A First Introduction to the Algebra of Sentences

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These notes cover the content of a basic course in propositional algebraic logic given by the author at the italian School of Logic held in Cesena, September 18-23, 2000. They are addressed to people having few background in Symbolic Logic and they are mostly intended to develop algebraic methods for establishing basic metamathematical results (like representation theory and completeness, finite model property, disjunction property, Beth definability and Craig interpolation). For the sake of simplicity, only the case of propositional intuitionistic logic is covered, although the methods we are explaining apply to other logics (e.g. modal) with minor modifications. The purity and the conceptual clarity of such methods is our main concern; for these reasons we shall feel free to apply basic mathematical tools from category theory.

The choice of the material and the presentation style for a basic course are always motivated by the occasion and by the lecturer's taste (that's the only reason why they might be stimulating) and these notes follow such a rule. *They are expecially intended to provide an introduction to category theoretic methods, by applying them to topics suggested by propositional logic.*

Unfortunately, time was not sufficient to the author to include all material he planned to include; for this reason, important and interesting developements are mentioned in the final Section of these notes, where the reader can however only find suggestions for further readings. Lack of time is also responsible of the fact that sometimes only key points of proofs are provided, so that the reader is assumed to cooperate 'with pencil and paper' to fill the missed (hopefully not difficult) details.

The categorical background required for these notes is rather limited: students need only the definitions of category, functor, natural transformation, monomorphism, epimorphism, (co)equalizer, pullback, pushout and adjoint functors. All this can be found e.g. in [CWM], [Ro] (or in the more comprehensive handbook [Bo]). First definitions from universal algebra (like congruences, etc.) are also needed and can be found e.g. in [BS]. Notice however that, for the pourposes of these notes, such textbooks should be used just for consultation, whenever basic unknown notions must be introduced. Although not needed for comprehension, some more information on intuitionistic logic could be useful: this is provided e.g. in [vD] or in [Dr].

Sections 1 and 2 are propaedeutic; in a first reading, students should however concentrate only on key points of Section 2, leaving out details. Section 3 is not needed for the remaining ones; on the contrary, Sections 4 and 5 bring essential information for the final Sections, which contain more interesting developements of the theory (still at a basic introductory level).

One word about notation. We indicate the composition of two arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in a category simply as fg. This has the advantage of directly following pictures, the disadvantage is that, in case such arrows are functions among sets and we want to apply fg to an element $a \in A$, we need to use the rule $(fg)(a) = g(f(a))$.

1 Posets, Lattices and Heyting Algebras

In this Section we recall the main algebraic structures which will be investigated within these notes. They are structures that provide an algebraic conceptualization of propositional logics and they are usually obtained by enriching posets by some algebraic operations. We are mainly interested in Heyting algebras i.e. those algebras that provide algebraic counterparts of intuitionistic propositional theories.

A partially oredered set (*poset*, for short) is a set P equipped with a reflexive, transitive and antisymmetric binary relation $\leq$. For such a poset, the *infimum* (resp. *supremum*) of a family $\{a_i\}_{i \in I}$ of elements of P is an element (it may or may not exists, but if it exists it is unique) $\bigwedge_i a_i \in P$

(resp. $\bigvee_i a_i \in P$) such that for all $b \in P$, we have

$$(\forall i \in I \quad b \leq a_i) \qquad \text{iff} \qquad b \leq \bigwedge_i a_i$$

(or

$$(\forall i \in I \quad a_i \leq b) \qquad \text{iff} \qquad \bigvee_i a_i \leq b,$$

respectively). In case the index I is empty, the above conditions say that the infimum of the empty set is the maximum element of P and the supremum of the empty set is just the minimum.

We recall some facts about *adjoints among posets*; although they can be deduced from the general results about categories, it is worth having a direct knowledge of what happens in this special case. The right adjoint f_* (resp. left adjoint f^*) to an order-preserving map $f : P \to Q$ among posets, is an order-preserving map in the opposite direction, satisfying

$$f(a) \leq b \qquad \text{iff} \qquad a \leq f_*(b)$$

(or

$$b \leq f(a) \qquad \text{iff} \qquad f^*(b) \leq a$$

respectively) for all $a \in P, b \in Q$. Such a right (left) adjoint may not exists, but if it exists it is unique. It is easily seen that left adjoints preserve existing suprema and right adjoints existing infima: the latter, for instance, is shown by an easy chain of equivalences as follows

$$\begin{array}{c} a \leq f_*(\bigwedge_i b_i) \\ \hline f(a) \leq \bigwedge_i b_i \\ \hline \forall i \ f(a) \leq b_i \\ \hline \forall i \ a \leq f_*(b_i) \\ \hline a \leq \bigwedge_i f_*(b_i) \end{array}$$

yieldying $f_*(\bigwedge_i b_i) = \bigwedge_i f_*(b_i)$ as a is arbitrary. If P is complete (i.e. iff all suprema -or equivalently all infima- exist), then any order-preserving map $f : P \to Q$ has a right adjoint iff it preserves suprema and has a left adjoint iff it preserves infima. Such adjoints are easily seen to be given by the following formulas:

$$f_*(b) = \bigvee_{f(a) \leq b} a \qquad f^*(b) = \bigwedge_{b \leq f(a)} a$$

for all $b \in Q$.

A (meet) *semilattice* is a commutative idempotent monoid, i.e. a structure $(M, \wedge, \top)$ satisfying the equations

$$a \wedge b = b \wedge a, \quad a \wedge \top = a, \quad a \wedge a = a, \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c \tag{1}$$

for all $a, b, c \in M$. Putting

$$a \leq b \qquad \text{iff} \qquad a \wedge b = a$$

we can define a partial order in any semilattice; the operation $\wedge$ turns out to be the infimum (also called *meet*) of the set $\{a, b\}$ and $\top$ turns out to be the maximum element. In fact, one can equivalently define a semilattice as a partially ordered set in which infima exist for all finite families of elements (this includes the maximum element, seen as the infimum over the empty family).

Many important further operations can be characterized with respect to the partial order so introduced: in order to obtain the notion of a *lattice*[1] one simply has to require that also suprema (called *joins* as well) exist for all finite families; equivalently, a lattice is a semilattice with another binary operation $\vee$ and another constant $\perp$ satisfying equations (1) (with $\wedge, \top$ replaced by $\vee, \perp$ respectively) and moreover the following absorption laws

$$a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a.$$

A lattice is said to be *distributive* iff it satisfies one of the two (equivalent) equations

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

In a given semilattice M it may happens that for $a, b \in M$ the supremum of the set $\{c \mid a \wedge c \leq b\}$ exists; such an element is called the *pseudocomplement of a relative to b* (or the *implication* of a and b, using logical terminology) and is written as $a \to b$. Otherwise said, $a \to b$, if it exists, it is the unique element satisfying the condition

$$a \wedge c \leq b \qquad \text{iff} \qquad c \leq a \to b$$

[1]Notice that we always require the presence of $\perp$ and $\top$ in a lattice (this is different from some common literature).

for all c. A *Brouwerian semilattice* is a semilattice in which all implications among pairs of elements exist and a *Heyting algebra* is a Brouwerian semilattice which is also a distributive lattice. Brouwerian semilattices (hence also Heyting algebras) may be equivalently introduced for instance through the equations

$$a \wedge (a \to b) = a \wedge b \qquad b \wedge (a \to b) = b$$
$$a \to (b \wedge c) = (a \to b) \wedge (a \to c) \qquad a \to a = \top.$$

An important example of a Heyting algebra is given by the open sets $O(T)$ of a topological space T; here the partial order is inclusion, finite meets and joins are intersections and unions, whereas implication of the open subsets a and b is the interior of $a' \cup b$ (where a' is the complement of a). The most important example for us is given by the downward closed subsets P^* of a poset P ($a \subseteq P$ is downward closed iff $p \in a$ and $q \leq p$ imply $q \in a$): here the partial order, joins and meets are again inclusion, intersections and unions, respectively, whereas the implication of a and b is

$$a \to b \ = \ \{p \in P \mid \forall q \leq p \ (q \in a \Rightarrow q \in b)\}.$$

Localizations provide a general method to build new Heyting algebras from a given one. A *Lawvere local operator* on a semilattice M is a function

$$\ddagger : M \longrightarrow M$$

satisfying the equations

$$a \leq \ddagger a \qquad \ddagger\, a = \ddagger\, \ddagger\, a \qquad \ddagger\, (a \wedge b) = \ddagger a \wedge \ddagger b.$$

It turns out that $\ddagger M = \{a \in M \mid a = \ddagger a\}$ is a subsemilattice of M; in case M is a Brouwerian semilattice, so is $\ddagger M$ and in case M is a distributive lattice so is $\ddagger M$ (however, in this case, $\ddagger M$ is not a sublattice of M because join of $\{a_1, \ldots, a_n\} \subseteq \ddagger M$ is $\ddagger(a_1 \vee \cdots \vee a_n)$, not just $a_1 \vee \cdots \vee a_n$ which may not belong to $\ddagger M$). Consquently, if M is a Heyting algebra, so is $\ddagger M$.

A finite distributive lattice is always a Heyting algebra, because a finite distributive lattice is complete and, thanks to distributivity, for any element a, the order preserving map $a \wedge (-)$ preserves suprema, so that it has a right adjoint $a \to (-)$. For the same reason, a finite Brouwerian semilattice is always a Heyting algebra: in fact joins exists and are distributive as $a \wedge (-)$ preserves them (being a left adjoint).

In a Heyting algebra H, negation is introduced through

$$\neg a = a \to \bot;$$

such operation satisfies many usual laws, but not all of them (for instance, only three of the four De Morgan identities hold). A *Boolean algebra* is a Heyting algebra in which we have $\neg\neg a = a$ (or, equivalently, $a \vee \neg a = \top$) for all a.

Distributive lattices and Boolean algebras (also Brouwerian semilattices, but this fact is less trivial) are *locally finite varieties*, namely varieties in which finitely generated algebras are finite; this is easily seen, e.g. in the case of Boolean algebras, from the fact that if the set G generates the algebra B, then every element of B admits a representation of the kind $\bigwedge_i \bigvee_j x_{ij}$ where i, j range over finite sets of indices and where x_{ij} is either g or $\neg g$ for some $g \in G$. Local finiteness is not true for Heyting algebras, since even an one-element generated Heyting algebra can be infinite.

Let us mention how to describe quotients in Heyting algebras. The central notion to this respect is the notion of a *filter*, which makes sense at the level of a semilattice R (although it becomes fully operative only when there are implications): this is a subset F of R satisfying the following requirements

- $\top \in F$;
- if $a_1, a_2 \in F$, then $a_1 \wedge a_2 \in F$;
- if $a_1 \in F$ and $a_1 \leq a_2$, then $a_2 \in F$.

Given a subset $S \subseteq R$, there exists the minimum filter $[S]$ containing S, which is given by

$$[S] \;=\; \{b \in R \,|\, \exists n \geq 0, \exists a_1, \dots, a_n \in S \text{ s.t. } a_1 \wedge \cdots \wedge a_n \leq b\}.$$

In particular, the minimum (or principal) filter containing an element a is just $[a] = \{b \,|\, a \leq b\}$.

In Heyting algebras, the lattice of filters and the lattice of congruences are isomorphic; given a congruence $\simeq$, we can associate to it the filter $\{a \,|\, a \simeq \top\}$ and given a filter F we can associate to it the congruence given by

$$a \simeq b \quad \text{iff} \quad a \leftrightarrow b \in F$$

(where $a \leftrightarrow b$ is $(a \to b) \wedge (b \to a)$). The two correspondences are inverse each other.

Given that congruences can be replaced by filters, we shall call *kernel* of a morphism $f : A \longrightarrow B$ among Heyting algebras directly the filter $\{a \in A \,|\, f(a) = \top\}$ (and not - as it is in the general case of universal algebra - the congruence given by $a \simeq b$ iff $f(a) = f(b)$). Also, notice that in order to check that a morphism $f : A \longrightarrow B$ among Heyting algebras is injective, it is sufficient to show that $[\top]$ is its kernel (i.e. that $f(a) = \top$ implies $a = \top$ for all $a \in A$).

The correspondence between filters and congruences can be used to get easily the so-called *congruence extension property* for Heyting algebras, namely the fact that the pushout of a monomorphism along a quotient is again a monomorphism. In fact, if $m : A \longrightarrow A'$ is mono and F is a filter of A with quotient map $q : A \longrightarrow A/F$, we can build a pushout square as follows

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "q"'] & A' \arrow[d, "q'"] \\ A/F \arrow[r, "m'"'] & A'/F' \end{tikzcd}$$

where F' is the filter $\{b \in A' \,|\, \exists a \in F \text{ s.t. } m(a) \leq b\}$. As F is precisely the kernel of mq', the morphism $m' : A/F \longrightarrow A'/F'$ exists and is mono.[2]

[2] We take this occasion to recall the following simple but very important *universal property of quotients*. Given algebras A, B, C and morphisms $f : A \longrightarrow B$, $q : A \longrightarrow C$ such that q is a quotient (i.e. it is surjective), we have that there exists a (necessarily unique) morphism $\bar{f} : C \longrightarrow B$ making the triangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \\ C \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

to commute iff the kernel of q is included in the kernel of f. Moreover $\bar{f}$ is injective iff the kernel of q coincides with the kernel of f.

A key notion is the notion of a *prime filter* in a distributive lattice D: it is just a filter $\mathbf{p}$ satisfying the further two requirements:

- $\perp \notin \mathbf{p}$;
- if $a_1 \vee a_2 \in \mathbf{p}$, then either $a_1 \in \mathbf{p}$ or $a_2 \in \mathbf{p}$.

Prime filters can be equivalently defined as the preimages of 1 along morphisms $D \longrightarrow \mathbf{2}$ (here $\mathbf{2} = \{1, 0\}$ is the two-element Boolean algebra). Prime filters in Boolean algebras are usually called *ultrafilters* and can equivalently be introduced as maximal proper filters.

Existence of enough prime filters is guaranteed by the following very important *extension/exclusion Lemma*, whose proof depends in an essential way on the axiom of choice. If S, T are subsets of a distributive lattice, notation $S \leq T$ means that there are $a_1, \dots, a_n \in S$ ($n \geq 0$) and $b_1, \dots, b_m \in T$ ($m \geq 0$) such that $a_1 \wedge \cdots \wedge a_n \leq b_1 \vee \cdots \vee b_m$.

Lemma 1.1 *Let D be a distributive lattice and let S, T be subsets of its such that $S \not\leq T$. Then there is a prime filter $\mathbf{p}$ such that $S \subseteq \mathbf{p}$ and $\mathbf{p} \cap T = \emptyset$.*

Proof. First notice that we can suppose that T is closed under finite joins (otherwise we simply close it, without loosing the condition $S \not\leq T$). We take the family $\mathcal{F}$ of filters including S and disjoint from T (this is not empty as it contains $[S]$). We order such family by set-theoretic inclusion; now the union of a chain of filters disjoint from T is also a filter disjoint from T, hence we are in the good conditions to apply Zorn lemma. We show that if M is maximal in $\mathcal{F}$, then it is prime. If not, there are $a_1, a_2 \notin M$ such that $a_1 \vee a_2 \in M$. Take $F_i = [M \cup \{a_i\}]$ ($i = 1, 2$); by maximality, these two filters cannot be in $\mathcal{F}$ hence they are not disjoint from T. It follows that there are $m_1, m_2 \in M$ and $b_1, b_2 \in T$ such that

$$m_1 \wedge a_1 \leq b_1 \quad \text{and} \quad m_2 \wedge a_2 \leq b_2.$$

Taking $m = m_1 \wedge m_2 \in M$ and $b = b_1 \vee b_2 \in T$, we get

$$m \wedge a_1 \leq b \quad \text{and} \quad m \wedge a_2 \leq b,$$

hence also (by the distributivity law)

$$m \wedge (a_1 \vee a_2) \leq b,$$

that is M is not disjoint from T, contradiction. $\square$

We shall use the above extension/exclusion Lemma quite often; notice that the condition $S \not\leq T$ appearing in it has obvious simplifications in case, say, S is closed under finite meets and T is closed under finite joins (in such a case $S \not\leq T$ holds iff for no $a \in S$ and $b \in T$ we have $a \leq b$). This and similar obvious simplifications will be tacitly adopted whenever possible.

2 Lindenbaum Algebras

The content of this Section *is not needed* for the subsequent formal developments, which rely only on the definitions from Section 1 and on general facts from category theory. For this reason, *we shall omit or merely sketch proofs* of the basic facts we are going to establish, thus avoiding dispersive not really deep details.

On the other hand, this Section is *quite crucial* for the correct understanding of the *logical meaning* of all the results we shall establish in these notes.

We briefly recall the syntax of intutionistic propositional calculus (IPC). Given a set X, we can introduce the set $Form(X)$ of well formed formulas over the alphabet X (of propositional letters) in the standard way (we use $\wedge, \vee, \rightarrow$ as connectives and an additional constant symbol $\bot$).[3] Letters $\varphi, \psi, \ldots$ denote such formulas and letters $\Gamma, \Delta, \ldots$ denote sets of formulas. We sometimes write $\varphi(X)$ (resp. $\Gamma(X)$) to emphasize the fact that the formula φ (resp. the set of formulas Γ) is built up from X (i.e. it does not contain propositional letters other than those belonging to X).

To make the presentation quick, we use a Hilbert-style calculus for IPC. We take as logical axioms the following schemata:

$$\varphi \rightarrow (\psi \rightarrow \varphi)$$
$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$
$$\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$$
$$\varphi \wedge \psi \rightarrow \varphi$$
$$\varphi \wedge \psi \rightarrow \psi$$
$$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi))$$

[3] $\top$ stands for $\bot \rightarrow \bot$, $\neg\varphi$ stands for $\varphi \rightarrow \bot$ and $\varphi \leftrightarrow \psi$ stands for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

$$\varphi \to \varphi \vee \psi$$
$$\psi \to \varphi \vee \psi$$
$$\bot \to \varphi$$

Modus ponens

$$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

is the only inference rule.

Given a *theory* (i.e. a set of formulas) Γ and given a formula φ, the notation

$$\vdash_\Gamma \varphi$$

means that there is a Γ-derivation of φ, i.e. that there is a list of formulas $\varphi_0, \varphi_1, \ldots, \varphi_n = \varphi$ such that for each i

- either φ_i is a logical axiom (i.e. it is an instance of one of the above axiom schemata);
- or φ_i is a proper axiom (i.e. it belongs to Γ);
- or, finally, there are $i_1, i_2 < i$ such that φ_{i_1} is $\varphi_{i_2} \to \varphi_i$ (i.e. φ_i is obtained from previous formulas in the derivation by applying modus ponens).

By

$$\psi \vdash_\Gamma \varphi$$

we mean that $\vdash_\Gamma \psi \to \varphi$ (or, equivalently, by the deduction theorem, that $\vdash_{\Gamma \cup \{\psi\}} \varphi$). What is important in the sequel is the following fact:

Proposition 2.1 *Let Γ be a theory (an alphabet X is supposed to be fixed); for formulas $\varphi(X), \psi(X)$ define the equivalence relation*

$$\varphi \sim_\Gamma \psi \quad \text{iff} \quad \vdash_\Gamma \varphi \leftrightarrow \psi.$$

Then $Form(X)/{\sim_\Gamma}$ is a Heyting algebra with respect to the obvious operations

$$\top = [\top]$$

$$
\begin{aligned}
\bot &= [\bot] \\
[\varphi] \wedge [\psi] &= [\varphi \wedge \varphi] \\
[\varphi] \vee [\psi] &= [\varphi \vee \varphi] \\
[\varphi] \rightarrow [\psi] &= [\varphi \rightarrow \varphi]
\end{aligned}
$$

Such an algebra is denoted $\mathcal{F}(X)/\Gamma$ and called the Lindenbaum algebra over X and Γ. $\square$

In case Γ is empty, we shall write $\vdash \varphi$, $\psi \vdash \varphi$, $\sim$, $\mathcal{F}(X)$, etc. instead of $\vdash_\emptyset \varphi$, $\psi \vdash_\emptyset \varphi$, $\sim_\emptyset$, $\mathcal{F}(X)/\emptyset$, respectively.

For an alphabet X, we have an (injective) set-theoretic map

$$\eta_X : X \longrightarrow \mathcal{U}(\mathcal{F}(X))$$

associating with $x \in X$ the equivalence class $[x]$ in $\mathcal{F}(X)$ (here $\mathcal{U}$ is the forgetful functor from the category **H** of Heyting algebras into the category **Set** of sets). The universal property of $(\eta_X, \mathcal{F}(X))$ is explained in next Theorem, saying that $\mathcal{F}(X)$ is the free Heyting algebra over X:

Theorem 2.2 *Given an alphabet X and given a set-theoretic function*

$$I : X \longrightarrow \mathcal{U}(A)$$

(where A is any Heyting algebra), there is a unique morphism

$$[I^e] : \mathcal{F}(X) \longrightarrow A$$

such that the following triangle

$$
\begin{tikzcd}
 & X \arrow[dl, "\eta_X"'] \arrow[dr, "I"] & \\
\mathcal{U}(\mathcal{F}(X)) \arrow[rr, "\mathcal{U}([I^e])"'] & & \mathcal{U}(A)
\end{tikzcd}
$$

commutes.

Proof. (Sketch) We first extend I to a set-theoretic function

$$I^e : Form(X) \longrightarrow \mathcal{U}(A)$$

as follows

$$\begin{gathered} I^e(x) = I(x), \quad \text{for } x \in X \\ I^e(\perp) = \perp \\ I^e(\varphi \wedge \psi) = I^e(\varphi) \wedge I^e(\psi) \\ I^e(\varphi \vee \psi) = I^e(\varphi) \vee I^e(\psi) \\ I^e(\varphi \to \psi) = I^e(\varphi) \to I^e(\psi). \end{gathered}$$

By induction on IPC-derivations, one easily shows that

$$\vdash \varphi \quad \Rightarrow \quad I^e(\varphi) = \top.$$

Consequently, if $\varphi \sim \psi$, then $I^e(\varphi) = I^e(\psi)$. This means that the map

$$[I^e] : \mathcal{F}(X) \longrightarrow A$$

given by

$$[I^e]([\varphi]) = I^e(\varphi)$$

is well-defined. It is indeed the unique Heyting algebras morphism making the above triangle commutative. $\square$

The above data, namely the functions $\eta_X : X \longrightarrow \mathcal{U}(\mathcal{F}(X))$, can be used to define a functor

$$\mathcal{F} : \mathbf{Set} \longrightarrow \mathbf{H}$$

(which turns out to be left adjoint to the forgetful functor $\mathcal{U}$) by the following purely categorical standard procedure. We already know the value of $\mathcal{F}$ on objects (i.e. on sets); for a function $f : X \longrightarrow Y$, we let $\mathcal{F}(f)$ to be the unique morphism such that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ \mathcal{U}(\mathcal{F}(X)) \arrow[r, "\mathcal{U}(\mathcal{F}(f))"'] & \mathcal{U}(\mathcal{F}(Y)) \end{tikzcd}$$

commutes. The fact that $\mathcal{F}$ is left adjoint to $\mathcal{U}$ means that we have natural bijections between the following entities:

$$\frac{X \longrightarrow \mathcal{U}(A)}{\mathcal{F}(X) \longrightarrow A.}$$

We shall establish now a suitable universal property for an *arbitrary* Lindenbaum algebra $\mathcal{F}(X)/\Gamma$. Suppose that $I_\Gamma : X \longrightarrow \mathcal{U}(A)$ is a function such that $I^e_\Gamma(\varphi) = \top$ holds for all $\varphi \in \Gamma$; we call such a function a Γ-model (in A).[4]

By Theorem 2.2 and the related proof, such an I_Γ uniquely extends to a morphism

$$[I^e_\Gamma] : \mathcal{F}(X) \longrightarrow A$$

such that $[I^e_\Gamma]([\varphi]) = \top$ holds for all φ such that $\vdash_\Gamma \varphi$. Now the kernel of the surjective morphism

$$q_\Gamma : \mathcal{F}(X) \longrightarrow \mathcal{F}(X)/\Gamma$$

is precisely formed by all $[\varphi]$ such that $\vdash_\Gamma \varphi$: this is again by Theorem 2.2 and its proof, because q_Γ can be defined as the unique extension of the set-theoretic map $X \longrightarrow \mathcal{U}(\mathcal{F}(X)/\Gamma)$ asssociating with $x \in X$ the equivalence class $[x]$ in $\mathcal{F}(X)/\Gamma$ (consequently we have $q_\Gamma([\varphi]) = [\varphi]$ for all $\varphi \in Form(X)$). By the universal property of quotients, we have a unique morphism $\alpha(I_\Gamma)$ such that the triangle

$$\begin{tikzcd} & \mathcal{F}(X) \arrow[dl, "q_\Gamma"'] \arrow[dr, "{[I^e_\Gamma]}"] & \\ \mathcal{F}(X)/\Gamma \arrow[rr, "\alpha(I_\Gamma)"'] & & A \end{tikzcd}$$

commutes.[5] Thus Γ-*models in a Heyting algebra* A

$$I_\Gamma : X \longrightarrow \mathcal{U}(A)$$

[4]Notice that by induction on IPC-derivations, we easily get that $I^e_\Gamma(\varphi) = \top$ is true for all φ such that $\vdash_\Gamma \varphi$.

[5]Notice that $\alpha(I_\Gamma)$ is given by $\alpha(I_\Gamma)([\varphi]) = I^e_\Gamma(\varphi)$; moreover, I_Γ can be recovered from $\alpha(I_\Gamma)$ by $I_\Gamma(x) = \alpha(I_\Gamma)([x])$.

bijectively corresponds to morphisms

$$\alpha(I_\Gamma) : \mathcal{F}(X)/\Gamma \longrightarrow A.$$

This bijective correspondence α is *natural in* A, in the sense that for $g : A \longrightarrow A'$ we have[6]

$$(\alpha(I_\Gamma))g \;=\; \alpha(I_\Gamma \mathcal{U}(g)).$$

The existence of such a natural bijection between Γ-models in A and morphisms $\mathcal{F}(X)/\Gamma \longrightarrow A$ characterizes the Heyting algebra $\mathcal{F}(X)/\Gamma$ uniquely up to isomorphism.

As a first application of this fact, let us show that *any Heyting algebra* B *is the Lindenbaum algebra of an intuitionistic theory* (in a suitable language). Let in fact $I : X \longrightarrow \mathcal{U}(B)$ be a function whose image $I(X)$ generates B as a Heyting algebra (we may take e.g. as I the identity function $\mathcal{U}(B) \longrightarrow \mathcal{U}(B)$). As usual, such a function extends to a map

$$I^e : Form(X) \longrightarrow \mathcal{U}(B)$$

which is surjective (the fact that I^e is surjective may be taken as a synonimous of the fact that $I(X)$ generates B). Let Γ be the theory $\{\varphi \in Form(X) \,|\, I^e(\varphi) = \top\}$. We have that

$$B \;\simeq\; \mathcal{F}(X)/\Gamma$$

because it is possible to find a natural bijective correspondence between morphisms $B \longrightarrow A$ and Γ-models in A for every Heyting algebra A (we leave the details to the reader).

The fact that every Heyting algebra is a Lindenbaum algebra says that Heyting algebras *must be regarded essentially as syntactic objects*. Better, they are *invariant* syntactic objects, in the sense that, when considering them, conventional linguistic choices have been forgotten as irrelevant: notice that, from this point of view, given a Heyting algebra B, the choice of a *presentation* for B (i.e. of X and Γ such that $B \simeq \mathcal{F}(X)/\Gamma$) is not univocally determined.

The previous remark, however, does not prevent us from regarding *specific kinds* of Heyting algebras as *semantic universes* (where models are taken into). More precisely:

[6] This naturality may be equivalently stated by the equation $(\alpha^{-1}(f))\mathcal{U}(g) = \alpha^{-1}(fg)$, for $g : A \longrightarrow A'$ and $f : \mathcal{F}(X)/\Gamma \longrightarrow A$.

- Heyting algebras of the kind P^*, for a poset P, are called *Kripkean* semantic universes;
- Heyting algebras of the kind $O(T)$, for a topological space T, are called *topological* semantic universes;
- Heyting algebras of the kind $\ddagger P^*$, for a poset P and a local operator $\ddagger : P^* \longrightarrow P^*$, are called *Beth-Grothendieck* semantic universes.

Thus, for a Heyting algebra B, we may define a *Kripke model* for B as a morphism of the kind

$$B \longrightarrow P^*.$$

Analogously, *topological models* for B are morphisms of the kind

$$B \longrightarrow O(T)$$

and *Beth-Grothendieck models* for B are morphisms of the kind

$$B \longrightarrow \ddagger P^*.$$

This terminology coincides with the standard terminology from textbooks in logic. Let us illustrate this in the case of Kripke models $B \longrightarrow P^*$. If $B \simeq \mathcal{F}(X)/\Gamma$, we know that Kripke models corresponds to Γ-models $I : X \longrightarrow \mathcal{U}(P^*)$ such that $I^e(\varphi) = \top$ holds for all $\varphi \in \Gamma$. Now functions $I : X \longrightarrow \mathcal{U}(P^*)$ are standard Kripke evaluations (they associate with each propositional letter $x \in X$ a downward closed subset of P).[7] The extension $I^e : Form(X) \longrightarrow \mathcal{U}(P^*)$ of I to all formulas is nothing but the usual extension of Kripke forcing from propositional letters to all formulas. In fact, if we write $I \models \varphi$ for $p \in I^e(\varphi)$, we have (taking into account the definition of I^e and the way the Heyting algebras operations are defined in P^*):

$$\begin{array}{ll}
I \models_p x & \text{iff} \;\; p \in I(x) \\
I \models_p \top & \\
I \not\models_p \bot & \\
I \models_p \varphi_1 \wedge \varphi_2 & \text{iff} \;\; I \models_p \varphi_1 \text{ and } I \models_p \varphi_2
\end{array}$$

[7] Most textbooks prefer to consider upward closed subsets.

$$I \models_p \varphi_1 \vee \varphi_2 \quad \text{iff} \quad I \models_p \varphi_1 \text{ or } I \models_p \varphi_2$$
$$I \models_p \varphi_1 \to \varphi_2 \quad \text{iff} \quad \forall q \leq p \ (I \models_q \varphi_1 \ \Rightarrow \ I \models_q \varphi_2).$$

Now the fact that, if $B \simeq \mathcal{F}(X)/\Gamma$, any morphism $B \longrightarrow P^*$ corresponds to a Γ-model $I : X \longrightarrow \mathcal{U}(P^*)$, means in term of forcing that we must have for such an I

$$I \models_p \varphi \quad \text{for all } \varphi \in \Gamma \text{ and } p \in P$$

(because this is the same as the Γ-model condition $I^e(\varphi) = \top$ for all $\varphi \in \Gamma$). Thus we are justified in calling a morphism $B \longrightarrow P^*$ a Kripke model for (any theory giving a presentation of) B.

The same observations apply to topological and Beth-Grothendieck models, where however the truth clauses for forcing are changed according to the new Heyting algebras operations. For topological models, only truth clause for implication is changed; the new one is

$$I \models_p \varphi_1 \to \varphi_2 \qquad \text{iff} \qquad \exists N \in \mathcal{N}(p) \text{ s.t. } \forall q \in N \ (I \not\models_q \varphi_1 \text{ or } I \models_q \varphi_2)$$

(here $\mathcal{N}(p)$ is the set of neighborhoods of the point p).

For Beth-Grothendieck models, only clauses for $\perp$ and disjunction are changed; the new ones are

$$I \models p \qquad \text{iff} \qquad \emptyset \text{ covers } p$$

and

$$I \models_p \varphi_1 \vee \varphi_2 \qquad \text{iff} \qquad \{q \leq p \,|\, I_q \models \varphi_1 \text{ or } I_q \models \varphi_2\} \text{ covers } p$$

(here we say that a downward set S covers a point p iff $p \in \ddagger S$).

We have seen the logical meaning of a single Heyting algebra B (it is a theory) and of a morphism with domain B and codomain specific kinds of Heyting algebras (it is a model of a suitable kind). It remains to give a logical meaning to *any morphism*

$$f : B \longrightarrow A$$

among Heyting algebras (such a meaning will be purely syntactical, as B and A are themselves syntactic objects). Suppose that $B \simeq \mathcal{F}(X)/\Gamma$ and $A \simeq$

$\mathcal{F}(Y)/\Delta$; then we know that f is the same as a Γ-model $I : X \longrightarrow \mathcal{F}(Y)/\Delta$. Given such a Γ-model, let us pick for every $x \in X$ a formula $\hat{x} \in Form(Y)$ such that $[\hat{x}] = I(x)$. The function $x \mapsto \hat{x}$ can be inductively extended to all formulas in $Form(X)$ in the obvious way (i.e. we put $(\varphi_1 * \varphi_2)\hat{} = \hat{\varphi}_1 * \hat{\varphi}_2$ for every connective $*$) so that we have $[\hat{\varphi}] = I^e(\varphi)$ for all φ. In this way (taking into consideration the definition of a Γ-model) we get that

$$\varphi \in \Gamma \quad \Rightarrow \quad \vdash_\Delta \hat{\varphi}.$$

This is what in logic is commonly called a *syntactic interpretation* of the theory Γ into the theory Δ. Notice that the syntactic interpretation $(\hat{-})$ is *conservative* (that is, $\vdash_\Delta \hat{\varphi}$ implies $\vdash_\Gamma \varphi$ for all $\varphi \in Form(X)$) precisely when f is a monomorphism. Moreover f is surjective iff we can arrange the presentations in such a way that $X = Y$, $\Gamma \subseteq \Delta$ and $\hat{\varphi} = \varphi$ for all φ. Thus:

- morphisms correspond to syntactic interpretations (in invariant sense);
- monomorphisms correspond to conservative syntactic interpretations (or, to put it in a slightly different way, to enlargements of the original theory into a theory in a bigger language not proving new theorems in the old language);
- quotients correspond to strenghtening of the original theory with new axioms within the same language.

It should be noticed that it is always possible, given a morphism

$$f : B \longrightarrow A,$$

to arrange a presentation

$$\mathcal{F}(X)/\Gamma \longrightarrow \mathcal{F}(Y)/\Delta$$

$$[\varphi] \longmapsto [\hat{\varphi}]$$

for it in such a way that we have $X \subseteq Y$, $\Gamma \subseteq \Delta$ and $\hat{\varphi} = \varphi$ for all $\varphi \in Form(X)$.[8] This is achieved e.g. in the following way, starting from any presentation $\mathcal{F}(X)/\Gamma$ of the domain algebra B. First, let Y to be $X \cup$

[8]This is not surprising, syntactic interpretations are subject to conventions, including the conventions used to build the source and the target theory.

$(\mathcal{U}(A)\backslash Im(f))$ and let J be the function $J : Y \longrightarrow \mathcal{U}(A)$ associating $f([x])$ to $x \in X$ and x itself to any $x \in \mathcal{U}(A)\backslash Im(f)$. Let finally Δ be the theory $\{\varphi \in Form(Y) \mid J^e(\varphi) = \top\}$. Now the following triangle

$$\begin{tikzcd} & B \simeq \mathcal{F}(X)/\Gamma \arrow[dl, "\alpha(I_X)"'] \arrow[dr, "f"] & \\ \mathcal{F}(Y)/\Delta \arrow[rr, "\alpha(J)"'] & & A \end{tikzcd}$$

commutes (here I_X is the Γ-model associating $[x]$ with $x \in X$) and $\alpha(J)$ is injective and surjective (i.e. an isomorphism).[9] In this way Δ is a theory in a language larger than the language of Γ, however Δ needs not to be a conservative extension of Γ (unless f is injective).

The above observation for presentations can be often used in order to understand the logical meaning of certain algebraic constructions. For Sections 6 and 7, a crucial construction is that of a pushout

$$\begin{tikzcd} A_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \arrow[d] \\ A_2 \arrow[r] & A \end{tikzcd}$$

Using the above schema for presentations, we can present A_0, A_1, A_2 as (let us write $\mathcal{F}(Y_0, Y_i)$ instead of $\mathcal{F}(Y_0 \cup Y_i)$)

$$A_0 \simeq \mathcal{F}(Y_0)/\Gamma_0, \qquad A_1 \simeq \mathcal{F}(Y_0, Y_1)/\Gamma_1 \qquad A_0 \simeq \mathcal{F}(Y_0, Y_2)/\Gamma_2$$

in such a way that f_i (for $i = 1, 2$) is the morphism associating the equivalence class $[\varphi(Y_0)]$ with $[\varphi(Y_0)]$ itself (seen as an equivalence class in $\mathcal{F}(Y_0, Y_i)/\Gamma_i$). Now, provided we keep Y_1 and Y_2 disjoint, a presentation for the pushout algebra A can be achieved by observing that the square

[9]For the involved verifications, just previously observe that $f([\varphi]) = J^e(\varphi)$ holds for all $\varphi \in Form(X)$.

$$\begin{tikzcd}
\mathcal{F}(Y_0)/\Gamma \arrow[r] \arrow[d] & \mathcal{F}(Y_0, Y_1)/\Gamma_1 \arrow[d] \\
\mathcal{F}(Y_0, Y_2)/\Gamma_2 \arrow[r] & \mathcal{F}(Y_0, Y_1, Y_2)/\Gamma_1 \cup \Gamma_2
\end{tikzcd}$$

is indeed a pushout. So taking pushout of two theories (over a given one) means *putting together languages and axioms*.

Presentations like the above one for pushouts *are not useful* for proofs (proofs should only rely on universal properties), however they are indispensable to get the correct logical intuition and to give logical meaning to algebraic constructions (in the relevant cases, there always is such a meaning).

3 The Glueing Construction

One of the most important properties of intuitionistic logic is disjunction property, saying that a disjunction of two formulas is provable iff one of the two disjuncts is provable. In algebraic terms this means that free Heyting algebras are *prime* (we say that a Heyting algebra A is prime iff for all $a, b \in A$, we have that $\top = a \vee b$ implies that either $\top = a$ or $\top = b$).

Let A_1, A_2 be Heyting algebras and let $f : A_1 \longrightarrow A_2$ be *a morphism of the underlying semilattices.* Consider the set

$$\gamma(f) = \{(a_1, a_2) \in A_1 \times A_2 \mid a_2 \leq f(a_1)\}.$$

Notice that if $(a_1, a_2), (a_1', a_2') \in \gamma(f)$, then the pairs

$$(a_1 \wedge a_1', a_2 \wedge a_2'), \quad (a_1 \vee a_1', a_2 \vee a_2'), \quad (\top, \top), \quad (\bot, \bot)$$

belongs to $\gamma(f)$ too. With these operations, $\gamma(f)$ is easily seen to be a distributive lattice. More is true indeed:

Proposition 3.1 *$\gamma(f)$ is a Heyting algebra and the first projection (restricted in its domain) is a Heyting algebra morphism $p : \gamma(f) \longrightarrow A_1$.*

Proof. We take

$$(a_1 \to a_1',\ (a_2 \to a_2') \wedge f(a_1 \to a_1'))$$

as implication of (a_1, a_2) and (a_1', a_2'). For $(a, b) \in \gamma(f)$ we have the following chain of equivalences:

$$\frac{\dfrac{(a, b) \wedge (a_1, a_2) \leq (a_1', a_2')}{a \wedge a_1 \leq a_1' \ \ \& \ \ b \wedge a_2 \leq a_2'}}{a \leq a_1 \to a_1' \ \ \& \ \ b \leq a_2 \to a_2'}$$

We claim that last two inequations are equivalent to

$$a \leq a_1 \to a_1' \ \ \& \ \ b \leq (a_2 \to a_2') \wedge f(a_1 \to a_1').$$

One side is trivial, for the other one, just notice that $b \leq f(a)$ and $f(a) \leq f(a_1 \to a_1')$ immediately imply that $b \leq f(a_1 \to a_1')$. $\square$

Lemma 3.2 *Let A be a Heyting algebra and let $c_A : A \longrightarrow \mathbf{2}$ be the characteristic function of $\{\top\}$; then the Heyting algebra $\gamma(c_A)$ is prime.*

Proof. $\gamma(c_A)$ contains all pairs $(a, 0)$; it contains $(a, 1)$ iff $a = \top$. It is then clear that $\gamma(c_A)$ has a penultimate element (namely $(\top, 0)$), so it is prime. $\square$

Theorem 3.3 *Free Heyting algebras are prime.*

Proof. Let $\mathcal{F}(X)$ be a free Heyting algebra and let $(\hat{-})$ be the unique morphism from $\mathcal{F}(X)$ into $\gamma(c_{\mathcal{F}(X)})$ such that $\hat{x} = (x, 0)$ holds for all $x \in X$. We have a commutative triangle

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[rr, "(\hat{-})"] \arrow[dr, "id"'] & & \gamma(c_{\mathcal{F}(X)}) \arrow[dl, "p"] \\
& \mathcal{F}(X) &
\end{tikzcd}$$

showing that $\widehat{(-)}$ is injective. For $a, b \in \mathcal{F}(X)$ we then have the following chain of equivalences

$$\frac{\frac{\frac{\top \leq a \vee b}{\top = \hat{\top} \leq \hat{a} \vee \hat{b}}}{\hat{\top} \leq \hat{a} \;\text{ or }\; \hat{\top} \leq \hat{b}}}{\top \leq a \;\text{ or }\; \top \leq b}$$

where in last two passages we used the previous Lemma and the injectivity of $\widehat{(-)}$. $\square$

4 Basic Adjunction

In Section 2 we saw how to give logical meaning to our algebras; in this and in next Section we see how to get appropriate *geometric* intuition about them, by representing them as suitable 'spaces'. We first begin with distributive lattices and connect them to posets.

Given a poset $(P, \leq)$ (to be indicated as usual simply as P), we can form the distributive lattice P^* of *sieves* (i.e. of downward closed subsets) of P, as mentioned in Section 1. We recall that Joins and Meets (i.e. suprema and infima) in P^* are just set-theoretical unions and intersections. Given an order-preserving map $\mu : Q \longrightarrow P$ among posets, we define $\mu^* : P^* \longrightarrow Q^*$ by taking inverse image (i.e. for $a \in P^*$, $\mu^*(a)$ is the sieve $\{q \mid \mu(q) \in a\}$). Notice that μ^* preserves all Joins and Meets. As this definition is clearly functorial, we have in fact a functor

$$(-)^* : \mathbf{P}^{op} \longrightarrow \mathbf{D}$$

where $\mathbf{P}$ is the category of posets and order-preserving maps and $\mathbf{D}$ is the category of distributive lattices and related morphisms.

There is a kind of reverse correspondence, given by the spectrum construction. For a distributive lattice D, let D^* be the poset of prime filters of D, ordered by reverse inclusion. For a distributive lattice morphism $f : E \longrightarrow D$, we can define $f^* : D^* \longrightarrow E^*$ by taking inverse image once again, i.e. we have for $\mathbf{p} \in D^*$

$$f^*(\mathbf{p}) \;=\; \{a \in E \mid f(a) \in \mathbf{p}\}.$$

Notice that $f^*(\mathbf{p})$ is a prime filter, moreover f^* triviallly preserves partial order. Thus we have a functor

$$(-)^* : \mathbf{D} \longrightarrow \mathbf{P}^{op}$$

(we give it the same name as the previous one).

Theorem 4.1 *Functors* $(-)^*$ *are adjoints.*

Proof. We need a natural bijection between entities of the following kinds

$$\frac{D \xrightarrow{f} P^*}{D^* \xleftarrow{\mu} P}$$

In fact, given $f : D \longrightarrow P^*$, define $\mu : P \longrightarrow D^*$ by

$$\mu(p) = \{a \in D \,|\, p \in f(a)\}.$$

Conversely, given $\mu : P \longrightarrow D^*$, define $f : D \longrightarrow P^*$ by

$$f(a) = \{p \,|\, a \in \mu(p)\}.$$

Bijectivity and naturality of this correspondence are easy. □

For a distributive lattice D, let us call D^* the *dual space* of D. It should be noticed that it is impossible in general to recover the distributive lattice D from its dual space (in particular, D^{**} is quite larger than D itself). To do this, one should enrich D^* with a topological structure (suitably connected with the partial order relation), leading to the notion of a *Priestley space.*[10]

The above functors $(-)^*$ can be described in a slightly different but suggestive way. Notice that the Boolean algebra $\mathbf{2}$ can be considered both as a poset and as a distributive lattice. Prime filters of a distributive lattice D are just distributive lattice morphisms $D \longrightarrow \mathbf{2}$. On the other hand, sieves of a poset P are just order-preserving maps $P \longrightarrow \mathbf{2}$. Thus for a distributive lattice D, D^* (as a set) is just $Hom_{\mathbf{D}}(D, \mathbf{2})$ and for a poset P, P^* (as a set)

[10]A Priestley space is a compact topological space T endowed with a partial order relation $\leq$ such that whenever we have $p \not\leq q$ (for $p, q \in T$), then there is a clopen sieve S containing q and not containing p. The category of Prietley spaces and continuous order-preserving maps is dual to $\mathbf{D}$.

is just $Hom_{\mathbf{P}}(P, \mathbf{2})$. The same observation applies to arrows (i.e. to distributive lattices morphisms and to order-preserving maps), hence our functors $(-)^*$ are, in a sense, both represented by $\mathbf{2}$; due to this double role it plays, $\mathbf{2}$ is called a *schizophrenic object.*

The natural bijection of Theorem 4.1, applied to identity maps $D^* \longrightarrow D^*$, gives distributive lattices morphisms

$$\eta_D : D \longrightarrow D^{**}$$

(which are the components of a natural transformation, more precisely of the unity of the adjointness).

Proposition 4.2 *η_D is an injective distributive lattice morphism for every D.*

Proof. We recall that η_D is so defined

$$\eta_D(a) = \{\mathbf{p} \,|\, a \in \mathbf{p}\}.$$

If $a \not\leq b$, by the extension/exclusion Lemma, there is $\mathbf{p} \in D^*$ such that $a \in \mathbf{p}$ and $b \notin \mathbf{p}$, that is we have $\eta_D(a) \not\subseteq \eta_D(b)$. $\square$

Notice that the above Proposition (as the extension/exclusion Lemma) depends on choice axiom.

Next, we characterize a subcategory of $\mathbf{D}$ which is dual to $\mathbf{P}$ via $(-)^*$. An element a of a distributive lattice D is said to be *Join-irreducible* iff for every family $\{b_i\}_{i\in I}$ of elements from D such that $\bigvee_{i\in I} b_i$ exists in D, we have that

$$a \leq \bigvee_i b_i \quad \Rightarrow \exists i \in I \text{ s.t. } a \leq b_i$$

(notice that this implies $a \neq \bot$, taking as I the empty set of indices). We let $J(D)$ to be the poset of Join-irreducible elements of D. A distributive lattice D is said to be *J-generated* iff for every $c \in D$ we have that

$$c \;=\; \bigvee_{a\in J(D), a\leq c} a.$$

Let $\mathbf{D}_{cJg}$ be the subcategory of $\mathbf{D}$ formed by complete and J-generated distributive lattices with complete morphisms (a morphism is complete iff it preserves all Joins and Meets).

Clearly, for every poset P, we have that P^* is complete and J-generated (Join-irreducible elements are just *cones*, i.e. sieves of P of the kind $\downarrow p = \{q \in P \,|\, q \leq p\}$ for some $p \in P$). Moreover, for an order-preserving map $\mu : Q \longrightarrow P$, we have that $\mu^* : P^* \longrightarrow Q^*$ is a complete morphism. We can so restrict the functor $(-)^*$ to $\mathbf{D}_{cJg}$ in the codomain (we shall call this restriction $(-)^*$ again).

Proposition 4.3 *The functor*

$$(-)^* : \mathbf{P}^{op} \longrightarrow \mathbf{D}_{cJg}$$

establishes an equivalence of categories.

Proof. For simplicity, we prove that $(-)^*$ is full, faithful and essentially surjective (leaving the reader to determine explicitly, if he likes so, the appropriate functor in the opposite direction).

Let $f : P^* \longrightarrow Q^*$ be a complete morphism; as such, it has a left adjoint $\exists_f : Q^* \longrightarrow P^*$. $\exists_f$ maps Join-irreducible elemets to Join-irreducible elements, because if $a \in Q^*$ happens to be Join-irreducible, we have for every family $\{b_i\}_i$ from P^*:

$$\begin{array}{c} \exists_f(a) \leq \bigvee_i b_i \\ \hline a \leq f(\bigvee_i b_i) \\ \hline a \leq \bigvee_i f(b_i) \\ \hline \exists i \;\; a \leq f(b_i) \\ \hline \exists i \;\; \exists_f(a) \leq b_i. \end{array}$$

Given that Join-irreducible elements are just cones, we have that for every $q \in Q$ there is $p \in P$ such that $\exists_f(\downarrow q) = \downarrow p$ (such p is indeed unique because $\downarrow p = \downarrow p'$ implies $p = p'$ by antisymmetry). Thus we can define $\mu : Q \longrightarrow P$ in such a way that $\exists_f(\downarrow q) = \downarrow \mu(q)$. This is order-preserving, as $q \leq q'$ implies $\exists_f(\downarrow q) \subseteq \exists_f(\downarrow q')$, thus $\downarrow \mu(q) \subseteq \downarrow \mu(q')$ and finally $\mu(q) \leq \mu(q')$. Moreover $\mu^* = f$, because for $q \in Q$ and $a \in P^*$ we have

$$\begin{array}{c} q \in \mu^*(a) \\ \hline \mu(q) \in a \\ \hline \downarrow \mu(q) \subseteq a \\ \hline \exists_f(\downarrow q) \subseteq a \\ \hline \downarrow q \subseteq f(a) \\ \hline q \in f(a). \end{array}$$

This proves that $(-)^*$ is full. For faithfulness, consider $\mu_1 : Q \longrightarrow P$ and $\mu_2 : Q \longrightarrow P$ such that $\mu_1^* = \mu_2^*$. We have for $q \in Q$

$$\frac{\frac{\frac{\frac{\frac{\mu_1(q) \leq \mu_2(q)}{\mu_1(q) \in \downarrow \mu_2(q)}}{q \in \mu_1^*(\downarrow \mu_2(q))}}{q \in \mu_2^*(\downarrow \mu_2(q))}}{\mu_2(q) \in \downarrow \mu_2(q)}}{\mu_2(q) \leq \mu_2(q)}$$

which holds trivially. Similarly, we have $\mu_2(q) \leq \mu_1(q)$, hence $\mu_1 = \mu_2$.

Essential surjectivity is reduced to the fact that if D is complete and J-generated, then $D \simeq J(D)^*$. Indeed we have maps

$$\alpha : D \longrightarrow J(D)^* \quad \text{and} \quad \beta : J(D)^* \longrightarrow D$$

given by (for $a \in D$ and $c \in J(D)^*$):

$$\alpha(a) = \{b \in J(D) \,|\, b \leq a\} \quad \text{and} \quad \beta(c) = \bigvee_{b \in c} b.$$

α and β are both order-preserving, so in order to prove that they establish an isomorphism of complete lattices, it is sufficient to observe that they are inverse each other. In fact, for $a \in D$, we have (thanks to the fact that D is J-generated)

$$\beta(\alpha(a)) = \bigvee \{b \in J(D) \,|\, b \leq a\} = a$$

and for $c \in J(D)^*$ we have

$$\alpha(\beta(c)) = \{d \in J(D) \,|\, d \leq \bigvee_{b \in c} b\} = \{d \in J(D) \,|\, \exists b \in c \;\; d \leq b\} = c.$$

This completes the proof of the Proposition. $\square$

By restricting $(-)^*$ to finite posets, we get the following finite duality Theorem:

Theorem 4.4 *The category of finite posets and order-preserving maps is dual to the category of finite distributive lattices.*

Proof. Clearly every finite distributive lattice D is complete, hence it is sufficient to show that it is also J-generated. This is proved by well-founded induction on the strict part $<$ of the partial order relation associated with D. Let $a \in D$; if a is itself Join-irreducible, it is clearly the Join of all Join-irreducible elements below it. Otherwise, we have a family $\{b_i\}_i$ (necessarily finite) such that $a \leq \bigvee_i b_i$ but $a \not\leq b_i$ for every i. Taking meet with a and applying distributivity, we can suppose that $b_i \leq a$ holds for every i (thus, in particular, $a = \bigvee_i b_i$). Hence we have $b_i < a$ for all i and so, by induction hypothesis, there are $c_{ij} \in J(D)$ such that $b_i = \bigvee_j c_{ij}$; we conclude that $a = \bigvee_{ij} c_{ij}$, as wanted. $\square$

Notice that in this Section we gave the same name $(-)^*$ to three different kinds of functors, namely to the dual space functor

$$\mathbf{D} \longrightarrow \mathbf{P}^{op},$$

to its adjoint

$$\mathbf{P}^{op} \longrightarrow \mathbf{D}$$

and to the restriction of the latter in the codomain

$$\mathbf{P}^{op} \longrightarrow \mathbf{D}_{cJg}.$$

We believe that this apparent notational confusion is harmless, because the context always clarifies which functor we are currently talking about.

5 Representation Theory

We now analyze what happens when passing from $\mathbf{D}$ to the category $\mathbf{H}$ of Heyting algebras.

Theorem 5.1 *For every Heyting algebra A, the distributive lattices injective morphism*

$$\eta_A : A \longrightarrow A^{**}$$

is a Heyting algebras morphism.

Proof. First notice that any distributive lattices morphism $f : B \longrightarrow C$ among Heyting algebras semi-preserves implications, namely we always have that the inequality

$$f(a \to b) \leq f(a) \to f(b)$$

holds for all $a, b \in B$. Thus in our case we only need to show that

$$\eta_A(a) \to \eta_A(b) \subseteq \eta_A(a \to b)$$

holds for all $a, b \in A$. In other words, given $\mathbf{p} \in A^*$ such that $a \to b \notin \mathbf{p}$, we have to find $\mathbf{q} \in A^*$ such that $\mathbf{p} \subseteq \mathbf{q}$, $a \in \mathbf{q}$ and $b \notin \mathbf{q}$. By the extension/exclusion Lemma it is sufficient to observe that

$$\mathbf{p} \cup \{a\} \not\leq \{b\}$$

(in fact, if $c \wedge a \leq b$ holds for some $c \in \mathbf{p}$, then we have $c \leq a \to b$, yieldying $a \to b \in \mathbf{p}$, contradiction). $\square$

In terms of Lindenbaum algebras (i.e. giving A a presentation of the kind $\mathcal{F}(X)/\Gamma$), we have shown that there is a Kripke model for every theory Γ which is *generic* for Γ (in the sense that if $\Gamma \not\vdash \varphi$, then there is a point of the model not forcing φ - this corresponds to the injectivity of η_A). We really proved a little more; if Δ is a set of formulas closed under disjunctions and such that $\Gamma \not\vdash \varphi$ holds for all $\varphi \in \Delta$, then there is a point in the model for Γ we just built which does not force all formulas in Δ simoultaneously (to see it, just apply the extension/exclusion Lemma once again). In this form, what we actually proved is called in the literature a *strong completeness theorem* with respect to Kripke semantics (for intuitionistic logic).

Let us now consider the functor

$$(-)^* : \mathbf{P}^{op} \longrightarrow \mathbf{D}$$

from the previous Section. Given $\mu : Q \longrightarrow P$ in $\mathbf{P}$, when does it happens that $\mu^* : P^* \longrightarrow Q^*$ preserves implications?

Proposition 5.2 *For μ as above, we have that μ^* is a morphism in $\mathbf{H}$ iff μ is open,*[11] *that is iff it satisfies the following condition, for all $q \in Q, p \in P$:*

$$p \leq \mu(q) \quad \Rightarrow \quad \exists q' \leq q \text{ s.t. } \mu(q') = p.$$

[11]The choice of the name is due to the fact that if we consider posets as topological spaces having sieves as open sets, then open maps (in the topological sense) between them are just open maps in our sense.

Proof. Let us first observe that any $a \in P^*$ can be represented as

$$a = \bigcup_{p \in a} \downarrow p$$

as well as

$$a = \bigcap_{p \notin a} p^c$$

(where $p^c = \{q \in P \mid p \not\leq q\}$). Moreover, implication in Heyting algebras satisfies the following identities

$$(\bigvee_i c_i) \to d = \bigwedge_i (c_i \to d)$$

$$c \to (\bigwedge_i d_i) = \bigwedge_i (c \to d_i)$$

whenever the mentioned Joins and Meets exist (and in our case they surely exist as we are dealing with complete Heyting algebras). Consequently, μ^* preserves implications iff the following identity holds for all $p_1, p_2 \in P$

$$\mu^*(\downarrow p_1) \to \mu^*(p_2^c) \subseteq \mu^*(\downarrow p_1 \to p_2^c).$$

This means that μ^* preserves implications iff for all $p_1, p_2 \in P, q \in Q$ we have that

- if there is $p \leq \mu(q)$ such that $p \leq p_1$ and $p_2 \leq p$, then there is also $q' \leq q$ such that $\mu(q') \leq p_1$ and $p_2 \leq \mu(q')$.

This condition is certainly true in case μ is open and it actually implies openness of μ (taking $p = p_1 = p_2$). $\square$

If we call **OP** the category of posets and open maps and $\mathbf{H}_{cJg}$ the category of complete J-generated Heyting algebras and complete morphisms, we immediately get the following refinements of Proposition 4.3 and Theorem 4.4:

Theorem 5.3 **OP** *is dual to* $\mathbf{H}_{cJg}$*; moreover the category of finite Heyting algebras is dual to the category of finite posets and open maps.* $\square$

Let us now investigate the dual space functor $(-)^* : \mathbf{D} \longrightarrow \mathbf{P}^{op}$ and prove that it restricts to a functor from **H** into $\mathbf{OP}^{op}$:

Proposition 5.4 *If $f : A \longrightarrow B$ is a Heyting algebras morphism, then $f^* : B^* \longrightarrow A^*$ is an open map.*

Proof. Suppose that $f^*(\mathbf{p}) \subseteq \mathbf{q}$ holds, for $\mathbf{p} \in B^*$ and $\mathbf{q} \in A^*$. We must find $\mathbf{p}' \supseteq \mathbf{p}$ such that $f^*(\mathbf{p}') = \mathbf{q}$. This is achieved by applying extension/exclusion Lemma to $\mathbf{p} \cup \{f(a) \,|\, a \in \mathbf{q}\}$ and $\{f(b) \,|\, b \notin \mathbf{q}\}$. In fact, we cannot have

$$c \wedge f(a) \leq f(b)$$

for $c \in \mathbf{p}$, $a \in \mathbf{q}$, $b \notin \mathbf{q}$, otherwise we would get $c \leq f(a \to b)$, i.e. $a \to b \in f^*(\mathbf{p}) \subseteq \mathbf{q}$, contradiction. $\square$

To sum up: *we have two functors*

$$(-)^* : \mathbf{H} \longrightarrow \mathbf{OP}^{op}$$

$$(-)^* : \mathbf{OP}^{op} \longrightarrow \mathbf{H}$$

obtained from the restriction of the corresponding functors for distributive lattices and posets. They are *not* adjoint anymore, however (from Theorem 5.1 and Propositions 5.2 and 5.4) we know that

$$\eta : Id \longrightarrow (-)^{**}$$

is still a natural transformation,[12] which means that the following commutative squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ A^{**} \arrow[r, "f^{**}"'] & B^{**} \end{tikzcd}$$

entierely lie in $\mathbf{H}$. These data should be kept in mind for the subsequent Sections.

We notice that, even if we have lost adjointness, we have now a better correspondence between injective/surjective morphisms. It can be shown

[12]On the contrary, it is easily seen that the components $P \longrightarrow P^{**}$ of the counity may not be open maps.

indeed (and we leave it as an exercice to the reader) that any morphism $f : A \longrightarrow B$ in $\mathbf{H}$ is injective (resp. surjective) iff $f^* : B^* \longrightarrow A^*$ in $\mathbf{OP}$ is surjective (resp. injective); moreover $\mu : Q \longrightarrow P$ in $\mathbf{OP}$ is injective (resp. surjective) iff $\mu^* : P^* \longrightarrow Q^*$ in $\mathbf{H}$ is surjective (resp. injective). Not all these properties, on the other hand, hold in the original case of $\mathbf{D}$ and $\mathbf{P}$. Of course, injectivity and surjectivity are not categorical conditions (at least by themselves), however there is a categorical point not far from all that: we shall see in next Section that monos and epis are all regular in $\mathbf{H}$ (a fact which is easily seen to fail in $\mathbf{D}$).

We conclude this Section by proving a *finite model property* of IPC with respect to Kripke semantics: this property says that whenever a formula is not provable, it fails in a finite Kripke model. In view of the above Theorem 5.3 (saying in particular that all finite Heyting algebras are of the kind P^* for a finite poset P), finite model property follows immediately provided we show that any free Heyting algebra embeds into a product of finite Heyting algebras. We first need a Lemma:

Lemma 5.5 *Let A, B be Heyting algebras such that A is a sublattice of B. For $a, b \in A$, if the implication of a and b (taken in B) is equal to an element $c \in A$, then c is also the implication of a and b (taken in A).*

Proof. This is trivial: 'c is the implication of a and b' is equivalent to the statement

$$\forall d \; (a \wedge d \leq b \;\Leftrightarrow\; d \leq c).$$

If this statement is true with the quantifier 'for all d' ranging over B, then it is certainly true with the quantifier 'for all d' ranging only over $A \subseteq B$. $\square$

Theorem 5.6 *Every free Heyting algebra $\mathcal{F}(X)$ embeds into a product of finite Heyting algebras.*

Proof. We need to prove that for every formula $\varphi(X)$ such that in $\mathcal{F}(X)$ we have $[\varphi] \neq \top$, there exists a morphism $f : \mathcal{F}(X) \longrightarrow A$ such that A is finite and $f([\varphi]) \neq \top$. Let A be the distributive sublattice of $\mathcal{F}(X)$ generated by elements of the kind $[\psi(X)]$, where ψ is a subformula of φ. This is certainly finite and consequently it is a Heyting algebra (we saw why in Section 1). Consider now the function

$$I : X \longrightarrow \mathcal{U}(A)$$

associating $[x]$ with any $x \in X$ occurring in φ (we leave $I(x)$ to be arbitrary for propositional letters not occurring in φ). Using notation from the proof of Theorem 2.2 and keeping in mind the previous Lemma, we have that $I^e(\psi) = [\psi]$ holds for all subformulas ψ of φ, hence in particular $I^e(\varphi) \neq \top$. Thus $[I^e] : \mathcal{F}(X) \longrightarrow A$ is the desired morphism. $\square$

Notice that the statement of last Theorem easily extends to all finitely presented Heyting algebras (but not to arbitrary Heyting algebras).

6 Coregular Factorizations

In this and in the next Section we study the opposite $\mathbf{H}^{op}$ of the category of Heyting algebras. We prefer not to develop a heavy duality theory in the style of Priestley duality for distributive lattices, however we should take in mind that reversing direction of arrows is not a purely formal operation: to get the right intuition, we shall think now of $\mathbf{H}^{op}$ *as a category of spaces.* For spaces, it makes sense to speak of good factorization properties, similar to the image factorization we have in **Set**. The fact that image factorization is good in sets is expressed by the fact that **Set** is a regular category. We recall that a category $\mathbf{C}$ is *regular* iff

- $\mathbf{C}$ has finite limits.
- Every kernel pair[13] in $\mathbf{C}$ has a coequalizer.

[13]The parallel pair of arrows

$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \end{tikzcd}$$

is a *kernel pair* iff there is a morphism f such that the diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "h"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback.

- Regular epis (i.e. arrows that happen to be coequalizers of some pair of parallel arrows) in **C** are pullback stable, i.e. pullback of a regular epi along any morphism is a regular epi.

It can be shown that in a regular category every arrow $A \xrightarrow{f} B$ factors as a regular epi followed by a mono; this factorization (called the *regular* or the *image factorization* of f)

$$A \xrightarrow{q} Q \xrightarrow{m} B$$

is obtained by taking as q the coequalizer of the kernel pair of f (then the arrow m given by the universal property of coequalizers can be proved to be mono by using the axioms of regular category, see [Bo]).

H trivially is regular, this is not a very interesting fact. However, as **H** has colimits and limits, we can reproduce the kernel pair/quotient procedure in $\mathbf{H}^{op}$ (thus getting what we call the *coregular factorization* of a morphism). Do we still get a regular category? Surprisingly, this fact is strictly related to *purely logical properties* of intuitionistic logic, as we shall see.

Let $\Gamma(X,Y)$ be a set of formulas and let $\varphi(X,Y)$ be a single formula. *Beth's Definability Theorem* says that whenever we have

$$\Gamma(X,Y) \cup \Gamma(X,Y') \vdash \varphi(X,Y) \leftrightarrow \varphi(X,Y') \tag{2}$$

we also have that

$$\text{there exists } \ \psi(X), \quad \text{such that} \quad \Gamma(X,Y) \vdash \varphi(X,Y) \leftrightarrow \psi(X) \tag{3}$$

(here Y' is a disjoint copy of Y).[14]

In order to find the appropriate algebraic conceptualization of Beth's Theorem, let us take any morphism $f : A \longrightarrow B$ among Heyting algebras. In order to get its coregular factorization, we first consider the pushout of f with itself

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & B +_A B \end{tikzcd}$$

[14]Strictly speaking, standard Beth's Theorem requires such property only for the case in which $Y = \{y\}$ is a singleton set and $\psi(X,Y)$ is the formula y itself. In such a weak form, Beth's Theorem corresponds algebraically to the fact that epimorphisms among finitely presented Heyting algebras are regular, i.e. onto.

and then the equalizer $E \xrightarrow{e} B$ of i_1, i_2; by the universal property of equalizers there is a unique q such that the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "q"'] & & B \\
& E \arrow[ur, "e"'] &
\end{tikzcd}$$

commutes. Let us now take a suitable presentation of the morphism $f : A \longrightarrow B$ in terms of Lindenbaum algebras associated to theories; recall from Section 2 that we can present A as $\mathcal{F}(X)/\Delta(X)$ and B as $\mathcal{F}(X,Y)/\Gamma(X,Y)$, in such a way that the morphism f corresponds to the map associating the equivalence class of each $x \in X$ with itself (more precisely, with the equivalence class of x in $B \simeq \mathcal{F}(X,Y)$). In this way, $B +_A B$ gets the presentation $\mathcal{F}(X,Y,Y')/\Gamma(X,Y) \cup \Gamma(X,Y')$ and the map i_1 (resp. i_2) associates the equivalence classes of $x \in X, y \in Y$ with the equivalence classes of x and y (resp. with the equivalence classes of x and y'). The equalizer E turns out to be the B-subalgebra of (equivalence classes of) sentences $\varphi(X,Y)$ satisfying (2); the image of f is on the other hand the B-subalgebra formed by (the equivalence classes of) sentences satisfying (3). The latter is clearly smaller, they coincide just in case Beth's Theorem holds, i.e. when the morphism q is onto. Thus Beth's Theorem holds iff *the coregular factorization* (q,e) *is nothing but the usual image (or regular) factorization in* $\mathbf{H}$.

Our aim is to prove that it is indeed so. First we need a Lemma:

Lemma 6.1 *Let $f : A \longrightarrow B$ be a Heyting algebras morphism and let b be an element not belonging to the image of f. Then there are prime filters $\mathbf{p}_b, \mathbf{q}_b \in B^*$ such that $b \in \mathbf{p}_b$, $b \notin \mathbf{q}_b$ and $f^*(\mathbf{p}_b) = f^*(\mathbf{q}_b)$.*

Proof. We need two applications of extension/exclusion Lemma. By the first application we can find $\mathbf{q}_b$ such that $b \notin \mathbf{q}_b$ and $\mathbf{q}_b \supseteq \{f(a) \,|\, b \leq f(a)\}$. Now notice that

$$\{b\} \cup \{f(a) \,|\, f(a) \in \mathbf{q}_b\} \not\leq \{f(a) \,|\, f(a) \notin \mathbf{q}_b\}$$

(otherwise there are a_1, a_2 such that $f(a_1) \in \mathbf{q}_b$, $f(a_2) \notin \mathbf{q}_b$ and $b \wedge f(a_1) \leq f(a_2)$, yeldying $b \leq f(a_1 \to a_2)$, contradiction because $f(a_1) \to f(a_2)$ cannot belong to $\mathbf{q}_b$).[15] By a second application of extension/exclusion Lemma we find $\mathbf{p}_b$ such that $b \in \mathbf{p}_b$ and $f^*(\mathbf{q}_b) = f^*(\mathbf{p}_b)$. $\square$

Lemma 6.2 *The pullback in* $\mathbf{P}$ *of an open map along any map is open (also, the pullback in* $\mathbf{P}$ *of an open surjective map along any map is open surjective).*

Proof. Easy. $\square$

Theorem 6.3 *The coregular factorization of a morphism* $f : A \longrightarrow B$ *in* $\mathbf{H}$ *coincides with its image factorization.*

Proof. Take the cokernel pair

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & B +_A B \end{tikzcd}$$

We need to prove that if b is not in the image of f, then $i_1(b) \neq i_2(b)$ (that is, b is not in the equalizer of i_1 and i_2). By the universal property of pushouts, it is sufficient to show that $g_1(b) \neq g_2(b)$, where g_1, g_2 is a pair of parallel morphisms having domain B whose composites with f are equal. Applying $(-)^*$ and taking pullbacks in the category of posets, we get a commutative square:

$$\begin{tikzcd} B^* \times_{A^*} B^* \arrow[r, "p_1"] \arrow[d, "p_2"'] & B^* \arrow[d, "f^*"] \\ B^* \arrow[r, "f^*"'] & A^* \end{tikzcd}$$

where we recall that $B^* \times_{A^*} B^*$ is the set of pairs of prime filters $(\mathbf{q_1}, \mathbf{q_2})$ such that $f^*(\mathbf{q_1}) = f^*(\mathbf{q_2})$ and p_1, p_2 are the two projections (restricted in their domains). As f^* is open, so are p_1, p_2 (by Lemma 6.2) hence we can come back to $\mathbf{H}$ and get the commutative diagram

[15] Notice that this argument does not work for distributive lattices, as it requires existence and preservation of implications.

$$\begin{tikzcd} A^{**} \arrow[r, "f^{**}"] & B^{**} \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] & (B^* \times_{A^*} B^*)^* \end{tikzcd}$$

The naturality of η ensures that the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ A^{**} \arrow[r, "f^{**}"'] & B^{**} \end{tikzcd}$$

commutes, hence the diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "\eta_B p_1^*"] \arrow[r, shift right, "\eta_B p_2^*"'] & (B^* \times_{A^*} B^*)^* \end{tikzcd}$$

commutes too. Now unravelling the definitions, we have for $i = 1, 2$ that $p_i^*(\eta_B(b)) = \{(\mathbf{q_1}, \mathbf{q_2}) \in B^* \times_{A^*} B^* \mid b \in \mathbf{q_i}\}$, consequently $p_1^*(\eta_B(b)) \neq p_2^*(\eta_B(b))$ by Lemma 6.1. $\square$

The following Corollary is an immediate consequence:

Corollary 6.4 *Monomorphisms are all regular and epimorphisms are also all regular in* $\mathbf{H}$.

Proof. For a monomorphism f, just observe that the first component of the coregular factorization of f is both onto (by the previous Theorem) and mono (as a first component of a mono), hence it is an isomorphism. Thus f equalizes its cokernel pair.

For an epimorphism f, the second component of its regular factorization is a regular mono (we just saw why) and also an epi (as second component of an epi), hence it is an isomorphism. Thus f coincides (up to an isomorphism) with the first component of its regular factorization. $\square$

7 Regularity of $\mathbf{H}^{op}$

We have seen in the previous Section that every epimorphism in $\mathbf{H}^{op}$ is regular. Thus, in order to establish that $\mathbf{H}^{op}$ is a regular category, we need to show that epis are pullback stable. We first have an easy Lemma:

Lemma 7.1 *If a morphism $m : A \longrightarrow B$ in $\mathbf{D}$ is injective, then m^* is a surjective map. If an order-preserving map $\mu : Q \longrightarrow P$ in $\mathbf{P}$ is surjective, then μ^* is injective.*

Proof. The second part of the claim is easy, being a purely set-theoretical fact. For the first part, given a prime filter $\mathbf{p} \in A^*$, it is sufficient to observe that

$$\{f(a) \,|\, a \in \mathbf{p}\} \not\leq \{f(b) \,|\, b \notin \mathbf{p}\}$$

and to apply extension/exclusion Lemma in order to get $\mathbf{q}$ such that $f^*(\mathbf{q}) = \mathbf{p}$. $\square$

Theorem 7.2 *The pushout of a monomorphism in $\mathbf{H}$ along any morphism is again a monomorphism.*

Proof. By the universal property of pushouts and by the fact that the first component of a mono is mono, it is enough, given a monomorphism $m : A \longrightarrow B$ and an arbitrary morphism $f : A \longrightarrow C$, to find a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & C \arrow[d, "m'"] \\ B \arrow[r, "f'"'] & D \end{tikzcd}$$

such that m' is a monomorphism. Taking pullback in $\mathbf{P}$ of m^* along f^*, we get a commutative square

$$\begin{tikzcd}
B^* \times_{A^*} C^* \arrow[r, "p"] \arrow[d, "q"'] & B^* \arrow[d, "m^*"] \\
C^* \arrow[r, "f^*"'] & A^*
\end{tikzcd}$$

entierely formed by open maps in which q is also surjective (by Lemmas 7.1 and 6.2). Dualizing again, we get a commutative square in $\mathbf{H}$

$$\begin{tikzcd}
A^{**} \arrow[r, "f^{**}"] \arrow[d, "m^{**}"'] & C^{**} \arrow[d, "q^*"] \\
B^{**} \arrow[r, "p^*"'] & (B^* \times_{A^*} C^*)^*
\end{tikzcd}$$

in which q^* is mono (by Lemma 7.1 again). It is now sufficient to glue the above square with the following commutative squares (provided by the naturality of η)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] & C \arrow[d, "\eta_C"] \\
A^{**} \arrow[r, "f^{**}"'] & B^{**}
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\
A^{**} \arrow[r, "m^{**}"'] & B^{**}
\end{tikzcd}$$

in order to obtain the square

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A \arrow[r, "f"] \arrow[d, "m"'] & C \arrow[d, "\eta_C q^*"] \\
B \arrow[r, "\eta_B p^*"'] & (B^* \times_{A^*} C^*)^*
\end{tikzcd}$$

in which $\eta_C q^*$ is mono, as required. $\Box$

We now ask for the logical meaning of the previous Theorem. This is the well-known *Craig's Interpolation Theorem*, saying that for formulas $\varphi(X, Y), \psi(Y, Z)$, we have

- *if* $\varphi(X, Y) \vdash \psi(Y, Z)$, *then there is a formula* $\theta(Y)$ *such that (*$\varphi(X, Y) \vdash \theta(Y)$ *and* $\theta(Y) \vdash \psi(Y, Z)$*).*

Craig's Theorem is proved from stability of monomorphisms under pushouts as follows:

Theorem 7.3 *Let*

$$\begin{tikzcd}
A_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \arrow[d, "g_1"] \\
A_2 \arrow[r, "g_2"'] & A
\end{tikzcd}$$

be a pushout square in $\mathbf{H}$. *If, for* $a_1 \in A_1, a_2 \in A_2$, *we have* $g_1(a_1) \leq g_2(a_2)$, *then there exists* $a_0 \in A_0$ *such that* $a_1 \leq f_1(a_0)$ *and* $f_2(a_0) \leq a_2$.

Proof. We first observe that Craig's Theorem follows from this Theorem, by considering the special case concerning the pushouts squares obtained by applying the free algebra functor to the set-theoretic pushouts of the kind

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & X+Y \arrow[d] \\
Y+Z \arrow[r] & X+Y+Z
\end{tikzcd}$$

(recall that left adjoints preserve colimits).

In order to prove the Theorem, let us introduce filters F_0 in A_0, F_1 in A_1, F_2 in A_2 as follows:

- F_1 is the filter generated by a_1;
- F_0 is the filter $\{b_0 \mid a_1 \leq f_1(b_0)\}$;
- F_2 is the filter $\{b_2 \mid \exists a_0 \in F_0 \text{ s.t. } f_2(a_0) \leq b_2\}$.

Let A'_0, A'_1, A'_2 be the quotient algebras A_0/F_0, A_1/F_1, A_2/F_2 and let

$$q_0 : A_0 \longrightarrow A'_0, \quad q_1 : A_1 \longrightarrow A'_1, \quad q_2 : A_2 \longrightarrow A'_2$$

be the canonical quotient maps. By the universal property of quotients, we can form commutative squares

$$\begin{tikzcd} A_0 \arrow[r, "f_1"] \arrow[d, "q_0"'] & A_1 \arrow[d, "q_1"] \\ A'_0 \arrow[r, "\bar{f}_1"'] & A'_1 \end{tikzcd}$$

$$\begin{tikzcd} A_0 \arrow[r, "f_2"] \arrow[d, "q_0"'] & A_2 \arrow[d, "q_2"] \\ A'_0 \arrow[r, "\bar{f}_2"'] & A'_2 \end{tikzcd}$$

Notice also that $\bar{f}_1$ is mono, as F_0 is precisely the kernel of $f_1 q_1$. This means that, taking pushout of $\bar{f}_1$ along $\bar{f}_2$

$$\begin{tikzcd} A'_0 \arrow[r, "\bar{f}_2"] \arrow[d, "\bar{f}_1"'] & A'_2 \arrow[d, "k"] \\ A'_1 \arrow[r, "h"'] & B \end{tikzcd}$$

we have that k is mono, by Theorem 7.2. As we have

$$f_1 q_1 h = q_0 \bar{f}_1 h = q_0 \bar{f}_2 k = f_2 q_2 k$$

by the universal property of pushouts, there is $s : A \longrightarrow B$ such that $g_1 s = q_1 h$ and $g_2 s = q_2 k$. Hence from $s(g_1(a_1)) \leq s(g_2(a_2))$ and from $h(q_1(a_1)) = h(\top) = \top$, we get $\top = s(g_2(a_2)) = k(q_2(a_2))$. As k is mono, we have $\top = q_2(a_2)$, which means that $a_2 \in F_2$. From the definitions of F_0 and F_2, we realize that there exists $a_0 \in A_0$ such that $f_2(a_0) \leq a_2$ and $a_1 \leq f_1(a_0)$, as wanted. $\square$.

To finish, let us make a couple of observations. First, the condition "monos are stable under pushouts" of Theorem 7.2, is equivalent to *amalgamation property* in case we have the congruence extension property (which is our case, see Section 1): this is because we can take image factorization and compose pushouts. We recall that amalgamation property says that any pair of monomorphisms $m_1 : A_0 \longrightarrow A_1$ and $m_2 : A_0 \longrightarrow A_2$ fills into a commutative square

$$\begin{tikzcd} A_0 \arrow[r, "m_1"] \arrow[d, "m_2"'] & A_1 \arrow[d] \\ A_2 \arrow[r] & A \end{tikzcd}$$

entierely formed by monomorphisms; notice that, as pushouts always exist for our algebras and as, once again, first component of a monomorphism is a monomorphism, amalgamation property simply says that the pushout of a monomorphism along a monomorphism is a monomorphism.

Second, we have seen that amalgamation property entails Craig's Interpolation Theorem. The proof we gave is completely general, e.g. it works by replacing **H** by any subvariety of its. Conversely, if we know that Craig's Theorem holds (e.g. in any variety of Heyting algebras), then *amalgamation property follows.* To see this, given 'inclusions' monomorphisms

$$A_0 \hookrightarrow A_1 \qquad A_0 \hookrightarrow A_2$$

give them the following presentations in terms of Lindenbaum algebras (see Section 2)

$$A_0 \hookrightarrow A_1 \quad \simeq \quad \mathcal{F}(X_0)/\Gamma_0 \longrightarrow \mathcal{F}(X_0, X_1)/\Gamma_1$$

$$A_0 \hookrightarrow A_2 \quad \simeq \quad \mathcal{F}(X_0)/\Gamma_0 \longrightarrow \mathcal{F}(X_0, X_2)/\Gamma_2$$

(where the monomorphisms to the right send the equivalence class of a formula $\psi(X_0)$ into the equivalence class of ψ itself and where Γ_i is conservative over Γ_0). The pushout of the two inclusions $A_0 \hookrightarrow A_1$ and $A_0 \hookrightarrow A_2$ can thus be presented as

$$\mathcal{F}(X_0, X_1, X_2)/\Gamma_1 \cup \Gamma_2.$$

To see that e.g. $\mathcal{F}(X_0, X_1)/\Gamma_1 \longrightarrow \mathcal{F}(X_0, X_1, X_2)/\Gamma_1 \cup \Gamma_2$ is mono, notice that if for $\varphi(X_0, X_1)$ there are (for $i = 1, 2$) formulas $\psi_i(X_0, X_i)$ such that $\vdash_{\Gamma_i} \psi_i(X_0, X_i)$ and such that

$$\psi_1(X_0, X_1) \wedge \psi_2(X_0, X_2) \vdash \varphi(X_0, X_1),$$

then the statement of the Interpolation Theorem (together with conservativity of Γ_2 over Γ_0) applied to

$$\psi_2(X_0, X_2) \vdash \psi_1(X_0, X_1) \to \varphi(X_0, X_1)$$

shows that $\vdash_{\Gamma_1} \varphi$.

8 Final Remarks and Further Readings

We make here some comments about the content of the present notes, trying to give some relevant credits and to suggest at the same time some further readings (sometimes going in very different directions with respect to the topics we covered in the previous Sections).

The material presented in Sections 1 and 2 is quite standard (see some classical textbook like [RS], [Ra] or also [BD], from the purely lattice-theoretic side); it should be noticed however that we followed the categorical logic point of view rather than the algebraic logic tradition. According to the former, theories are seen as (small) categories with structure, models are structure preserving functors into some special (large) categories, morphisms among models are natural transformations, etc. (see e.g. the textbook [MR1]). In the propositional case, most things simplify, however we followed the above schema when introducing Lindenbaum algebras and models. For a discussion about the invariant point of view in algebra and logic, see [La].

The glueing construction is due to P. Freyd; it should be noticed that this is a quite powerful technique, going much beyond propositional logic (see [LS]).

Sections 4 and 5 also cover more or less standard topics; we followed the presentation of [GM1], where basic adjunction is emphasized and extended to modal logic through the notion of weak (or continuous) morphism, drawn from topology (where inverse image along a continuous map only semi-preserves the interior operation). Representation Theorem 5.1 through the canonical model method (i.e. through Stone embedding η) extends to many subvarieties of Heyting algebras: a syntactic sufficient condition is given in [GM2], where Sahlqvist-style results [Sa], [SV2] are extended to intermediate logics by a new "constructive" technique. It should be noticed that tools from Sections 4 and 5 can also be used to get negative results about existence of a Representation Theorem for subvarieties of Heyting algebras (this is done in [GMi]).

Priestley Duality (adapting to distributive lattices the standard Stone duality for Boolean algebras) is explained in [Pr], its extension to Heyting algebras is common folklore; duality for modal logic is thoroughly investigated in [SV1]. For a deep duality theory in the context of first order classical logic, see [M1].

Finite model property immediately entails solvability of word problem for Heyting algebras, however the algorithm it suggests is impracticable. Good results can be achieved for instance by tableaux methods (see e.g. [MMO], where a duplication-free calculus is introduced). It should be noticed however that word problem for Heyting algebras is highly complex, being PSPACE-complete [St]; recent improvements on space complexity can be found in [Hu].

Finite model property can be used to refine the Representation Theorem 5.1 in the special case of finitely generated free algebras; this gives rise to the so-called effective (or definable) embeddings investigated in [Be] and in various papers from the russian school. The related construction have been conceptualized from the categorical point of view (in a general context) in [Gh2]. There is also a direct description of finitely generated free Heyting algebras in [Ur] (see [Gh1] for some simplifications and for a proof of the fact that the opposite lattice of a finitely generated free Heyting algebra is also a Heyting algebra). For free Brouwverian semilattices, see [Kö].

The strong Beth property considered in Section 6 and its categorical conceptualization are due, as far as we know, to M. Makkai. Quite recently, such a strong Beth property has been reconsidered (in an equivalent formulation) by L.L. Maksimova in [Ma4], where the few subvarieies of Heyting algebras enjoying it are determined. It sould be noticed that, on the contrary, standard Beth property holds in all intermediate logics (by a nice trick suggested by G. Kreisel in [K]). The topics we studied in Section 6, once suitably reformulated in the appropriate two-dimensional context, give rise in the case of first order logic to the so-called descent theory [M2], [Z] (both papers originated from the important descent theorem for toposes of [JT]). Relevant problems in the area concerning Heyting pretoposes (i.e. first order intuitionistic theories) are still open. Conceptual completeness for Heyting pretoposes is investigated in [Pi3].

Equivalence of amalgamability and interpolation is first established in the classical paper [Ma1], where it is proved the remarkable fact that there are only 8 subvarities of Heyting algebras enjoying it. The key point in the amalgamation proof for **H** we gave in Section 7 is the same as Maksimova's one; the general context, however, is rather different. There is also another quite interesting proof in [Pi1], which is constructive in the sense that it works in any topos. For an extension of these proofs to first order intuitionistic logic, see [Pi2], [M3]. Of course, interpolation (as well as other relevant metamathematical properties like disjunction property from Section 3) can be obtained also by proof-theoretic techniques like in [Gi]. For deep results on interpolation, amalgamation and superamalgamation in modal logic, see [Ma2], [Ma3]. Notice that interpolation implies the strong Beth property we considered in Section 6 (but is not equivalent to it, see [Ma4]) in all intermediate logics, as the reader may see by himself ; nevertheless, the two properties have a different algebraic status, so we prefered to deal with them separately.

There are basically two directions in which the material contained in these notes can be developed. There is first the *non-classical logics tradition*, where intermediate and also modal logics are investigated and classified with respect to the various metamathematical properties we met. The recommended reading for these problems is the recent book [CZ], covering large part of the relevant literature on the subject (an alternative recent reference for modal logic is [Kr]).

The second direction is the *topos theoretic tradition.* In [Jo] the theory of locales (i.e. of complete Heyting algebras) is developed and compared with standard theory of topological spaces. For the relevance of Heyting algebras to topos theory, we recommend the comprehensive graduate level textbook [CWL]. It should be expecially remarked (because it is not widely appreciated) that also modal $S4$-logic has a very nice interpretation in terms of geometric morphisms among toposes, see [MR2]. Paper [RZ] contains a first analysis of these aspects in propositional logic.

Let us finally mention that research concerning propositional intuitionistic logic itself is far from being concluded. We just mention two recent developements in which the author of the present notes is involved. Interpolation Theorem can be strenghtened to *uniform interpolation*, saying that there are the greatest and the smallest interpolants among formulas $\varphi(X, Y)$ and $\psi(Y, Z)$ such that $\varphi \vdash \psi$. This turns out to be an immediate consequence of a surprising theorem by Pitts saying that second order propositional intuitionistic logic can be interpreted into ordinary intuitionistic propositional logic. Pitts' theorem was first established by proof-theoretic techniques in [Pi4] and later on obtained through semantic methods in [GZ1] and [Vi]. In categorical terms, Pitts' theorem says that the opposite of the category of finitely presented Heyting algebras is a Heyting category; there is also a model-theoretic version of its, saying that the first order theory of Heyting algebras *admits a model completion* [GZ2]. Such topics are all addressed (and extended to modal logic, where however Pitts' theorem often fails [GZ2]) in the book [GZ4], where still open problems on the subject are mentioned.

The second recent developement concerns *unification theory* for Heyting (and also modal) algebras [Gh3], [Gh4]. This topic is suggested by automated deduction, however it has both logical relevance (for it gives a new solution to the problem of effectiveness of admissible inference rules [Ry1], [Ry2] and it solves some problems related to deJongh exactness [dJ] of formulas) and algebraic consequences (for it extends the well-known characterization [BH] of projective Heyting algebras from the finite to the finitely presented case). Unification theory suggests new algorithmic questions (first approached in [Gh5]), where tableaux techniques are mixed with classical resolution.

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