr}\,y \Rightarrow N_2 : C}$$

Similar to above.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\,M)\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_1 : C}$

$$\dfrac{\dfrac{\Gamma \vdash M \longrightarrow M' : A}{\Gamma \vdash \mathsf{inl}\, M \longrightarrow \mathsf{inl}\, M' : A+B}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow \mathsf{case}\,(\mathsf{inl}\, M')\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 : C}$$

$\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M')\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M'/x]N_1 : C$ by definition
$\Gamma \vdash [M/x]N_1 \longrightarrow^* [M'/x]N_1 : C$ by Lemma A.20 (5)

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_1 : C}$

$$\dfrac{\Gamma, x : A \vdash N_1 \longrightarrow N_1' : C}{\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow \mathsf{case}\,(\mathsf{inl}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1' \mid \mathsf{inr} y \Rightarrow N_2 : C}$$

$\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1' \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_1' : C$ by definition
$\Gamma \vdash [M/x]N_1 \longrightarrow^* [M/x]N_1' : C$ by Lemma A.5

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : B \in \mathsf{sn} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inr}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/y]N_2 : C}$

$$\dfrac{\Gamma, y : B \vdash N_2 \longrightarrow N_2' : C}{\Gamma \vdash \mathsf{case}\,(\mathsf{inr}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow \mathsf{case}\,(\mathsf{inr}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2' : C}$$

Similar to above.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A \in \mathsf{sn} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_1 : C}$

$$\dfrac{\Gamma, y : B \vdash N_2 \longrightarrow N_2' : C}{\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow \mathsf{case}\,(\mathsf{inl}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2' : C}$$

$\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2' \longrightarrow_{\mathsf{sn}} [M/x]N_1 : C$ by definition
$\Gamma \vdash [M/x]N_1 \longrightarrow^* [M/x]N_1 : C$ by definition

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : B \in \mathsf{sn} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inr}\, M)\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/y]N_2 : C}$

$$\dfrac{\Gamma, x : A \vdash N_1 \longrightarrow N_1' : C}{\Gamma \vdash \mathsf{case}\,\mathsf{inr}\, M\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1 \mid \mathsf{inr} y \Rightarrow N_2 \longrightarrow \mathsf{case}\,\mathsf{inr}\, M\,\mathsf{of}\ \mathsf{inl} x \Rightarrow N_1' \mid \mathsf{inr} y \Rightarrow N_2 : C}$$

Similar to above.

□

Lemma A.25 (Backward closure of sn)**.**

1. *If* $\Gamma \vdash M : A+B \in \mathsf{sn}$, $\Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn}$, $\Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}$, $\Gamma \vdash M \longrightarrow^* M' : A+B$, *and* $\Gamma \vdash \mathit{case} M'\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2 \in \mathsf{sn}$, *then* $\Gamma \vdash \mathit{case} M\ \mathit{of}\ \mathit{inl} x \Rightarrow N_1 \mid \mathit{inr} y \Rightarrow N_2 \in \mathsf{sn}$.
2. *If* $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$ *and* $\Gamma \vdash M' : A \in \mathsf{sn}$ *then* $\Gamma \vdash M : A \in \mathsf{sn}$.

Proof.

1. If $\Gamma \vdash M : A+B \in \mathsf{sn}$, $\Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn}$, $\Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}$, $\Gamma \vdash M \longrightarrow^* M' : A+B$, and $\Gamma \vdash \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \in \mathsf{sn}$, then $\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \in \mathsf{sn}$.

By induction on $\Gamma, x : A \vdash N_1 : C \in \mathsf{sn}$ and $\Gamma, y : B \vdash N_2 : C \in \mathsf{sn}$ and $\Gamma \vdash M : A+B \in \mathsf{sn}$.

Assume $\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow Q : C$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow M'' : A+B}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, M''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C}$

$\Gamma \vdash M \longrightarrow M'' : A+B$ by assumption
$\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A+B$ by assumption
$\Gamma \vdash M' = M''$ or $\exists P$ s.t. $\Gamma \vdash M'' \longrightarrow_{\mathsf{sn}} P : A \Rightarrow B$ and $\Gamma \vdash M' \longrightarrow^* P : A \Rightarrow B$ by Conf. Lemma A.24

Sub-case $\Gamma \vdash M' = M''$

$\Gamma \vdash \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$ by assumption
$\Gamma \vdash \mathsf{case}\, M''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$ since $M' = M''$

Sub-case $\exists P$ s.t. $\Gamma \vdash M'' \longrightarrow_{\mathsf{sn}} P : A \Rightarrow B$ and $\Gamma \vdash M' \longrightarrow^* P : A \Rightarrow B$

$\Gamma \vdash \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow^* \mathsf{case}\, P\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : A+B$ by Lemma A.20 (4)
$\Gamma \vdash \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$ by assumption
$\Gamma \vdash \mathsf{case}\, P\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$ by Lemma A.8 using $\Gamma \vdash M' \longrightarrow^* P : A \Rightarrow B$
$\Gamma \vdash M'' \longrightarrow_{\mathsf{sn}} P : A+B$ by assumption
$\Gamma \vdash M \longrightarrow M'' : A+B$ by assumption
$\Gamma \vdash M'' : A+B \in \mathsf{sn}$ using $\Gamma \vdash M : A \Rightarrow +B \in \mathsf{sn}$ and $\Gamma \vdash M \longrightarrow M'' : A+B$
$\Gamma \vdash \mathsf{case}\, M''\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$ by IH (since $\Gamma \vdash M'' : A+B \in \mathsf{sn}$ is smaller)

Case $\mathscr{D} = \dfrac{\Gamma, x : A \vdash N_1 \longrightarrow N_1' : C}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 : C}$

$\Gamma, x : A \vdash N_1 \longrightarrow N_1' : C$ by assumption
$\Gamma \vdash \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2$ by E-CASE-L
$\Gamma \vdash \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$ by assumption
$\Gamma \vdash \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{sn}$ by definition of sn
$\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A+B$ by assumption
$\Gamma \vdash M : A+B \in \mathsf{sn}$ by assumption
$\Gamma, x : A \vdash N_1' : C \in \mathsf{sn}$ using $\Gamma, x : A \vdash N_1 : C \in \mathsf{sn}$ and $\Gamma, x : A \vdash N_1 \longrightarrow N_1' : C$
$\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2 : B \in \mathsf{sn}$ by IH (since $\Gamma, x : A \vdash N_1' : C \in \mathsf{sn}$ is smaller)

Case $\mathscr{D} = \dfrac{\Gamma, y : B \vdash N_2 \longrightarrow N_2' : C}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2' : C}$

Similar to above.

Case $\mathscr{D} = \dfrac{}{\Gamma \vdash \mathsf{case}\,\mathsf{inl}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow [M/x]N_1 : C}$

Contradiction with $\Gamma \vdash \mathsf{inl}\ M \longrightarrow_{\mathsf{sn}} M' : A+B$.

Case $\mathscr{D} = \dfrac{}{\Gamma \vdash \mathsf{case}\,\mathsf{inr}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow [M/y]N_2 : C}$

Contradiction with $\Gamma \vdash \mathsf{inr}\ M \longrightarrow_{\mathsf{sn}} M' : A+B$.

2. If $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$ and $\Gamma \vdash M' : A \in \mathsf{sn}$ then $\Gamma \vdash M : A \in \mathsf{sn}$.

By induction on $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$. We highlight the cases for disjoint sums.

Case $\mathscr{D} = \dfrac{\Gamma, x : A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y : B \vdash N_2 : C \in \mathsf{sn} \quad \Gamma \vdash M : A \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\,\mathsf{inl}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_1 : C}$

$\Gamma \vdash [M/x]N_1 : C \in \mathsf{sn}$ by assumption
$\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\,M)\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C \in \mathsf{sn}$ by Lemma A.21 (1)

Case $\mathscr{D} = \dfrac{\Gamma, x : A \vdash N_1 : C \in \mathsf{sn} \quad \Gamma, y : B \vdash N_2 : C \in \mathsf{sn} \quad \Gamma \vdash M : B \in \mathsf{sn}}{\Gamma \vdash \mathsf{case}\,\mathsf{inr}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_2 : C}$

$\Gamma \vdash [M/x]N_2 : C \in \mathsf{sn}$ by assumption
$\Gamma \vdash (\mathsf{case}\,\mathsf{inr}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2) : C \in \mathsf{sn}$ by Lemma A.21 (2)

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A+B \quad \Gamma, x : A \vdash N_1 : C \quad \Gamma, y : B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C \longrightarrow_{\mathsf{sn}} \Gamma \vdash \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C}$

$\Gamma \vdash \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C \in \mathsf{sn}$ by assumption
$\Gamma \vdash M' : A+B \in \mathsf{sn}$ by Lemma A.21 (3)
$\Gamma \vdash M : A+B \in \mathsf{sn}$ by IH
$\Gamma, x : A \vdash N_1 : C \in \mathsf{sn}$ by Lemma A.21 (3)
$\Gamma, y : B \vdash N_2 : C \in \mathsf{sn}$ by Lemma A.21 (3)
$\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C \in \mathsf{sn}$ by Property (1)

□

Lemma A.26. *If* $\Gamma \vdash M : A \in \mathit{SNe}$ *then* $\Gamma \vdash M : A$ ne.

Proof. By induction on $\Gamma \vdash M : A \in \mathsf{SNe}$. We highlight the cases for disjoint sums.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A+B \in \mathsf{SNe} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{SN}}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C \in \mathsf{SNe}}$

$\Gamma \vdash M : A+B \in \mathsf{SNe}$ by assumption
$\Gamma \vdash M : A+B$ ne by IH
$\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C$ ne by definition of neutral terms

□

Theorem. [Soundness of SN]

1. If $\Gamma \vdash M : A \in \mathsf{SN}$ then $\Gamma \vdash M : A \in \mathsf{sn}$.
2. If $\Gamma \vdash M : A \in \mathsf{SNe}$ then $\Gamma \vdash M : A \in \mathsf{sn}$.
3. If $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A$ then $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$.

Proof. By mutual structural induction on the given derivations using the closure properties. We highlight the cases for disjoint sums.

1. If $\Gamma \vdash M : A \in \mathsf{SN}$ then $\Gamma \vdash M : A \in \mathsf{sn}$.

Induction on $\Gamma \vdash M : A \in \mathsf{SN}$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A \in \mathsf{SN}}{\Gamma \vdash \mathsf{inl}\, M : A + B \in \mathsf{SN}}$

$\Gamma \vdash M : A \in \mathsf{sn}$ by IH (1)
$\Gamma \vdash \mathsf{inl}\, M : A + B \in \mathsf{sn}$ by Lemma A.21 (1)

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : B \in \mathsf{SN}}{\Gamma \vdash \mathsf{inr}\, M : A + B \in \mathsf{SN}}$

Similar to above.

2. If $\Gamma \vdash M : A \in \mathsf{SNe}$ then $\Gamma \vdash M : A \in \mathsf{sn}$.

Induction on $\Gamma \vdash M : A \in \mathsf{SNe}$.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A + B \in \mathsf{SNe} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{SN}}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 : C \in \mathsf{SNe}}$

$\Gamma \vdash M : A + B \in \mathsf{sn}$ by IH (2)
$\Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn}$ by IH (1)
$\Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}$ by IH (1)
$\Gamma \vdash M : A + B\ \mathsf{ne}$ by Lemma A.26
$\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \in \mathsf{sn}$ by Lemma A.23 (2)

3. If $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A$ then $\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A$.

Induction on $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A$

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A \in \mathsf{SN} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{SN}}{\Gamma \vdash \mathsf{case}\, (\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow_{\mathsf{SN}} [M/x]N_1 : C}$

$\Gamma \vdash M : A \in \mathsf{sn}$ by IH (1)
$\Gamma, x{:}A \vdash N_1 : C \in \mathsf{sn}$ by IH (1)
$\Gamma, y{:}B \vdash N_2 : C \in \mathsf{sn}$ by IH (1)
$\Gamma \vdash \mathsf{case}\, (\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} [M/x]N_1 : C$ by def. of $\longrightarrow_{\mathsf{sn}}$

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : B \in \mathsf{SN} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{SN}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inr}\,M)\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{SN}} [M/y]N_2 : C}$

Similar to above.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A + B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{SN}} \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C}$

$\Gamma \vdash M \longrightarrow_{\mathsf{sn}} M' : A + B$ by IH (3)
$\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{sn}} \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C$ by def. of $\longrightarrow_{\mathsf{sn}}$ □

A.5.2 *Properties of the inductive definition of SN*

Lemma A.27 (SN and SNe characterize well-typed terms)**.**

1. *If* $\Gamma \vdash M : A \in SN$ *then* $\Gamma \vdash M : A$.
2. *If* $\Gamma \vdash M : A \in SNe$ *then* $\Gamma \vdash M : A$.
3. *If* $\Gamma \vdash M \longrightarrow_{SN} M' : A$ *then* $\Gamma \vdash M : A$ *and* $\Gamma \vdash M' : A$.

Proof. By induction on the definition of SN, SNe, and $\longrightarrow_{\mathsf{SN}}$. □

Lemma A.28 (Renaming)**.**

1. *If* $\Gamma \vdash M : A \in SN$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma' \vdash [\rho]M : A \in SN$
2. *If* $\Gamma \vdash M : A \in SNe$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma' \vdash [\rho]M : A \in SNe$
3. *If* $\Gamma \vdash M \longrightarrow_{SN} N : A$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma' \vdash [\rho]M \longrightarrow_{SN} [\rho]N : A$.

Proof. By induction on the first derivation.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A \in \mathsf{SN}}{\Gamma \vdash \mathsf{inl}\,M : A + B \in \mathsf{SN}}$

$\Gamma' \vdash [\rho]M : A \in \mathsf{SN}$ by IH (1)
$\Gamma' \vdash [\rho](\mathsf{inl}\,M) \in \mathsf{SN}$ by def. of SN and subst.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : B \in \mathsf{SN}}{\Gamma \vdash \mathsf{inr}\,M : A + B \in \mathsf{SN}}$

Similar to above.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A + B \in \mathsf{SNe} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{SN}}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C \in \mathsf{SNe}}$

$\Gamma' \vdash [\rho]M : A + B \in \mathsf{SNe}$ by IH (2)
$\Gamma', x{:}A \leq_{\rho, x/x} \Gamma, x{:}A$ by def. of $\leq_\rho$
$\Gamma', x{:}A \vdash [\rho, x/x]N_1 : C \in \mathsf{SN}$ by IH (1)
$\Gamma', y{:}B \leq_{\rho, y/y} \Gamma, y{:}B$ by def. of $\leq_\rho$
$\Gamma', y{:}B \vdash [\rho, y/y]N_2 : C \in \mathsf{SN}$ by IH (1)
$\Gamma' \vdash [\rho](\mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2) \in \mathsf{SNe}$ by def. of SNe and subst.

Case $\mathcal{D} = \dfrac{\Gamma \vdash M : A \in \mathsf{SN} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{SN}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\,M)\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{SN}} [M/x]N_1 : C}$

$\Gamma' \vdash [\rho]M : A \in \mathsf{SN}$ by IH (1)
$\Gamma', x{:}A \leq_{\rho,x/x} \Gamma, x{:}A$ by def. of $\leq_\rho$
$\Gamma', x{:}A \vdash [\rho, x/x]N_1 : C \in \mathsf{SN}$ by IH (1)
$\Gamma', y{:}B \leq_{\rho,y/y} \Gamma, y{:}B$ by def. of $\leq_\rho$
$\Gamma', y{:}B \vdash [\rho, y/y]N_2 : C \in \mathsf{SN}$ by IH (1)
$\Gamma' \vdash \mathsf{case}\,([\rho](\mathsf{inl}\,M))\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow [\rho, x/x]N_1 \mid \mathsf{inr}\,y \Rightarrow [\rho, y/y]N_2 \longrightarrow_{\mathsf{SN}} [\rho, [\rho]M/x]N_1 : C$ by def. of $\longrightarrow_{\mathsf{SN}}$
$\Gamma' \vdash [\rho](\mathsf{case}\,(\mathsf{inl}\,M)\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2) \longrightarrow_{\mathsf{SN}} [\rho]([M/x]N_1) : C$ by def. of subst.

Case $\mathcal{D} = \dfrac{\Gamma \vdash M : A \in \mathsf{SN} \quad \Gamma, x{:}A \vdash N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash N_2 : C \in \mathsf{SN}}{\Gamma \vdash \mathsf{case}\,(\mathsf{inr}\,M)\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{SN}} [M/y]N_2 : C}$

Similar to above.

Case $\mathcal{D} = \dfrac{\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A + B \quad \Gamma, x{:}A \vdash N_1 : C \quad \Gamma, y{:}B \vdash N_2 : C}{\Gamma \vdash \mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 \longrightarrow_{\mathsf{SN}} \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2 : C}$

$\Gamma' \vdash [\rho]M \longrightarrow_{\mathsf{SN}} \rho[M'] : A + B$ by IH (3)
$\Gamma' \vdash \mathsf{case}\,[\rho]M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow [\rho, x/x]N_1 \mid \mathsf{inr}\,y \Rightarrow [\rho, y/y]N_2 \longrightarrow_{\mathsf{SN}} \mathsf{case}\,[\rho]M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow [\rho, x/x]N_1 \mid \mathsf{inr}\,y \Rightarrow [\rho, y/y]N_2$ by def. of $\longrightarrow_{\mathsf{SN}}$
$\Gamma' \vdash [\rho](\mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2) \longrightarrow_{\mathsf{SN}} [\rho](\mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow N_1 \mid \mathsf{inr}\,y \Rightarrow N_2)$ by def. of subst.

□

Lemma A.29 (Anti-Renaming)**.**

1. *If* $\Gamma' \vdash [\rho]M : A \in \mathit{SN}$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma \vdash M : A \in \mathit{SN}$
2. *If* $\Gamma' \vdash [\rho]M : A \in \mathit{SNe}$ *and* $\Gamma' \leq_\rho \Gamma$ *then* $\Gamma \vdash M : A \in \mathit{SNe}$
3. *If* $\Gamma' \vdash [\rho]M \longrightarrow_{\mathit{SN}} N' : A$ *and* $\Gamma' \leq_\rho \Gamma$ *then there exists* N *s.t.* $\Gamma \vdash M \longrightarrow_{\mathit{SN}} N : A$ *and* $[\rho]N = N'$*.*

Proof. By induction on the first derivation.

Case $\mathcal{D} = \dfrac{\Gamma \vdash [\rho]M : A \in \mathsf{SN}}{\Gamma \vdash [\rho](\mathsf{inl}\,M) : A + B \in \mathsf{SN}}$

$\Gamma \vdash M : A \in \mathsf{SN}$ by IH (1)
$\Gamma \vdash \mathsf{inl}\,M \in \mathsf{SN}$ by def. of SN

Case $\mathcal{D} = \dfrac{\Gamma \vdash [\rho]M : B \in \mathsf{SN}}{\Gamma \vdash [\rho](\mathsf{inr}\,M) : A + B \in \mathsf{SN}}$

Similar to above.

Case $\mathscr{D} = \dfrac{\Gamma \vdash [\rho]M : A+B \in \mathsf{SNe} \quad \Gamma, x{:}A \vdash [\rho, x/x]N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash [\rho, y/y]N_2 : C \in \mathsf{SN}}{\Gamma \vdash [\rho](\mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2) : C \in \mathsf{SNe}}$

$\Gamma \vdash M : A+B \in \mathsf{SNe}$ by IH (2)
$\Gamma', x{:}A \leq_{\rho,x/x} \Gamma, x{:}A$ by def. of $\leq_\rho$
$\Gamma, x{:}A \vdash N_1 : C \in \mathsf{SN}$ by IH (1)
$\Gamma', y{:}B \leq_{\rho,y/y} \Gamma, y{:}B$ by def. of $\leq_\rho$
$\Gamma, y{:}B \vdash N_2 : C \in \mathsf{SN}$ by IH (1)
$\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \in \mathsf{SNe}$ by def. of SNe

Case $\mathscr{D} = \dfrac{\Gamma \vdash [\rho]M : A \in \mathsf{SN} \quad \Gamma, x{:}A \vdash [\rho, x/x]N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash [\rho, y/y]N_2 : C \in \mathsf{SN}}{\Gamma \vdash [\rho](\mathsf{case}\,(\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2) \longrightarrow_{\mathsf{SN}} [\rho, [\rho]M/x]N_1 : C}$

$\Gamma \vdash M : A \in \mathsf{SN}$ by IH (1)
$\Gamma', x{:}A \leq_{\rho,x/x} \Gamma, x{:}A$ by def. of $\leq_\rho$
$\Gamma, x{:}A \vdash N_1 : C \in \mathsf{SN}$ by IH (1)
$\Gamma', y{:}B \leq_{\rho,y/y} \Gamma, y{:}B$ by def. of $\leq_\rho$
$\Gamma, y{:}B \vdash N_2 : C \in \mathsf{SN}$ by IH (1)
$\Gamma \vdash \mathsf{case}\,(\mathsf{inl}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow_{\mathsf{SN}} [M/x]N_1 : C$ by def. of $\longrightarrow_{\mathsf{SN}}$

Case $\mathscr{D} = \dfrac{\Gamma \vdash [\rho]M : A \in \mathsf{SN} \quad \Gamma, x{:}A \vdash [\rho, x/x]N_1 : C \in \mathsf{SN} \quad \Gamma, y{:}B \vdash [\rho, y/y]N_2 : C \in \mathsf{SN}}{\Gamma \vdash [\rho](\mathsf{case}\,(\mathsf{inr}\, M)\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2) \longrightarrow_{\mathsf{SN}} [\rho, [\rho]M/y]N_2 : C}$

Similar to above.

Case $\mathscr{D} = \dfrac{\Gamma \vdash [\rho]M \longrightarrow_{\mathsf{SN}} M' : A+B}{\Gamma \vdash [\rho](\mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2) \longrightarrow_{\mathsf{SN}} \mathsf{case}\, M'\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2' : C}$

and $N_1' = [\rho, x/x]N_1$, $N_2' = [\rho, y/y]N_2$

$[\rho](\mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2) = \mathsf{case}\,[\rho]M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1' \mid \mathsf{inr}\, y \Rightarrow N_2'$ by def. of subst. $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M_0 : A+B$ and $[\rho]M_0 = M'$ by IH (3)
$\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2 \longrightarrow_{\mathsf{SN}} \mathsf{case}\, M_0\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow N_1 \mid \mathsf{inr}\, y \Rightarrow N_2$ by def. of $\longrightarrow_{\mathsf{SN}}$

□

A.5.3 Reducibility Candidates

Theorem.

1. CR1: If $\Gamma \vdash M \in \mathscr{R}_C$ then $\Gamma \vdash M : C \in \mathsf{SN}$.
2. CR2: If $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : C$ and $\Gamma \vdash M' \in \mathscr{R}_C$ then $\Gamma \vdash M \in \mathscr{R}_C$.
3. CR3: If $\Gamma \vdash M : C \in \mathsf{SNe}$ then $\Gamma \vdash M \in \mathscr{R}_C$.

Proof. Mutually, by induction on the structure of types C. We highlight the case for disjoint sums.

CR 1. If $\Gamma \vdash M \in \mathscr{R}_C$ then $\Gamma \vdash M : C \in \mathsf{SN}$.

Case $C = A + B$

$\Gamma \vdash M \in \mathscr{R}_{A+B}$ by assumption

We consider different subcases and prove by an inner induction on the closure defining $\mathscr{R}_{A+B}$ that $\Gamma \vdash M : A + B \in \mathsf{SN}$.

Subcase $\Gamma \vdash M \in \{\mathsf{inl}\, N \mid \Gamma \vdash N \in \mathscr{R}_A\}$

$M = \mathsf{inl}\, N$ and $\Gamma \vdash N \in \mathscr{R}_A$ by assumption
$\Gamma \vdash N : A \in \mathsf{SN}$ by IH (CR 1)
$\Gamma \vdash \mathsf{inl}\, N : A + B \in \mathsf{SN}$ by definition of SN

Subcase $\Gamma \vdash M \in \{\mathsf{inr}\, N \mid \Gamma \vdash N \in \mathscr{R}_B\}$

Similar to the case above.

Subcase $\Gamma \vdash M : A + B \in \mathsf{SNe}$

$\Gamma \vdash M : A + B \in \mathsf{SN}$ by definition of SN

Subcase $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A + B$ and $\Gamma \vdash M' \in \mathscr{R}_{A+B}$

$\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A + B$ and $\Gamma \vdash M' \in \mathscr{R}_{A+B}$ by assumption
$\Gamma \vdash M' : A + B \in \mathsf{SN}$ by inner IH
$\Gamma \vdash M : A + B \in \mathsf{SN}$ by definition of SN

CR 2. If $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : C$ and $\Gamma \vdash M' \in \mathscr{R}_C$ then $\Gamma \vdash M \in \mathscr{R}_C$.

Case $C = A + B$

$\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A + B$ and $\Gamma \vdash M' \in \mathscr{R}_{A+B}$ by assumption
$\Gamma \vdash M \in \mathscr{R}_{A+B}$ by definition of $\mathscr{R}_{A+B}$

CR 3. If $\Gamma \vdash M : C \in \mathsf{SNe}$ then $\Gamma \vdash M \in \mathscr{R}_C$.

Case $C = A + B$

$\Gamma \vdash M : A + B \in \mathsf{SNe}$ by assumption
$\Gamma \vdash M \in \mathscr{R}_{A+B}$ by definition of $\mathscr{R}_{A+B}$

□

A.5.4 Proving strong normalization

Lemma. [Fundamental lemma] If $\Gamma \vdash M : C$ and $\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$ then $\Gamma' \vdash [\sigma]M \in \mathscr{R}_C$.

Proof. By induction on $\Gamma \vdash M : C$. We highlight the cases involving disjoint sums.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{inl}\, M : A + B}$

$\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$ by assumption
$\Gamma' \vdash [\sigma]M \in \mathscr{R}_A$ by IH
$\Gamma' \vdash \mathsf{inl}\ [\sigma]M \in \mathscr{R}_{A+B}$ by definition of $\mathscr{R}_{A+B}$
$\Gamma' \vdash [\sigma]\mathsf{inl}\ M \in \mathscr{R}_{A+B}$ by subst. definition

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : B}{\Gamma \vdash \mathsf{inr}\, M : A + B}$

Similar to the case above.

Case $\mathscr{D} = \dfrac{\Gamma \vdash M : A+B \qquad \Gamma, x{:}A \vdash M_1 : C \qquad \Gamma, y{:}B \vdash M_2 : C}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow M_1 \mid \mathsf{inr}\, y \Rightarrow M_2 : C}$

$\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$ by assumption
$\Gamma' \vdash [\sigma]M \in \mathscr{R}_{A+B}$ by IH

We consider different subcases and prove by an inner induction on the closure defining $\mathscr{R}_{A+B}$ that $\Gamma' \vdash [\sigma](\mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow M_1 \mid \mathsf{inr}\, y \Rightarrow M_2) \in \mathscr{R}_C$.

Subcase $\Gamma' \vdash [\sigma]M \in \{\mathsf{inl}\, N \mid \Gamma' \vdash N \in \mathscr{R}_A\}$
$[\sigma]M = \mathsf{inl}\, N$ for some $\Gamma' \vdash N \in \mathscr{R}_A$ by assumption
$\Gamma' \vdash N : A \in \mathsf{SN}$ by CR 1
$\Gamma' \vdash \mathsf{inl}\, N : A + B \in \mathsf{SN}$ by definition
$\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$ by assumption
$\Gamma' \vdash [\sigma, N/x] \in \mathscr{R}_{\Gamma, x:A}$ by definition
$\Gamma' \vdash [\sigma, N/x]M_1 \in \mathscr{R}_C$ by IH
$\Gamma', x{:}A \vdash x \in \mathscr{R}_A$ by definition
$\Gamma', x{:}A \vdash [\sigma, x/x] \in \mathscr{R}_{\Gamma, x:A}$ by definition
$\Gamma', x{:}A \vdash [\sigma, x/x]M_1 \in \mathscr{R}_C$ by IH
$\Gamma', x{:}A \vdash [\sigma, x/x]M_1 : C \in \mathsf{SN}$ by CR 1
$\Gamma', y{:}B \vdash y \in \mathscr{R}_B$ by definition
$\Gamma', y{:}B \vdash [\sigma, y/y] \in \mathscr{R}_{\Gamma, y:B}$ by definition
$\Gamma', y{:}B \vdash [\sigma, y/y]M_2 \in \mathscr{R}_C$ by IH
$\Gamma', y{:}B \vdash [\sigma, y/y]M_2 : C \in \mathsf{SN}$ by CR 1
$\Gamma' \vdash \mathsf{case}\,(\mathsf{inl}\, N)\,\mathsf{of}\ \mathsf{inl}\, x \Rightarrow [\sigma, x/x]M_1 \mid \mathsf{inr}\, y \Rightarrow [\sigma, y/y]M_2 \longrightarrow_{\mathsf{SN}} [\sigma, N/x]M_1 : C$ by $\longrightarrow_{\mathsf{SN}}$
$\mathsf{case}\,(\mathsf{inl}\, N)\,\mathsf{of}\ \mathsf{inl}\, x \Rightarrow [\sigma, x/x]M_1 \mid \mathsf{inr}\, y \Rightarrow [\sigma, y/y]M_2$
$= [\sigma](\mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow M_1 \mid \mathsf{inr}\, y \Rightarrow M_2)$ by subst. definition and $[\sigma]M = \mathsf{inl}\, N$
$\Gamma' \vdash [\sigma](\mathsf{case}\, M\, \mathsf{of}\ \mathsf{inl}\, x \Rightarrow M_1 \mid \mathsf{inr}\, y \Rightarrow M_2) \in \mathscr{R}_C$ by CR 2

Subcase $\Gamma' \vdash [\sigma]M \in \{\mathsf{inr}\, N \mid \Gamma' \vdash N \in \mathscr{R}_B\}$
Similar to the case above.

Subcase $\Gamma' \vdash [\sigma]M : A + B \in \mathsf{SNe}$.
$\Gamma' \vdash \sigma \in \mathscr{R}_\Gamma$ by assumption
$\Gamma', x{:}A \vdash x \in \mathscr{R}_A$ by definition

$\Gamma', y{:}B \vdash y \in \mathscr{R}_B$ by definition
$\Gamma', x{:}A \vdash [\sigma, x/x] \in \mathscr{R}_{\Gamma, x:A}$ by definition
$\Gamma', y{:}B \vdash [\sigma, y/y] \in \mathscr{R}_{\Gamma, y:B}$ by definition
$\Gamma', x{:}A \vdash [\sigma, x/x]M_1 \in \mathscr{R}_C$ by IH
$\Gamma', y{:}B \vdash [\sigma, y/y]M_2 \in \mathscr{R}_C$ by IH
$\Gamma', x{:}A \vdash [\sigma, x/x]M_1 : C \in \mathsf{SN}$ by CR 1
$\Gamma', y{:}B \vdash [\sigma, y/y]M_2 : C \in \mathsf{SN}$ by CR 1
$\Gamma' \vdash \mathsf{case}\,[\sigma]M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow [\sigma, x/x]M_1 \mid \mathsf{inr}\,y \Rightarrow [\sigma, y/y]M_2 : C \in \mathsf{SNe}$ by definition of SNe
$\Gamma' \vdash [\sigma](\mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow M_1 \mid \mathsf{inr}\,y \Rightarrow M_2) : C \in \mathsf{SNe}$ by substitution def.
$\Gamma' \vdash [\sigma](\mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow M_1 \mid \mathsf{inr}\,y \Rightarrow M_2) \in \mathscr{R}_C$ by CR 3

Subcase $\Gamma' \vdash [\sigma]M \longrightarrow_{\mathsf{SN}} M' : A + B$ and $\Gamma' \vdash M' \in \mathscr{R}_{A+B}$
$\Gamma' \vdash [\sigma]M \longrightarrow_{\mathsf{SN}} M' : A + B$ and $\Gamma' \vdash M' \in \mathscr{R}_{A+B}$ by assumption
$\Gamma' \vdash \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow [\sigma, x/x]M_1 \mid \mathsf{inr}\,y \Rightarrow [\sigma, y/y]M_2 \in \mathscr{R}_C$ by inner IH
$\Gamma' \vdash \mathsf{case}\,[\sigma]M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow [\sigma, x/x]M_1 \mid \mathsf{inr}\,y \Rightarrow [\sigma, y/y]M_2$
$\longrightarrow_{\mathsf{SN}} \mathsf{case}\,M'\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow [\sigma, x/x]M_1 \mid \mathsf{inr}\,y \Rightarrow [\sigma, y/y]M_2 : C$ by $\longrightarrow_{\mathsf{SN}}$
$\Gamma' \vdash [\sigma](\mathsf{case}\,M\,\mathsf{of}\ \mathsf{inl}\,x \Rightarrow M_1 \mid \mathsf{inr}\,y \Rightarrow M_2) \in \mathscr{R}_C$ by CR 2

□