Graph Theory

Basics

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Given a set S and any $k \in \{2, \dots, |S|\}$, $[S]^k$ is the collection of all k-element subsets of S. So, $\big[\{1,2,3\}\big]^2 = \big\{\{1,2\},\{1,3\},\{2,3\}\big\}$.

A graph $G = (V, E)$ has a finite **vertex set** V and a finite **edge set** $E \subseteq [V]^2$. We use i, j, u, v, w, x to denote vertices in V. The number $|V|$ of vertices is the **order** of G. The graph of order zero is empty. A graph is **non-trivial** if it has at least one edge. An element of E is denoted by e or (i, j). If $(i, j) \in E$, then i, j denote the endpoints of the edge (the order does not matter). Note that G has no self-loops (i, i) because $(i, i) \notin [V]^2$. Moreover, there can be at most one edge in G between any two pair of vertices. Such graphs are often called **simple**. A vertex i is **incident** with an edge e if $e = (i, j)$ for some $j \in V \setminus \{i\}$. Two vertices i, j are **adjacent** if $(i, j) \in E$. If $E \equiv [V]^2$, then G is the complete graph (or **clique**) on n vertices, denoted by K_n.

If $G' = (V', E')$ is such that $V' \subseteq V$ and $E' \subseteq [V']^2 \cap E$, then G' is a **subgraph** of G. If a subgraph G' is such that $E' \equiv [V']^2 \cap E$, then G' is called the subgraph induced by V' and denoted by $G[V']$.

The complement of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ such that $e \in E$ if and only if $e \notin \overline{E}$ for all $e \in [V]^2$.

Degrees. The **neighborhood** of a vertex v of G is the set $N(v)$ of vertices that are adjacent to v. The **degree** $d(v)$ of v is the cardinality of $N(v)$. A vertex with degree zero is **isolated**. The numbers $\delta(G) = \min\{d(v) : v \in V\}$ and $\Delta(G) = \max\{d(v) : v \in V\}$ are the minimum and maximum degree of G. If $\delta(G) = \Delta(G) = k$ then G is k**-regular**.

The **average degree** of G is

$$d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$$

and we obviously have $\delta(G) \le d(G) \le \Delta(G)$. A related quantity is the **edge density** $\varepsilon(G) = |E|/|V|$. Note that

$$|E| = \frac{1}{2} \sum_{v \in V} d(v) = \frac{1}{2} d(G)|V|$$

implying $\varepsilon(G) = d(G)/2$.

Fact 1 *The number of vertices of odd degree is always even in any graph.*

Proof. Since $|E| = \frac{1}{2}\sum_{v \in V} d(v)$ is integer, then $\sum_{v \in V} d(v)$ must be even. Therefore, the number of vertices v with odd degree $d(v)$ must be even. □

We already know that the edge density is half the average degree. Now note that the minimum degree can be larger than the edge density. For instance, in K_2 we have $\delta(G) = 1$ and $\varepsilon(G) = \frac{1}{2}$.

Fact 2 *Every G with at least one edge has an induced subgraph H such that $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$.*

PROOF. Construct a sequence of nested subgraphs $G \equiv G_0, G_1, \dots$ induced by the vertex sets $V = V_0 \supseteq V_1 \supseteq V_2 \cdots$ as follows. If V_i has a vertex v_i of degree $d(v_i) \leq \varepsilon(G_i)$ then $V_{i+1} \equiv V_i \setminus \{v_i\}$. Otherwise, stop and set $H = G_i$. If G_{i+1} is created, then

$$\varepsilon(G_{i+1}) = \frac{|E_{i+1}|}{|V_{i+1}|} = \frac{|E_i| - d(v_i)}{|V_i| - 1} \geq \frac{|E_i| - \varepsilon(G_i)}{|V_i| - 1} = \frac{|E_i|}{|V_i|} = \varepsilon(G_i)$$

When the procedure stops (say at $H \equiv G_k$ for some $k \geq 0$), then E_k is nonempty. Indeed, G_0 has at least one edge, and if $|E_{i+1}| = 0$ for some i, then all edges in E_i have v_i as endpoint. But then $d(v_i) = |E_i| > |E_i|/|V_i| = \varepsilon(G_i)$, implying that v_i is not removed. Hence, E_k is not empty and since the procedure stopped, it must be that $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$, concluding the proof. □

Paths and cycles. A **path** in $G = (V, E)$ of length $k \geq 0$ is a subgraph P_k containing $k+1$ distinct vertices $v_0, \dots, v_k \in V$ and k edges $e_1, \dots, e_k \in E$ such that $e_i = (v_{i-1}, v_i)$ for $i = 1, \dots, k$. If $k = 0$ then $P_0 = K_1$. A **cycle** C_k in G of length $k \geq 3$ is formed when a path P_{k-1} can be extended in G to include the edge $(v_{k-1}, v_0) \in E$. The length of a shortest cycle in G is the **girth** $g(G)$, while the length of a longest cycle in G is the **circumference**. A **chord** is any edge between two vertices of a cycle which is not itself an edge of the cycle.

If a graph has a large minimum degree, then it contains long paths and cycles.

Fact 3 *Every graph G with $\delta(G) \geq 2$ contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$.*

PROOF. Let $v_0, \dots, v_k$ be the vertices on any longest path P_k in G. Then $N(v_k)$ all belong to P_k (otherwise P_k is not the longest path). Therefore, $k \geq d(v_k) \geq \delta(G)$. Now let v_i the vertex of P_k with smallest index i such that $(v_i, v_k) \in E$. Then the vertices $v_i, \dots, v_k$ form a cycle of length at least $\delta(G) + 1$ because the degree of v_k is at least $\delta(G)$. □

The **distance** $d(i, j)$ betweeen two vertices i, j is the length of the shortest path between them (if no path exists between the two vertices, then their distance is infinite). The **diameter** $\operatorname{diam}(G)$ of G is the largest distance between any two vertices in G (note that the diameter can be infinite, for example when the graph has an isolated vertex). The **radius** $\operatorname{rad}(G)$ of a graph G is the smallest distance d such that there exists a vertex whose distance from any other vertex in G is at most d. Formally,

$$\operatorname{rad}(G) = \min_{i \in V} \max_{j \in V} d(i, j)$$

Clearly, $\operatorname{rad}(G) \leq \operatorname{diam}(G)$. Also, let $x \in V$ such that $d(x, v) \leq \operatorname{rad}(G)$ for all $v \in V$. Pick any two vertices $u, v \in V$ then $d(u, v) \leq d(u, x) + d(x, v) \leq 2 \operatorname{rad}(G)$. This shows that $\operatorname{diam}(G) \leq 2 \operatorname{rad}(G)$.

Girth and diameter are related as follows.

Fact 4 *Every graph G containing at least a cycle satisfies $g(G) \leq 2 \operatorname{diam}(G) + 1$.*

PROOF. Let C be the shortest cycle in G. If $g(G) \geq 2 \operatorname{diam}(G) + 2$, then C contains at least $2 \operatorname{diam}(G) + 2$ edges. Take any two vertices x, y at opposite extremes of C. Then x, y are connected by two paths in C, say P_1 and P_2, whose each length is at least $\operatorname{diam}(G) + 1$. On the other hand, the distance between x and y in G can be at most $\mathrm{diam}(G)$ by definition of diameter. Let P be a path joining x to y in G. Note that not all the edges of P can be in C (otherwise, C is not the shortest cycle in G). Then some edges of P are neither in P_1 nor in P_2 and, togheter with the shortest between P_1 and P_2, form a cycle shorter than C. Thus we have a contradiction. $\square$

Connectivity. A non-empty graph G is **connected** if any two of its vertices are linked by a path in G. A maximal connected subgraph of G is a **component** of G. Any non-empty graph corresponds to a set containing at least one component. G is **k-connected** if $|V| > k$ and for all $X \subset V$ with $|X| < k$, the subgraph induced by $V \setminus X$ is connected. In words, a graph is k-connected when we cannot disconnect the graph by removing any subset of $k-1$ vertices. Every non-empty graph is 0-connected, and the 1-connected graphs are precisely all the non-empty connected graphs except K_1 (which is not 1-connected because it does not have order 2).

The largest integer k such that G is k-connected is the **connectivity** $\kappa(G)$ of G. Thus $\kappa(G) = 0$ if and only if G is disconnected or $G \equiv K_1$, and $\kappa(K_n) = n - 1$ for all $n \geq 1$.

We now relate connectivity to minimum degree and to the existence of a set of edges whose removal disconnects the graph.

Theorem 5 *If G is non-trivial, then $\kappa(G) \leq |F| \leq \delta(G)$ where F is any smallest set of edges whose removal causes the graph to disconnect.*

Proof. Let $G = (V, E)$ be non-trivial and let v be any vertex with minimum degree $\delta(G)$. Then $|F| \leq \delta(G)$ because v can be disconnected by removing the edges that are incident with $N(v)$. We now show that $\kappa(G) \leq |F|$ by a case analysis. Let $G' = (V, E \setminus F)$ and note that G' has two connected components.

Case 1. G has a vertex v that is not incident with an edge in F. Let C be the component of G' that contains v and consider the set V_C of vertices of C that are incident with an edge of F. If we remove these vertices, then v is disconnected from the other component of G. Hence $\kappa(G) \leq |V_C|$. On the other hand, no edge in F can have both ends in C (otherwise, F is not minimal). Therefore, $|V_C| \leq |F|$.

Case 2. All vertices of G are incident with some edge in F. Pick an arbitrary vertex v and let C be the component of G' that contains v. Some $u \in N(v)$ are such that $(v, u) \in F$. The others nodes in the neighborhood of v must belong to C and are incident with distinct edges of F (otherwise, F is not minimal). Therefore $d(v) \leq |F|$, which implies $d(v) = |F| = \delta(G)$, because we already know that $|F| \leq \delta(G)$ and $d(v) \geq \delta(G)$ must be true. As removing $N(v)$ disconnects v, we conclude $\kappa(G) \leq \delta(G) = |F|$. $\square$

Bipartite graphs. A graph $G = (V, E)$ is bipartite if V admits a partition into 2 elements such that every edge has its ends in different elements: vertices in the same partition element must not be adjacent (i.e., they form an independent set). A bipartite graph in which every two vertices from different partition elements are adjacent is called complete. Clearly, K_1 and K_2 are bipartite (K_1 with a trivial partition element), but K_3 (the triangle) is not. We now show that this is due to the presence of an odd cycle (a cycle of odd length).

Fact 6 *A graph is bipartite if and only if it has no odd cycles.*

PROOF. First, suppose that G is bipartite. If G contains a cycle $v_0, \ldots, v_k$, then the even vertices $v_0, v_2, \ldots$ must be all in the same partition element, which must be different from the one which all the odd vertices $v_1, v_3, \ldots$ belong to. If the cycle is odd, then k is even and so v_0 and v_k belong to the same partition element. But they are connected by an edge, and so G cannot be bipartite. Now suppose that G is a nontrivial graph that has no odd cycles. It is enough to consider connected graphs since if G is bipartite, so is every component of G (and vice versa). Consider any vertex $z \in V$ and let $A = \{v : d(z, v) \text{ is even}\}$. Similarly, let $B = \{v : d(z, v) \text{ is odd}\}$. Clearly, A and B are a partition of V. We now show that A and B are independent sets. Assume that A is not independent and suppose that x and y are adjacent vertices of A. Now let P be a shortest path $z = v_0, v_1, \ldots, v_{2k} = x$ and Q a shortest path $z = u_0, u_1, \ldots, u_{2m} = y$. Note that y cannot be on P and x cannot be on Q. So the two paths must separate at some vertex w such that $w = v_j = u_j$. Now consider the cycle following P from w to x, then going through (x, y), and then following Q from y to w. This cycle has odd length, which contradicts the assumption that G has no odd cycles. This proves that A is an independent set. Similarly, we can prove the also B is an independent set. $\square$

Euler tours. A walk (resp., a closed walk) is a path (resp., cycle) whose vertices may not be all distinct. A closed walk in a graph is an Euler tour if it traverses every edge of the graph exactly once. A graph is **Eulerian** if it admits an Euler tour.

Theorem 7 (Euler, 1736) *A connected graph is Eulerian if and only if every vertex has even degree.*

PROOF. The degree condition is clearly necessary: a vertex appearing k times in an Euler tour (or $k + 1$ times, if it is the starting and finishing vertex and as such counted twice) must have degree $2k$. Conversely, we show by induction on $|E|$ that every connected graph $G = (V, E)$ with all degrees even has an Euler tour. The induction starts trivially with $|E| = 0$. Now let $|E| \geq 1$. Since all degrees are even, we can find in G a non-trivial closed walk that contains no edge more than once (see Exercise 4). Let W be such a walk of maximal length and let F be the set of its edges. We now show that $F \equiv E$, implying that W is an Euler tour. For the purpose of contradiction, suppose that $E' = E \setminus F$ has at least an edge. For every vertex $v \in V$, an even number of the edges $\{(v, u) : u \in N(v)\}$ lies in F (because W is a closed walk containing each edge only once), so the degrees of the subgraph $G' = (V, E')$ are again all even. Since G is connected, G' has an edge e incident with a vertex w on W. By the induction hypothesis, the component C of G' containing e has an Euler tour W'. Consider the closed walk starting at w that goes through W', comes back at w, and then goes through W coming back again at w. This closed walk in G contradicts the maximal length of W. $\square$

Euler tours can be found in time $\mathcal{O}(|E|)$ using Hierholzer's algorithm.

Hamilton cycles. A Hamilton cycle is a cycle that contains all vertices. A graph is Hamiltonian if it contains a Hamilton cycle. A trivial example of a Hamiltonian graph is K_n for any $n \geq 3$.

The following result provides a characterization of Hamiltonian graphs in terms of their closure. Let G be a graph of order n. Then the closure of G, written $[G]$, is constructed by adding edges that connect pairs of nonadjacent vertices u and v for which $d(u)+d(v) \geq n$. One continues recursively, adding one edge at the time until all nonadjacent pairs u, v satisfy $d(u)+d(v) < n$.

Theorem 8 (Bondy–Chvátal 1972) *A graph of order at least* 3 *is Hamiltonian if and only if its closure is Hamiltonian.*

Note that every graph G with $n \geq 3$ vertices and $\delta(G) \geq n/2$ is Hamiltonian because $[G] \equiv K_n$. More in general, any graph G of order at least 3 such that $d(u)+d(v) \geq n$ for every nonadjacent vertices u and v satisfies $[G] \equiv K_n$ and is therefore Hamiltonian.

A characterization of Hamiltonian graphs in terms of a simply checkable property, like the vertex degree for Eulerian graphs, is still missing.

The problem of determining whether an Hamiltonian path exists in a graph is NP-complete.

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Exercises.

1. Show that every 2-connected graph contains a cycle.
2. Show that every connected graph $G = (V, E)$ contains a path of length at least
$$\min\left\{2\delta(G), |V| - 1\right\}$$
3. Show that every tree is a bipartite graph.
4. Show that in every non-trivial connected graph whose each vertex has even degree there exists a non-trivial closed walk that contains no edge more than once.