Denoising in digital radiography: A total variation approach

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Images are corrupted by noise...

i) When measurement of some physical parameter is performed, noise corruption cannot be avoided.

ii) Each pixel of a digital image measures a number of photons.

Therefore, from i) and ii)...

...Images are corrupted by noise!
Gaussian noise

(not so useful for digital radiographs, but a good model for learning...)

- Measurement noise is often modeled as Gaussian noise...
- Let \( x \) be the measured physical parameter, let \( \mu \) be the noise free parameter and let \( \sigma^2 \) be the variance of the measured parameter (noise power); the probability density function for \( x \) is given by:

\[
p(x | \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right]
\]

Gaussian noise and likelihood

- Images are composed by a set of pixels, \( x \) (\( x \) is a vector!)
- How can we quantify the probability to measure the image \( x \), given the probability density function for each pixel?
- Let us assume that the variance is equal for each pixel;
- Let \( x_i \) and \( \mu_i \) be the measured and noiseless values for the \( i \)-th pixel;
- Likelihood function, \( L(x | \mu) \):

\[
L(x | \mu) = \prod_{i=1}^{N} p(x_i | \mu_i) = \prod_{i=1}^{N} \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x_i - \mu_i}{\sigma} \right)^2 \right]
\]

- \( L(x | \mu) \) describes the probability to measure the image \( x \), given the noise free value for each pixel, \( \mu \).
What about denoising???

- What is denoising then?

**Denoising = estimate μ from x.**

- How can we estimate μ?
- Maximize \( p(\mu|x) \) => this usually leads to an hard, inverse problem.
- It is easier to maximize \( p(x|\mu) \), that is => maximize the likelihood function (a “simple”, direct problem).
- But... Is maximization of \( p(\mu|x) \) different from that of \( p(x|\mu) \)?

Bayes and likelihood

- Bayes theorem:

\[
p(\mu | x)p(x) = p(x | \mu)p(\mu) \Rightarrow
\]

\[
\Rightarrow p(\mu | x) = \frac{p(x | \mu)p(\mu)}{p(x)}
\]

- In this case, maximizing \( p(\mu|x) \) or \( p(x|\mu) \) is the same!

A priori hypothesis on the estimated parameters \( \mu \). For the moment, let us suppose \( p(\mu) = \text{cost.} \)

Probability density function for the data \( x \)... Just a normalization factor!!!
So, let us maximize the likelihood...

- Instead of maximizing $L(x|\mu)$, it is easier to minimize $-\log[L(x|\mu)]$.
- When the noise is Gaussian, we get:

$$L(x|\mu) = \prod_{i=1}^{N} p(x_i | \mu_i) = \prod_{i=1}^{N} \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x_i - \mu_i}{\sigma} \right)^2 \right]$$

$$f(x | \mu) = -\ln[L(x | \mu)] = -\sum_{i=1}^{N} \ln\left( \frac{1}{\sigma \sqrt{2\pi}} \right) + \frac{1}{2\sigma} \sum_{i=1}^{N} (x_i - \mu_i)^2$$

- Maximize $L$ => Least squares problem!

However, what about noise in digital radiography?

- Noise in digital radiography is Poisson (photon counting noise)!
- Let $p_{n,i}$ be the noisy (measured) number of photons associated to pixel $i$, and $p_i$ the unnoisy number of photons. Then:

$$p(p_{n,i} | p_i) = \frac{p_i^{p_{n,i}} e^{-p_i}}{p_{n,i}!}$$
Gaussian noise: example

Poisson noise: example

Lower variance for low signal
**Likelihood for Poisson noise**

- Let us write the negative log likelihood for the Poisson case:
  \[ L(p_n | p) = \prod_{i=1}^{N} p(p_{n,i} | p_i) = \prod_{i=1}^{N} \frac{p_i^{p_{n,i}} e^{-p_i}}{p_{n,i}!} \]
  
  \[ f(p_n | p) = -\ln[L(x | \mu)] = -\sum_{i=1}^{N} [p_{n,i} \cdot \ln(p_i)] + \sum_{i=1}^{N} p_i + \sum_{i=1}^{N} \ln(p_{n,i}) \]
  
  \[ = \sum_{i=1}^{N} [p_i - p_{n,i} \cdot \ln(p_i)] \]

- \( L(p_n | p) \) is also known as Kullback-Leibler divergence (apart from a constant term), \( KL(p_n | p) \).

**Maximize \( L! \)**

- \( L \) is maximized \( \iff \) \( f \) is minimized;
  
  Optimization (Gaussian noise) can be performed posing:
  
  \[ \frac{\partial f(x | \mu)}{\partial \mu} = 0 \iff \frac{\partial f(x | \mu)}{\partial \mu_i} = 0, \forall i \Rightarrow \frac{\partial}{\partial \mu_i} \sum_{j=1}^{N} (x_j - \mu_j)^2 = 0, \forall i \Rightarrow \]
  
  \[ 2(x_i - \mu_i) = 0, \forall i \Rightarrow x_i = \mu_i, \forall i \]

- The noisy image gives the highest likelihood!!!
- This solution is not so interesting... The likelihood approach suffers from a severe overfitting problem.
Maximize $L$!

$L$ is maximized $\iff f$ is minimized;

- Optimization (Poisson noise) can be performed posing:

$$\frac{\partial f(p_n | p)}{\partial p} = 0 \iff \frac{\partial f(p_n | p)}{\partial p_i} = 0, \quad \forall i \Rightarrow \frac{\partial}{\partial p_i} \left[ \sum_{i=1}^{N} \left( p_i - p_{n,i} \cdot \ln(p_i) \right) \right] = 0, \quad \forall i \Rightarrow$$

$$1 - \frac{p_{n,i}}{p_i} = 0, \quad \forall i \Rightarrow p_i = p_{n,i}, \quad \forall i$$

- The noisy image gives the highest likelihood!!!
- This solution is not so interesting... The likelihood approach suffers from a severe overfitting problem.

Back to Bayes

- Bayes theorem:

$$\Rightarrow p(p | p_n) = \frac{p(p_n | p)p(p)}{p(p_n)}$$

- If we introduce a-priori knowledge about the solution $\mu$, we get a Maximum A Posteriori (MAP) solution - $p(p | p_n)$ is maximized!

A priori hypothesis on the estimated parameters $\mu$. Probability density function for the data $x$... Just a normalization factor!!!
What do we have to minimize now?

- We want to maximize \( p(p|p_n) \sim p(p_n|p)p(p) \), that is:

\[
- \ln[p(p|p_n)] = -\ln[p(p_n|p)p(p)] = -\ln \prod_{i=1}^{N} [p(p_{n,i}|p_{i}) \cdot p(p_{i})] = \\
- \sum_{i=1}^{N} \ln[p(p_{n,i}|p_{i}) \cdot p(p_{i})] = -\sum_{i=1}^{N} \ln p(p_{n,i}|p_{i}) - \sum_{i=1}^{N} \ln p(p_{i}) = \\
- \ln[L(p_n|p)] = \sum_{i=1}^{N} \ln p(p_{i})
\]

A priori term

- Let us call \( p_x \) and \( p_y \) the two components of the gradient of the image.
- These are easily computed, for instance as:
  - \( p_x = p(i,j) - p(i-1,j) \);
  - \( p_y = p(i,j) - p(i,j-1) \);
- The gradient (a vector!) will be indicated as \( \nabla p \);
- \( ||\nabla p|| \) indicates the norm of the gradient.
A priori term – image gradients (no noise)

\[ p_x = p(i, j) - p(i-1, j) \]

\[ p_y = p(i, j) - p(i, j-1) \]

A priori term – image gradients (noise)

\[ p_x = p(i, j) - p(i-1, j) \]

\[ p_y = p(i, j) - p(i, j-1) \]
A priori term – norm of image gradient

In the real image, most of the areas are characterized by an (almost) null gradient norm;

We can for instance suppose that $\|\nabla p\|$ is a random variable with Gaussian distribution, zero mean and variance equal to $\beta^2$.

[Note that, in the noisy image, the norm of the gradient assume higher values → low $\|\nabla p\|$ means low noise!]

MAP and regularization theory

- Poisson noise, normal distribution for the norm of the gradient:

$$f(p_n | p) = -\ln[L(p_n | p)] - \sum_{i=1}^{N} \ln p(p_i) =$$

$$= \sum_{i=1}^{N} [p_i - p_{ni} \cdot \ln(p_i)] - \sum_{i=1}^{N} \ln \left( \frac{1}{\beta \sqrt{2\pi}} \exp\left( -\frac{1}{2} \frac{\|\nabla p_i\|^2}{\beta^2} \right) \right) =$$

$$= \sum_{i=1}^{N} [p_i - p_{ni} \cdot \ln(p_i)] + N \ln(\beta \sqrt{2\pi}) + \frac{1}{2\beta^2} \sum_{i=1}^{N} \|\nabla p_i\|^2$$

Negative log likelihood

Regularization term (a priori information)
MAP and regularization theory

- We look for the minimum of \( f \)...
- ... The likelihood is maximized (data fitting term)...
- ... At the same time, the norm of the gradient is minimized (regularization term)...
- ... The regularization parameter \( (1/2\beta^2) \) balances between a perfect data fitting and very regular image...

\[
f(p_n | p) = \sum_{i=1}^{N} [p_i - p_n,i \cdot \ln (p_i)] + \frac{1}{2\beta^2} \sum_{i=1}^{N} ||\nabla p_i||^2
\]

For \((1/2\beta^2) = 0\) we get the maximum likelihood solution;
Increasing \((1/2\beta^2)\) we get a more regular (less noisy) solution;
For \((1/2\beta^2) \to \infty\), a completely smooth image is achieved.

Noise reduction.

Noise and edge reduction.
Fix the ideas

• A statistical based denoising filter is achieved minimizing:

\[ f = -\ln[L(p_n|p)] - \lambda \cdot \ln[p(p)] \]

• The **data fitting term** is derived from the noise statistical distribution (likelihood of the data); generally, the choice for this term is unquestionable.

• The **regularization term** is derived from a-priori knowledge regarding some properties of the solution; this term is generally user defined.

• Depending on the regularization parameter \( \lambda \), the first or the second term assume more or less importance. For \( \lambda \rightarrow 0 \), the maximum likelihood solution is obtained.

Gibbs prior

• Up to now, we assumed a normal distribution for the norm of the gradient, \( \rightarrow \) Tikhonov regularization (quadratic penalization).

• A more general framework is obtained considering:

\[ p(p) = \exp[-R(p)] \quad (\text{Gibb's prior}) \]

• \( R(p) \rightarrow \) Energy function \( \sim \) regularization term (note that \( -\ln \exp[-R(p)] = R(p)! \))

• Tikhonov assumes \( R(p) = -\frac{1}{2} (||\nabla p||/\beta)^2 \)
**Edge preserving denoising?**

- Tikhonov term penalizes the image edges (high gradient) more than the noise gradients.
- It is well known that Tikhonov regularization does not preserve edges.
- An edge preserving algorithm is obtained considering \( R(p) = ||\nabla p|| \) [Total variation, TV].

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**Tikhonov vs. TV (preview)**

- Tikhonov =>
- Original image
- TV =>
- Filtered image
- Difference
TV in digital radiography: starting point and problems

- $p_n$, noisy image affected by Poisson noise (likelihood $\Rightarrow$ KL);
- $p$, noise free image (unknown);
- $R(p) = ||\nabla p||$ (Total variation);
- Minimize $f(p|p_n) = KL(p_n,p) + \lambda \cdot \sum_{i=1..N} ||\nabla p_i||$.

- How to compute $||\nabla p||$? $\Rightarrow$ A compromise between computational efficiency and accuracy has to be achieved.
- How to minimize $f(p|p_n)$? $\Rightarrow$ An iterative optimization technique is required.

How to compute $||\nabla p||$?

$\nabla p_x = p(u,v) - p(u-1,v)$
$\nabla p_y = p(u,v) - p(u,v-1)$

$||p||_1 = |p_x| + |p_y|$ \textbf{L1 norm}
$||p||_2 = \sqrt{p_x^2 + p_y^2}$ \textbf{L2 norm}

The computational cost increases with the number of neighbours considered for computing the gradient;

The computational cost is higher for L2 norm with respect to L1 norm;

What about accuracy? $\Rightarrow$ See experimental results!
How to minimize \( f(p|p_n) \)?

- \( f(p|p_n) \) is strongly non linear; solving \( df(p|p_n)/dp = 0 \) directly is not possible => iterative optimization methods.

1) Steepest descent + line search (SD+LS)
2) Expectation – Maximization (damped with line search - EM)
3) Scaled gradient (SG)

Steepest descent + line search (SD+LS)

- \( p^{k+1} = p^k - \alpha \cdot df(p|p_n)/dp \)
- \( p^{k+1} = p^k - \alpha \cdot df(p|p_n)/dp \)

- The damping parameter \( \alpha \) is estimated at each iteration to assure convergence \( (f^{k+1} < f^k) \);

- Easy implementation;
- Slow convergence, the method has been damped (line search) to improve convergence \( (\alpha > 1) \).
**EM + line search (EM)**

- Consider the pixel i, then:

  \[ \frac{df(p|p_n)}{d_\pi} = 0 \Rightarrow \frac{dKL(p|p_n)}{d_\pi} + \frac{dR}{d_\pi} = 0 \Rightarrow p_i = p_{n,i} \cdot \beta \cdot \frac{dR}{d_\pi} + (\beta \cdot \frac{dR}{d_\pi} + 1) \]  
  [Fixed point iteration]

- Damped formula:  
  \[ p_i = \alpha \cdot (1 - \alpha) + \alpha \cdot \frac{p_{n,i}}{\beta \cdot \frac{dR}{d_\pi} + 1} \]

- The damping parameter \( \alpha \) is estimated at each iteration to assure convergence (\( \forall k < k' \)).

  +: easy implementation, fast convergence;
  -: the method has been damped to assure convergence (\( \alpha < 1 \), what happens when \( \beta \cdot \frac{dR}{d_\pi} + 1 \rightarrow 0 \)??)

**Scaled gradient (SG)**

- Consider the gradient method formula;

- Each component of the gradient is scaled to improve convergence (\( S \) is a diagonal matrix containing the scaling parameters):

  \[ p_{k+1} = p_k - \alpha \cdot S \cdot \frac{df(p|p_n)}{dp} \]

- The matrix \( S \) is computed from an opportune gradient decomposition and KKT conditions;

  +: easy implementation, fastest convergence; it can also be demonstrated that, for positive initial values, the estimated solution remains positive at each iteration!
  -: ???
Problems with $dR/dp_i$

- Independently from the optimization method, the term $dR/dp_i$ has to be computed at each iteration for any $i$;
- We have:

$$dR/dp_i = d[\Sigma_{i=1..N}(||v_i||_2)]/dp_i$$

XOR

$$dR/dp_i = d[\Sigma_{i=1..N}(||v_i||_2)]/dp_i$$

Let us compute it for $||.||_2 (R/dp_i = d[\Sigma_{i=1..N}(||v_i||_2)]/dp_i)$

$$\frac{dR}{dp_i} = \frac{d\sum_{i}^{N} (p_{i,j}^2 + p_{i,j}^2)}{dp_i} = \sum_{i}^{N} \frac{d}{dp_i} \left( \sqrt{(p(u,v)-p(u-1,v))^2 + (p(u,v)-p(u,v-1))^2} \right)$$

$$= \frac{2[p(u,v)-p(u-1,v)] + 2[p(u,v)-p(u,v-1)]}{\sqrt{(p(u,v)-p(u-1,v))^2 + (p(u,v)-p(u,v-1))^2}} + ... = 2 \frac{p_{i,j} + p_{i,j}}{\sqrt{p(u,v)}} + ...$$

To avoid division by zero:

$$\frac{dR}{dp_i} = 2 \frac{p_{i,j} + p_{i,j}}{\sqrt{p(u,v)}} + ... \rightarrow 2 \frac{p_{i,j} + p_{i,j}}{\sqrt{(p(u,v)-p(u-1,v))^2 + (p(u,v)-p(u,v-1))^2} + \delta} + ...$$
Problems with $dR/dp_i$

- Let us compute it for $\|l\|_2$ ($R/dp_i = d[\Sigma_{i=1..N}(\|l\|_1)]/dp_i$)

$$
\frac{dR}{dp_i} = \frac{d\sum_{i=1}\|l_i\|}{dp_i} = \left[ \sum_{i=1}^N \frac{d}{dp_i} \left( p(u,v) - p(u-1,v) \right) + \frac{p(u,v) - p(u,v-1)}{\sqrt{p(u,v)^2 - p(u-1,v)^2}} \right] + ... = \\
= \sum_{i=1}^N \left[ \text{sign}(p_{i..}) + \text{sign}(p_{..i}) \right] + ...
$$

- Here divisions by zero are automatically avoided — only “sign” is required -> computationally efficient!

Questions

- How many neighbor pixels do we have to consider to achieve a satisfying accuracy at low computational cost?

- Best norm, $\|l\|_1$ vs $\|l\|_2$?

- Best optimization method (SD+LS, EM, SG)?
TV in digital radiography...

Research in progress...

Results (answers)

- 75 simulated radiographs with different frequency content, corrupted by Poisson noise (max 15,000 photons).

- For any filtered image, measure:

  \[
  \text{MAE} = \frac{1}{N} \sum_{i=1}^{N} |p_i^{\text{noise free}} - p_i^{\text{filtered}}| \\
  \text{RMSE} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (p_i^{\text{noise free}} - p_i^{\text{filtered}})^2} \\
  \text{KL} = \sum_{i=1}^{N} [p_i^{\text{noise free}} \ln(p_i^{\text{noise free}}/p_i^{\text{filtered}}) + p_i^{\text{noise free}} - p_i^{\text{filtered}}]
  \]
2 neighbors (02010000) vs. 4 neighbors (11110000)

- MAE
  - 01010000
  - 11110000

- RMSE
  - 01010000
  - 11110000

- KLD
  - 01010000
  - 11110000

- TIME
  - 01010000
  - 11110000

II·II₂ vs. II·II₂

- MAE
  - norm 1
  - norm 2

- RMSE
  - norm 1
  - norm 2

- KLD
  - norm 1
  - norm 2

- TIME
  - norm 1
  - norm 2
EM vs. SD+LS

EM vs. SG
Convergence and iterations

Filter effect

Original

Filtered
Filter effect: before filtering