When will we ever use this?

Making Decisions Using Advanced Mathematics

A product of Project MINDSET www.mindsetproject.org

Volume I: Deterministic Modeling

A partnership between North Carolina State University, Wayne State University and University of North Carolina Charlotte.

Funded by the National Science Foundation (NSF).
NSF Directorate for Education and Human Resources Project # DRL-0733137
Version: May 2011

Project Principal Investigators

Dr. Kenneth Chelst
Professor, Department of Industrial & Manufacturing Engineering
Wayne State University

Dr. Thomas Edwards
Associate Professor, Teacher Education Division
Wayne State University

Dr. Karen Keene
Assistant Professor, Department of STEM Education
North Carolina State University

Dr. Karen Norwood
Associate Professor, Department of Math STEM Education
North Carolina State University

Dr. David Pugalee
Professor, Center for Science, Technology, Engineering, and Mathematics (STEM) Education
University of North Carolina - Charlotte

Dr. Robert Young
Professor, Edward P. Fitts Department of Industrial & Systems Engineering
North Carolina State University

Project Managers
Dr. Molly Purser
North Carolina State University

Mr. Jay Johnson
Wayne State University

Researchers
North Carolina State University
Ms. Amy Craig
Ms. Richelle Dietz
Ms. Krista Holstein
Mr. Bill Scott

Wayne State University
Mr. Saman Alaniazar
Mrs. Hatice Ucar

University of North Carolina – Charlotte
Ms. Sarah Johnson

MINDSET Project

Mathematics **IN**struction using **D**ecision **S**cience and **E**ngineering **T**ools

Finally, an answer to the question all mathematics teachers fear…

“How will we ever use this?”

M*athematics* ***IN****struction using* ***D****ecision* ***S****cience and* ***E****ngineering* ***T****ools* (MINDSET) is a NSF funded collaboration between mathematics educators, industry professionals, engineers, and mathematicians at NC State University, Wayne State University and UNC Charlotte. Partnering with high school teachers, the MINDSET project team is working to create, implement, and evaluate a new curriculum. The intent is to reinforce standard mathematics concepts using math-based decision-making tools from Operations Research (OR) and Industrial Engineering (IE) for a non-calculus fourth-year mathematics course. The curriculum is presented to high school students as a series of real-world problems with the purpose of making the underlying mathematics more relevant to them. One goal of MINDSET is to improve students' mathematical abilities and attitudes by building on skills learned in Algebra II in relatable problem contexts. A second goal is to improve their problem solving skills and, especially, to interpret the results of that problem solving activity.

The course is comprised of two sections, one section of deterministic content and the other probabilistic. Problem contexts covered in the deterministic curriculum include linear programming, the critical path method, facility location problems, transportation problems, and multi-criterion decision making. In the probabilistic curriculum, topics include probability distributions, decision trees, quality control, Program Evaluation Review Technique (PERT), Markov chains, and queuing theory. But…MINDSET is different. These techniques and tools are woven into real-world problem contexts, where the student starts with the contexts and proceeds to math skill development.

Acknowledgements

The following teachers have piloted the MINDSET materials and provided valuable feedback to improve the material. We are grateful for their support of the project.

Derek Blackwelder
Jamie Blanchard
James Bruckman
Robert Butler
Molly Charles
Suzanne Christopher
Krystle Corbin
Dean DiBasio (teacher writer)
Robin Dixon
Caroline Geel
Breanna Harrill
Amy Johnson
Jenese Jones (teacher writer)
Anna Kamphaus
George Lancaster (teacher writer)
Craig Lazarski
Jeffrey Loewen
Charles Ludwick
Margart Lumsden
Leon Martin
Cora McMillan
Kristen Meck (teacher writer)
Karen Mullins
Erica Nelson
Bethany Peters
Joseph Price
Angela Principato (teacher writer)
John Pritchett
Tyler Pulis
Paul Rallo
Tonya Raye
Kym Roberts
William Ruff
Amy Setzer
Rob Sharpe
Catherine Sitek
Cheryl Smith
Julie Smith
Susan Sutton
Greg Tucker
Laura Turner
Thad Wilhelm (teacher writer)
Tracey Walden
Tracey Weigold
Lanette Wood

WHAT IS OPERATIONS RESEARCH?

Operations research (OR) is a scientific way to analyze problems, make decisions, and improve processes. OR professionals try to provide a sound basis for decision-making. These decisions may focus on day-to-day operations that arise in a manufacturing plant. Or, they may involve long-range issues such as designing new environmental regulations or establishing minimum prison sentence guidelines.

Operations researchers attempt to understand and structure complex situations. They develop mathematical and computer models of a system of people, machines, and procedures. If you have ever played the game Sim City, you have manipulated a computer model.

Operations researchers often use numerical, algebraic, and statistical techniques. Then they manipulate their models to study the behavior of the system. They use this understanding to predict how the system will behave under different rules and policies. Then they work to improve system performance.

Unlike most disciplines, we can point to specific events that mark the birth of operations research. OR was born in the years just prior to World War II. The British anticipated an air war with Germany. In 1937, they began to test radar. By 1938, they were studying how to use the information radar provides to direct the operations of their fighter planes.

Until this time, the word experiment usually meant a scientist carrying out a controlled experiment in a laboratory. In contrast, this radar-fighter plane project used a multi-disciplinary team of scientists. They studied actual operating conditions in the field instead of in the laboratory. Then they designed experiments in the field of operations, and the new term “operations research” was born. Their goal was to understand the operation of the complete system of equipment, people, and environmental conditions (e.g. weather, nighttime). Then they tried to improve it. Their work was an important factor in winning the Battle of Britain. OR eventually spread to all of the military services. Several of the leaders of this effort eventually won Nobel Prizes in their original fields of study.

All branches of the US Armed Forces later formed similar groups of interdisciplinary scientists. These groups worked to protect naval convoys, search for enemy convoys, enhance anti-submarine warfare and improve the effectiveness of bombers. To do so, they collected data by directly observing operations. Then they built a mathematical model of the system. Next, they used the model to recommend improvements. Finally, they obtained feedback on the impact of the changes.

Today, every branch of the military has its own operations research group. These OR groups include both military and civilian personnel. They play a key role in long-term strategy and weapons development. They also direct the operation of actions such as Operation Desert Storm. In addition, the National Security Agency has its own Center for Operations Research.

In the 1950s, national professional organizations were formed. These organizations published research journals. Universities added OR departments. All of this raised operations research to the level of a profession. The use of OR expanded beyond the military to include other government organizations and private companies. The petroleum and chemical industries were early users of OR. They improved the performance of plants, developed natural resources and planned strategy. Today, OR plays important roles in industry and government such as in:

- airline industry – scheduling planes and crews, pricing tickets, taking reservations, and planning the size of the fleet;
- pharmaceutical industry – managing research and development and designing sales territories;
- delivery services – planning routes and developing pricing strategies;
- financial services – credit scoring, marketing, and internal operations;
- lumber industry – managing forests and cutting timber;
- local government – deploying emergency services; and
- policy studies and regulation – environmental pollution, air traffic safety, AIDS, and criminal justice policy.

As the field of OR grew, there was less stress on interdisciplinary teams. Today, the focus is on using mathematical models. These models can be deterministic or probabilistic. Mathematical programming, routing and network flow are examples of deterministic models. Examples of probabilistic models include queuing, simulation and decision trees. Mathematical modeling is the core of the OR curriculum. OR programs are found in either engineering or business schools. Most mathematics departments also offer introductory OR courses at the junior or senior undergraduate level.

Volume I: Deterministic Material

May 2011

Table of Contents

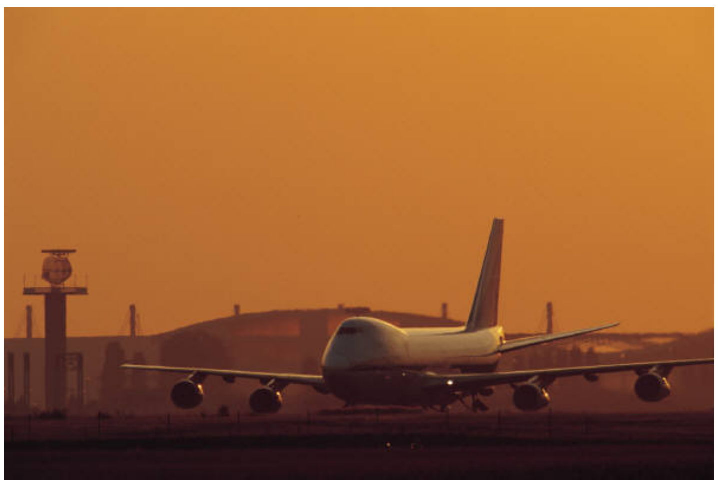
Section 1.0: Introduction to Making Hard Decisions

We all face decisions in our jobs, in our communities, and in our personal lives.

- Where should a new airport, manufacturing plant, power plant, or health care clinic be located?
- Which college should I attend, or which job should I accept?
- Which car, house, computer, stereo, or even health insurance plan should I buy?
- Which supplier or building contractor should I hire?

Decisions such as these involve comparing alternatives that have strengths or weaknesses with regard to multiple objectives of interest to the decision maker. For example, your objectives in buying health insurance might be to minimize cost *and* maximize protection. Sometimes multiple objectives like these get in each other's way.

Multi-attribute utility theory (MAUT) is one form of multi criteria decision making. MAUT is a structured methodology designed to handle the tradeoffs among multiple objectives. One of the first applications of MAUT involved a study of alternative locations for a new airport in Mexico City in the early 1970s. The factors that were considered included cost, capacity, access time to the airport, safety, social disruption, and noise pollution.



Utility theory is a systematic approach to quantifying an individual's preferences. It is used to rescale a numerical value of some measure of interest onto a 0–1 scale, with 0 representing the worst value of the measure and 1 representing the best. This allows the direct comparison of many diverse measures. In other words, with the right tool, it really is possible to compare apples to oranges! The end result of this process is an evaluation of the alternatives in a rank order that reflects the decision makers' preferences. An analogous situation arises when individuals, college sports teams, Master's in Business Administration degree programs, or even hospitals are ranked in terms of their performance on many diverse measures. Another example is the Bowl Championship Series (BCS) in college football that attempts to identify the two best college football teams in the United States to play in a national championship bowl game. This process has reduced, but not eliminated, the annual end-of-year arguments as to which college should be crowned national champion.

Section 1.1: Choosing a Cell Phone Plan

Isabelle Nueva is trying to help her mother and father decide on the best cell phone plan for her family. She and her friend, Angelo Franco, think that some of the things they have learned in their math class about making decisions will be helpful to them. They studied a process for making decisions that have multiple criteria.

1.1.1 Identify Criteria and Measures

They know that the first thing they must do is identify the **criteria** of a cell phone plan that are important to Isabelle's family. From discussions she has had with her mother and father, Isabelle knows that the criteria that are important to them are the cost and other important factors. Isabelle and Angelo know that they need to find at least one way to **measure** each of the criteria. For the cost criterion, they decide to use the monthly service charge, number of minutes per month and the minimum length of the contract as the measures. For the important factors criterion, they decide to use whether there is free unlimited texting and the quality of service as the measures.

The monthly service charge and the total minutes allowed per month could be any amount within a reasonable range. These are examples of **continuous** measures. Isabelle and Angelo decide that the data they collect for the other three measures can be grouped into a finite number of categories. For example, the measure "Free unlimited text messaging" has only two categories, "yes" and "no." Thus, "Free unlimited text messaging" is an example of a categorical measure.

Isabelle and Angelo similarly define categories for each of the remaining measures. For quality of service, they decide to use ratings from a consumer magazine. They are only considering plans rated good or better. To rate the quality of service, the magazine asked about common cell-phone problems. They used dropped or disconnected calls, static and interference, and voice distortion to rate quality.

Isabelle's parents were also concerned about being locked into paying for a plan for a long period of time. Therefore, Isabelle and Angelo decided to use minimum contract length as one of the measures of cost. Minimum contract length is the shortest time a customer must remain with a particular plan to avoid paying a fee to cancel the service. The plans under consideration had only three different minimum contract lengths, 6 months, 1 year, and 2 years. The categorical variables and their possible values are provided in Table 1.1.1. For each of the categorical measures, Isabelle and Angelo also assigned a number to each category. Those numbers also appear in Table 1.1.1.

Variable	Category	Numeric Value
Free Unlimited Texting	Yes No	1 0
Quality of Service	Excellent Very Good Good	2 1 0
Minimum Contract	6 months 1 year 2 years	2 1 0

Table 1.1.1: Categorical variables with categories and numeric values

1.1.2 Collect Data

Isabelle's parents are considering three telephone plans, Trot, ust&t, and Horizon. Isabelle and Angelo collect the data they will need to make their decision. Their data appears in Table 1.1.2.

Plan	Monthly Charge	Minutes per Month	Minimum Contract Length	Free Unlimited Texting	Quality of Service
Trot	$35	400	6 months	no	Good
ust&t	$50	500	2 years	yes	Excellent
Horizon	$60	600	1 year	yes	Very Good

Table 1.1.2: Isabelle and Angelo's cell phone data

1.1.3 Range of Each Measure

Next, Isabelle and Angelo recall that in their mathematics class they learned that it was important to specify the range for each measure. That range will be used to scale the values within the range. There are two continuous measures, monthly service charge and number of minutes allowed per month. For each measure, they decide to use the range of the data they collected. For the categorical variables, they decide to define the range as the specific possible values, because there are only two or three of them.

1.1.4 Rescale Measures to a Common Unit

For a number of reasons, it would be difficult to compare the three plans using this raw data. The first reason is that comparing a $10 difference in the monthly service charge to a minimum contract length of 2 years or 1 year is like comparing apples to oranges. In order to avoid that problem, operations researchers scale the range of each measure creating a common unit that varies from 0 to 1. Zero always represents the worst value and 1 the best value.

For both of the continuous measures, Isabelle and Angelo decide to use a proportional scale. For example, the range they have decided to use for the monthly service charge measure is from $35 to $60. This is a range of $25. The smallest possible value here is the best option, so $35 is converted to a common unit value of 1. Similarly, the largest possible value of the monthly service charge is the worst option, so $60 is converted to 0.

Now, what should be done with the $50 monthly service charge?

Isabelle and Angelo must assign an appropriate common unit score to the $50 monthly service charge. They realize that they need to decide where $50 lies when it is compared to the best and worst options for monthly service charge.

Q1. Is $50 closer to the best or the worst option?

Q2. How far from the best option is $50? How far from the worst?

Q3. How far apart are the best and worst options?

Using the answers to these three questions, Isabelle and Angelo solve a proportion to arrive at the common unit value for the monthly service charge of $50.

$$\frac{10}{25} = \frac{x}{1}$$
$$x = 0.4$$

If Isabelle and Angelo were considering a plan with a $45 monthly service charge, what is the common unit value for $45?

Isabelle and Angelo computed the common unit values for the number of minutes allowed each month in the same way.

For the three categorical measures, Isabelle and Angelo again assigned a common unit value of 0 to the worst and 1 to the best. There were only two possibilities for the free unlimited texting measure, so there was nothing else to do. However, for the other two categorical measures, there was something between the best and worst values. For each of these measures, the value in the middle was assigned a common unit value of 0.5.

Q4. Verify that 0.5 is the appropriate common unit value for a quality measure of very good and a minimum contract length of 1 year if the common unit is assigned proportionally in each case.

Table 1.1.3 contains all of the common unit values for each of the five measures for each of the three plans.

PLAN	Monthly service charge	Minutes per month	Free unlimited texting	Quality of service	Minimum length of contract
Trot	1	0	0	0	1
ust&t	0.4	0.5	1	1	0
Horizon	0	1	1	0.5	0.5

Table 1.1.3: Cell phone data converted to a common unit

When Isabelle and Angelo looked at these results, they noticed that each plan received the top common unit value of 1 on two of the measures. They also noticed that each plan received at least one bottom common unit value of 0. Since the Trot plan got 3 zeros, they thought they could eliminate that plan. However, they weren't sure how to choose between the other two plans.

Angelo thought about using the total of all of the common units to get a total score for each plan. Using his system, Angelo got the following scores:

Trot: **2**
ust&t: **2.9**
Horizon: **3**

Anna thought it would be more meaningful to compute the average common unit score for each plan. When she did so, she obtained the following averages:

Trot: **0.40**
ust&t: **0.58**
Horizon: **0.60**

Q5. Would it make any difference whether Anna and Angelo used the sum or the average? Explain.

In either case, on the basis of their work, Horizon would be slightly preferred over ust&t, with Trot a distant third. But then they worried about whether some measure was more important than another. For example, Anna remembered that her parents were really worried about the monthly service charge, and not as worried about the length of the contract. They decide that they need a system that does not treat all of the measures equally, as the sum and average do. They need a system that weights the measure according to how important they are to Anna's parents.

1.1.5 Conduct an Interview to Calculate Weights

In order to learn how important each measure is to Anna's parents, Anna and Angelo decide to interview them. First, they need to know which measure is the most important to them. To find out, they ask Anna's parents to rank the five measures in their order of importance. Table 1.1.4 shows their rank ordering of the measures.

MEASURE	Monthly charge	Minutes	Free texting	Quality	Contract length
RANK	1	4	2	3	5

Table 1.1.4: Rank-order of the measures according to Anna's parents

Then Anna and Angelo needed to assign weights to each measure that capture more than the order of importance. They also need a sense of how important the measures are with respect to one another. For example, if one measure is twice as another, then the assigned weights should reflect the strength of that difference. In their math class, they learned a technique of assigning points that can be used to determine the proper weights. The technique continues the interview.

Isabelle and Angelo ask Mr. and Mrs. Nueva to assign 100 points to the measure they ranked number 1. Then they asked them to assign a number of points less than 100 to the second-ranked measure, free unlimited texting. In doing so, they asked Isabelle's parents to pick a number that reflected how important it was compared to the number one ranked measure. Mr. and Mrs. Nueva chose to assign 80 points to free unlimited texting. The interview continued until a number of points had been assigned to each of the five measures. Table 1.1.5 shows the points assigned to each measure.

MEASURE	Monthly charge	Minutes	Free texting	Quality	Contract length
RANK	1	4	2	3	5
POINTS	100	60	80	70	40

Table 1.1.5: Points assigned to each of the measures

Now, Isabelle and Angelo totaled all of the assigned points and divided the point assignment for each measure by the total. This number is the weight of that measure. Table 1.1.6 shows how the weight of the monthly service charge was calculated, as well as the calculated weight for each of the other measures.

MEASURE	Monthly charge	Minutes	Free texting	Quality	Contract length	Total
RANK	1	4	2	3	5	
POINTS	100	60	80	70	40	350
WEIGHT	100/350=0.29	0.17	0.23	0.20	0.11	1.00

Table 1.1.6: A weight is calculated for each measure.

Q6. Verify that all of the weights in Table 1.1.6 are correct.

Q7. What is the largest weight? Which is the smallest?
Q8. What is the range of the weights, from largest to smallest?
Q9. What is the ratio of the largest weight to the smallest weight?
Q10. What should this ratio mean in the context of the decision?

1.1.6 Calculate Total Scores

Now, a total score for each plan can be calculated. The total score is an example of a **weighted average**. Each common unit value from Table 1.1.3 is multiplied by the corresponding weight from Table 1.1.6. Then for each plan, those products are added together to get the total score. Table 1.1.7 shows the results of these computations. Notice that this weighted average captures how important the various measures are to Isabelle's parents. Notice also that on the basis of this weighted average approach, ust&t is preferred over Horizon by Isabelle's parents.

PLAN	Monthly service charge (0.29)	Minutes per month (0.17)	Free unlimited texting (0.23)	Quality of service (0.20)	Minimum contract (0.11)	Total Score
Trot	1(.29)=0.29	0(.17)=0	0	0	0.11	0.40
ust&t	0.4(.29)=0.12	0.5(.17)=0.09	0.23	0.20	0	0.64
Horizon	0	0.17	0.23	0.5(.20)=0.10	0.5(.11)=0.06	0.56

Table 1.1.7: A weighted total score is computed for each plan.

Q11. Would everyone's score results lead to the same preferred choice? Explain.

1.1.7 Determine Strengths/Weaknesses and Make Final Decision

Isabelle and Angelo decide to examine their results, because the total scores of ust&t and Horizon were so close. They are also concerned, because their weighting system produced a different result.

Q12. For which measures did ust&t have a higher weighted score than Horizon? For which did Horizon outscore ust&t?

Q13. What were the ranks of the measures where ust&t scored higher than Horizon? What were the ranks for the measures where Horizon was higher?

When Isabelle and Angelo compare ust&t with Horizon, they see that ust&t had higher weighted scores for the first and third ranked measures. Horizon scored higher on the fourth and fifth ranked measures. ust&t and Horizon were tied on the second ranked measure. ust&t scored better on two of the three most important measures. In contrast, Horizon scored better only on the two least important measures. Therefore, Isabelle and Angelo believe that their weighting system did what it was supposed to do. They decide to recommend the ust&t plan to Isabelle's parents.

Section 1.2: Enrique Ramirez Chooses a College

Sooner or later, everyone faces an important decision in life. In this problem, we are going to follow Enrique Ramirez in his search for the right college to attend. Enrique has been accepted at four of the seven colleges that he applied to: Canisius College in Buffalo, NY; Clark University in Worcester, MA; Drexel University in Philadelphia, PA; and Suffolk University in Boston, MA. Now he must decide which one of the four to attend.

1.2.1 Applying a Process

Enrique has asked his friend Anna for help. Enrique and Anna have realized that there are many different issues to be considered when making this decision. They have also realized that the issues of interest to Enrique and their relative importance are not the same as those for Anna.
In this activity, you will read about a systematic process called multiple-criteria decision making that will help Enrique and Anna make an informed choice about which college he should choose to attend. The specific process they will use is multi-attribute utility theory (MAUT). Recall that the steps in this process are:

- Generate a list containing general criteria that are important to you in choosing a college. These criteria will be broad in nature and will be based on objective and subjective goals.
- Specify at least one measure for each criterion, and specify a reasonable scale for each measure.
- For each college, collect the data for each measure.
- Rescale each measure to common units from 0 to 1, with 0 being the worst alternative and 1 being the best alternative.
- Ask someone else to interview you in order to determine the relative importance of each measure. With that person's help, rank-order the measures, assign points from 0 to 100 to each measure, and calculate a proportional weight between 0 and 1 for each measure.

- Calculate a total weighted score for each college. These weights will yield a ranking of the colleges, allowing you to identify the best option based on your preferences.
- Last, review the results to understand the strengths and weaknesses of your top alternatives before finalizing your decision.

Specify Criteria and Measures
With Anna's help, Enrique has decided that academics, cost, location, and social life are the factors (criteria) most critical in his choice of a school. Next, Enrique and Anna have taken his list of four criteria and specified two or three measures for each criterion. His criteria and measures are given in Table 1.2.1

Criteria	**Measures**	
Academics	1) Average total SAT score of last year's freshman class	
	2) *U.S. News & World Report* ranking	
Cost	3) Room and board—annual	
	4) Tuition	
Location	5) Average daily high temperature	
	6) Proximity to home	
Social Life	7) Athletics	
	8) Reputation	
	9) Size	

Table 1.2.1: Enrique's criteria and measures

Scale Each Measure
The next task that Enrique and Anna face is choosing an appropriate scale for each of the nine measures Enrique has identified above. They also realize that some of the measures, such as average SAT score, have a natural scale (the combined score), while other measures, such as athletics, require the construction of a scale. Furthermore, some of the measures will be continuous (SAT score), while others are categorical. For example, Enrique and Anna have developed a three-category scale for athletics:

1. Division 1.
2. Division 2.
3. Division 3.

Still others will be converted from continuous into categorical. For example, distance from home has a natural continuous measure: miles. However, Enrique feels that exact mileage is not important, but rather broad ranges of mileage better represent his concerns.

Anna and Enrique also realize that the range of each scale is important. For example, the theoretical range of the average combined SAT score is 600–2400, but in actuality, the range of the average combined SAT score at the colleges Enrique is considering is much narrower. Enrique and Anna decide that it is much more realistic to use that that is close to the actual range, which is 1480–1750. The scale range and type of each measure are given in Table 1.2.2.

Measure	Scale range	Type
Average combined SAT score	1400–1800 (realistic)	Continuous-natural
U.S. News & World Report rank	1. Nationally ranked 2. Regionally ranked 3. Regionally tier 3 4. Regionally tier 4	Categorical-constructed
Room and board—annual	$8,000–$14,000 (realistic)	Continuous-natural
Tuition and fees—annual	$25,000–$35,000 (realistic)	Continuous-natural
Average daily high temperature	50˚–70˚F	Continuous-natural
Proximity to home	1. Within 1 hr. drive (50–100 mi.) 2. Within 4 hr. drive (101–200 mi.) 3. Within a day's drive (201–300 mi.)	Categorical-constructed
Athletics	1. Division 1 2. Division 2 3. Division 3	Categorical-constructed
Reputation	1. Seriously academic 2. Balanced academics and social life 3. Party school	Categorical-constructed
Size	1. Under 3,000 2. 3,001–6,000 3. 6,001–12,000 4. Over 12,000	Categorical-constructed

Table 1.2.2: Types and ranges of measures

Collect Data
After scaling each measure, Enrique and Anna collect the data in Table 1.2.3. They also place the specific values for mileage and size into the appropriate categories.

Measure	Canisius	Clark	Drexel	Suffolk
SAT score	1590	1750	1700	1480
U.S. News	22—rgnl.	91—natl.	109—natl.	Tier 3—rgnl.
Room & board	$10,150	$8,850	$12,135	$11,960
Tuition	$28,157	$33,900	$30,470	$25,850
Avg. daily high temp.	56°	56°	64°	59°
Nearness to home	297 mi.	157 mi.	81 mi.	191 mi.
Athletics	Division 1	Division 3	Division 1	Division 3
Reputation	Balanced	Seriously academic	Seriously academic	Balanced
Size	3,300	2,175	12,348	4,985

Table 1.2.3: Raw data for Enrique's four schools

Rescale to Common Units
Once Enrique and Anna collect the data, Anna reminds Enrique that if they compare the data in its current form, it would be like comparing apples to oranges. They decide to convert the data to common units. This means they first need to convert their raw numbers for categorical measures into the categorical values. The converted data is in Table 1.2.4.

Measure	Canisius	Clark	Drexel	Suffolk
SAT score	1590	1750	1700	1480
U.S. News	2	1	1	3
Room & board	10,150	$8,850	12,135	11,960
Tuition	28,157	$33,900	30,470	25,850
Avg. daily high temp.	56	56	64	59
Nearness to home	3	2	1	2
Athletics	1	3	1	3
Reputation	2	1	1	2
Size	2	1	4	2

Table 1.2.4: Converted data for Enrique's four schools

Then they have to assign 1 to the best value and 0 to the worst value in the range of each measure. For intermediate values, if the measure has a continuous scale, the common unit value may be assigned proportionally. For example, the average combined SAT score at Canisius is 1590. The range for this measure is 1400–1800, so 1400 would convert to 0, 1800 to 1, and the proportional value for Canisius is

$$(1590 - 1400)/(1800 - 1400) = 0.475.$$

Enrique and Anna decide to use proportional common units for each of the measures that have a continuous scale.

On the other hand, for categorical measures, after assigning the best value a 1 and the worst value a 0, Enrique and Anna have to decide how to apportion the common units. In some cases, apportionment might be proportional, while in other cases it might not. They decide to use proportional common units for athletics, reputation, and proximity to home. However, Enrique chooses "balanced" as the best value for reputation and "party school" as the worst. For the *U.S. News & World Report* ranking, they reason that there is a big difference between being ranked nationally or regionally. So, they decide to assign 1 to nationally ranked, 0.5 to regionally ranked, 0.25 to Tier 3, and 0 to Tier 4. Finally, Enrique really prefers a smaller school. Therefore, he assigns 1 to a size less than 3,000, 0.75 to a size between 3,000 and 6,000, 0.25 to a size between 6,000 and 12,000, and 0 to a size greater than 12,000.

To review, there are essentially three steps to rescale data to common units.

- Step 1: Assign 1 to the best value in the range.
  Assign 0 to the worst value in the range.
- Step 2: For continuous data, assign intermediate scores proportionally.
  For categorical data, assign intermediate scores proportionally or based on your own opinions and values.
- Step 3: Calculate scaled scores for proportional assignment.

$$\text{Scaled Score} = \frac{|\text{score} - \text{least preferred score}|}{\text{scale range}}$$

Table 1.2.5 contains the results of Enrique and Anna's rescaling of each measure to common units.

MEASURE	Canisius	Clark	Drexel	Suffolk
SAT score	0.475	0.875	0.750	0.200
U.S. News	0.50	1	1	0.25
Room & board	0.642	0.858	0.311	0.340
Tuition	0.684	0.110	0.453	0.915
Avg. daily high temp.	0.30	0.30	0.70	0.45
Nearness to home	1	0.5	0	0.5
Athletics	1	0	1	0
Reputation	1	0.5	0.5	1
Size	0.75	1	0	0.75

Table 1.2.5: Each measure rescaled to common units

From Table 1.2.3, Clark University has the highest average combined SAT score and the highest tuition and fees.

Q1. Why does it make sense in Table 1.2.5 that Clark has the highest common unit value on one of those measures, but the lowest common unit value on the other?

Conduct an Interview to Calculate Weights

Next, Enrique and Anna assign weights to each of the measures to reflect the relative importance Enrique attaches to each of them. They decide Anna will interview Enrique. She makes observations to ensure that Enrique understands the measures he chose and the effects of the weights he has assigned to each of them. As a reference tool during the interview, they create Table 1.2.6.

Anna: We have some measures and their ranges for making a decision about your college preference. Focus first on the column of least preferred values. Which one of the measures would you most want to increase from the least preferred value to its most preferred value? For example, is it more important to you to move the SAT score from 1100 to 1500 or reduce tuition from $20,000 to $4,000?

Enrique: Lower the tuition!

Anna: Are you sure that lowering the tuition to $4,000 is the most important improvement in the whole list?

Enrique: Yes, so I think we should rank tuition number one.

Anna: Enrique, what would be the next most important measure to move from least preferred to most preferred?

Enrique: *U.S. News & World Report* ranking is important, so let's rank that second, and SAT score third.

Criteria	Measure	Least preferred	Most preferred	Rank order	Points (0–100)	Weight (Points/Sum)
Academics	SAT score	1400	1800	3		
	U.S. News	Tier 4	Nat'l. Rank	2		
Cost	Room & board	$14,000	$8,000	4		
	Tuition	$35,000	$25,000	1		
Location	Avg. daily high temp.	50° F	70° F	9		
	Nearness to home	Within 1 hr.	Within 1 day	6		
Social life	Athletics	Div. 3	Div. 1	8		
	Reputation	Party	Balanced	7		
	Size	Over 12,000	Under 3,000	5		
				Sum:		

Table 1.2.6: Ranking and weighting the measures

The next task is to subjectively assign points from 0 to 100 for each measure based on the rank order so that the points assigned also reflect the relative importance of each measure. Here is what they do.

Anna: Let's start by assigning 100 points to the tuition range, which you've ranked first. Now, you've ranked *U. S. News & World Report* rating second. How important is this rating, from worst to best, compared to reducing the cost of tuition from $35,000 to $25,000? If it's close, you should use a number close to 100.

Enrique: I think it's about 90% as important, so let's use 90 points for that one, and SAT scores are almost as important, so we'll use 85 points for that range.

Table 1.2.7 contains the rest of the points Enrique has assigned to each of his measures.

Criteria	Measure	Least preferred	Most preferred	Rank order	Points (0–100)	Weight (Points/Sum)
Academics	SAT score	1400	1800	3	85	
	U.S. News	Tier 4	Nat'l. Rank	2	90	
Cost	Room & board	$14,000	$8,000	4	80	
	Tuition	$35,000	$25,000	1	100	
Location	Avg. daily high	50° F	70° F	9	20	
	Nearness to home	Within 1 hr.	Within 1 day	6	60	
Social life	Athletics	Div. 3	Div. 1	8	30	
	Reputation	Party	Balanced	7	50	
	Size	> 12,000	< 3,000	5	70	
				Sum :	585	

Table 1.2.7: Enrique's rank and point assignment

Anna: Enrique, what did you get for the total number of points for all your measures? Once you have the point total, you'll need to divide the points for each measure by this total to get the weight.

Enrique: I got 585 total points. Now I can calculate the weights.

The weights Enrique has calculated appear in Table 1.2.8.

Criteria	Measure	Least preferred	Most preferred	Rank order	Points (0–100)	Calculated weight (Points/Sum)
Academics	**SAT score**	1400	1800	3	85	0.145
	U.S. News	Tier 4	Nat'l. Rank	2	90	0.154
Cost	**Room & board**	$14,000	$8,000	4	80	0.137
	Tuition	$35,000	$25,000	1	100	0.171
Location	**Avg. daily high**	50° F	70° F	9	20	0.034
	Nearness to home	Within 1 hr.	Within 1 day	6	60	0.103
Social life	**Athletics**	Div. 3	Div. 1	8	30	0.051
	Reputation	Party	Balanced	7	50	0.085
	Size	> 12,000	< 3,000	5	70	0.120
				Sum =	585	1.000

Table 1.2.8: Enrique's assignment of weights to each measure

Anna: Enrique, what is the total weight for each criterion?

Enrique: I get a total of 0.299 for academics, 0.308 for cost, 0.137 for location, and 0.256 for social life.

Anna: Which criterion has the greatest weight assigned to it?

Enrique: It looks like cost, with 0.308.

Anna: Are there criteria with similar weights?

Enrique: It looks like academics and cost are almost the same.

Anna: Are these the criteria you feel are the most important criteria for choosing a college, and do you think they're about the same in importance?

Enrique: I didn't realize I placed so much importance on academics.

Anna: What did you expect to happen?

Enrique: I thought social life would be at the top of the list!

Anna: Well, you gave athletics only 30 points, reputation 50 points, and size 70 points. Do you want to change anything?

Enrique: No, I really think academics and cost are most important.

Calculate Total Scores

Enrique and Anna have determined which school is the best choice for him to attend.

They have used the data from Table 1.2.4, where common units were computed, and the weights calculated in the last column of Table 1.2.8 to calculate a score for each school on each measure. In Table 1.2.9 below, Enrique has calculated the product of the weight and the corresponding common unit: Score = Weight times Common Unit → Score = $W \times CU$. Then, by totaling the scores for each college in Table 1.2.9, Enrique learns which of his college choices best suits his needs.

MEASURE	WEIGHT	Canisius	Clark	Drexel	Suffolk
SAT score	0.145	**0.145 x 0.475 = 0.069**	0.127	0.109	0.029
U.S. News	0.154	0.077	0.154	0.154	0.038
Room & board	0.137	0.088	0.117	0.043	0.046
Tuition	0.171	0.117	0.019	0.077	0.156
Avg. daily high	0.034	0.010	0.010	0.024	0.015
Nearness to home	0.103	0.103	0.051	0	0.051
Athletics	0.051	0.051	0	0.051	0
Reputation	0.085	0.085	0.043	0.043	0.085
Size	0.120	0.090	0.120	0	0.090
Total Score	1.000	0.690	0.641	0.501	0.512

Table 1.2.9: Calculating the total scores of Enrique's schools

1.2.2 Interpreting the Results

Enrique reviews the results and notices that Drexel and Suffolk have scored much lower than his top-ranked choice and excludes them from further study. However, he decides to take a closer look at the relative strengths and weaknesses of Canisius, ranked first, and Clark, ranked second. There is only a 0.049 difference between the two, and he is not sure that it is enough evidence to make this critical life decision.

He decides to list all of the criteria for which one school or the other is better by at least 0.05 units. He notices that the largest difference is in the tuition, where Canisius holds a clear advantage. The other measures with a difference of at least 0.05 units are SAT scores and the *U. S. News & World Report* rankings, where Clark was better. Enrique looks back at his data in Table 1.2.3 to investigate further. He combines the tuition and room and board costs and notices the total cost for a year at Canisius will be $38,307 and a year at Clark will be $42,750. Clark's cost is 11.5% more than Canisius's. Based on Clark's superior scores on academics, Enrique decides that paying 11.5% more than he would at Canisius is a worthwhile investment. Based on the analysis above, Enrique has decided to attend Clark University.

Section 1.3: Judy Purchases a Used Car

Judy is trying to decide which used car to purchase from among four possibilities: a 2006 Honda Civic Hybrid, a 2006 Toyota Prius, and a 2007 Nissan Versa that she has found at dealerships, as well as a 2005 Ford Focus that Judy's uncle Roger is selling himself. Judy has asked her friend Dave to help her structure her thoughts in a consistent manner and use the steps in the process of multiple-criteria decision-making.

1.3.1 Applying the Process

Specify Criteria and Measures
With Dave's help, Judy has decided that the criteria most important for her choice of a used car are minimizing total cost and maximizing condition, accessories, and aesthetics. They have identified two measures for each criterion, as shown in Table 1.3.1.

Criterion	Measures
Total cost	1) Purchase price 2) Miles per gallon, based on the EPA rating when new
Condition	3) Odometer reading 4) Body condition
Accessories	5) Functional air conditioner and heater 6) Sound system
Aesthetics	7) Color 8) Body design

Table 1.3.1: Judy's list of criteria and measures

Scale Each Measure
Now that Judy and Dave have two measures for each of their criteria, they will establish a scale for each measure. The scales will be made to reflect Judy's personal preferences for the used car she will purchase.

Some measure scales, such as miles per gallon, are natural. Others, such as color, must be constructed. Both types of scale are completely subjective. Even the direction of the scale, which reflects an individual's preference, can differ. For example, one person may prefer a blue car, while another may prefer a red one. Thus, an individual's preferences will influence the scales as well as their ranges and directions. Because measures may be numeric or categorical, natural or constructed, four types of measures are theoretically possible: numeric-natural, categorical-natural, numeric-constructed, and categorical-constructed. Most of the time, however, only numeric-natural and categorical-constructed scales are used.

Miles per gallon is an example of a measure having a numeric-natural scale. If body condition is used as a measure and given a scale of 1 to 3 representing, fair, good, and excellent, this is an example of a categorical-constructed scale. Next, each of the four categories would need to be defined. For example, the definition of excellent condition at *Kelley Blue Book* online is as follows:

"Excellent" condition means that the vehicle looks new, is in excellent mechanical condition and needs no reconditioning. This vehicle has never had any paint or body work and is free of rust. The vehicle has a clean title history and will pass a smog and safety inspection. The engine compartment is clean, with no fluid leaks and is free of any wear or visible defects. The vehicle also has complete and verifiable service records. Less than 5% of all used vehicles fall into this category.

Table 1.3.2 displays each of Judy's measures, its type, and the range she has decided to use.

Measure	Type	Scale range
1) Purchase price	Numeric-natural	\$6,000–\$16,000
2) Miles per gallon (EPA rating when new)	Numeric-natural	20–50 mpg
3) Odometer reading	Numeric-natural	50,000–125,000 miles
4) Body condition	Categorical-constructed	1. Fair 2. Good 3. Excellent
5) Functional air conditioner and heater	Categorical-constructed	1. Neither works 2. Only one works 3. Both work
6) Sound system	Categorical-constructed	1. None 2. Radio only 3. Radio and CD player 4. Radio, CD, MP3
7) Color	Categorical-constructed	1. Blue 2. Red 3. Silver 4. White 5. Black
8) Body design	Categorical-constructed	1. Wagon 2. Hatchback 3. Sedan

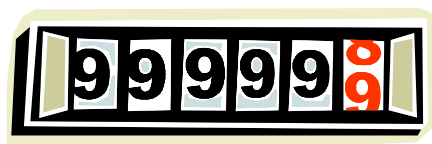
Table 1.3.2: The type and range for each of Judy's measures

Collect Data

Judy and Dave have collected the data in Table 1.3.3.

MEASURE	Honda Civic Hybrid	Toyota Prius	Ford Focus	Nissan Versa
Purchase price	$15,000	$15,500	$7,700	$11,000
Miles per gallon	43	46	25	33
Odometer reading	85,000	80,000	95,000	65,000
Body condition	2	2	2	3
A/C and heater	3	3	3	3
Sound system	3	3	4	3
Color	2	3	1	4
Body design	3	3	1	2

Table 1.3.3: Judy's data on four used cars



Rescale to Common Units

The raw data from each automobile for each measure must be rescaled, so that the different measures may be compared directly. Each measure is rescaled to a common unit, from 0 to 1, with 1 representing the most desired and 0 representing the least desired. This process varies depending on the nature of the measure. The common unit for measures that are numeric-natural may be calculated proportionally. For example, purchase price probably has an actual range of $1,000 to $25,000 or more. However, using the more realistic $6,000 to $16,000 range, common units are assigned based on where each car's purchase price falls within that $10,000 range. In this case, $6,000, the most preferred price, is assigned 1 point, and $16,000, the least preferred price, is assigned 0 points. Then, if the purchase price of a given car is *x*, where $\$6,000 \le x \le \$16,000$, then the common unit assigned to *x* would be $\frac{16,000 - x}{10,000}$. Of course, this analysis assumes that lower prices are desirable. On the other hand, we presume higher miles per gallon is desirable.

Q1. How should the common units be calculated in that case?

In the case of categorical-constructed measures, the assignment of the common units from 0 to 1 is completely subjective. This subjectivity reflects Judy's preferences for the criterion being measured. Sometimes, the common units may be assigned to a categorical-constructed measure proportionally. Other times, the common units may be assigned in a non-proportional manner that demonstrates a strong preference in one direction or the other on the scale. Judy has used proportional assignment for all of her categorical measures except for sound system. In that case, she has reasoned that having no sound system is highly undesirable and having any sound system at all is a significant improvement. Therefore, she has assigned the common unit values so that they are loaded toward the upper extreme (0, 0.5, 0.75, 1), rather than assigned proportionally (0, 0.33, 0.67, 1). Table 1.3.4 contains the range of the common units for each measure, and Table 1.3.5 contains the common units after rescaling the measures for each of the used cars Judy is considering purchasing.

Measure	Scale range	Range of common units
Purchase price	$6,000–$16,000	0–1, Proportional
Miles per gallon	20–50 mpg	0–1, Proportional
Odometer reading	50,000– 00,000	0–1, Proportional
Body condition	1. Fair 2. Good 3. Excellent	0 0.5 1
A/C and heater	1. Neither works 2. Only one works 3. Both work	0 0.5 1
Sound system	1. None 2. Radio only 3. Radio and CD player 4. Radio, CD, MP3	0 0.5 0.75 1
Color	1. Blue 2. Red 3. Silver 4. White 5. Black	0.75 0.5 1 0 0.25
Body design	1. Wagon 2. Hatchback 3. Sedan	0 0.5 1

Table 1.3.4: The range of Judy's common units for each measure

	Common Units (Data rescaled to between 0 and 1)			
Measure	**Honda Civic Hybrid**	**Toyota Prius**	**Ford Focus**	**Nissan Versa**
Purchase price	0.100	0.050	0.830	0.500
Miles per gallon	0.767	0.867	0.167	0.433
Odometer reading	0.300	0.400	0.100	0.700
Body condition	0.500	0.500	0.500	1.000
Air conditioner and heater	1.000	1.000	1.000	1.000
Sound system	0.750	0.750	1.000	0.750
Color	0.500	1.000	0.750	0.000
Body design	1.000	1.000	0.000	0.500

Table 1.3.5: Judy's measures rescaled to common units

The earlier discussion of a realistic range for each measure is a critical issue. For example, in the case of miles per gallon, the use of an appropriate range can affect the results. To see how that could happen, suppose we first work with a range of 10 to 60 mpg. Then the Ford Focus, at 25 mpg, converts to a common unit score of 0.30, while the Toyota Prius, at 46 mpg, would have a common unit score of 0.72. In that case, the common unit score for the Focus is a little less than half that of the Prius. If we use a more realistic range of 20 to 50 mpg, the corresponding common unit scores are 0.17 and 0.87, a far more dramatic difference.

Conduct an Interview to Calculate Weights

Dave should informally interview Judy to try to ensure that the weights they assign to the criterion measures are appropriate to the relative importance Judy places on them. In fact, such an interview is part of the multiple-criteria-decision-making process in every context. During the interview, Judy and Dave will rank-order her set of measures, assign points across the range of each measure, and calculate a weight for each measure. The interview should end with a review of the weights assigned to each measure to be sure that they reflect their relative importance to Judy. The weights of the measures within each criterion should also be summed and compared to ensure that the weights assigned to each criterion are appropriate to Judy's preferences. A table similar to Table 1.3.6 will help Judy and Dave keep track of the weighting process.

Criterion	Measures	Least preferred	Most preferred	Rank order	Points (0–100)	Weight
Total cost	Purchase price	$16,000	$6,000			
	Fuel economy	20 mpg	50 mpg			
Condition	Odometer reading	100,000 mi	50,000 mi			
	Body condition	1 (fair)	3 (excellent)			
Accessories	A/C and heater	1 (neither works)	3 (both work)			
	Sound system	1 (none)	4 (radio, CD, MP3)			
Aesthetics	Color	1 (blue)	5 (black)			
	Body design	1 (wagon)	3 (sedan)			
				Sum =		

Table 1.3.6: Rank ordering, point assignment, and weight calculation for each measure

Rank-Order Measures

The informal interview process helps to minimize ambiguity. At this point, Judy and Dave will need to work together. Included below is an interview between our fictitious students. Dave is the interviewer, and Judy is the interviewee. They are trying to determine how Judy should rank-order her measures. The rank-ordering process should pay explicit attention to the range of each measure. The goal of this task is to determine for which measure the range is most significant to Judy—that is, which measure Judy would most want to increase from the least preferred to the most preferred value.

Dave: You have some measures and their ranges for deciding which used car you prefer to buy. Look first in the column of least preferred values. Which one of the measures would you most want to move from the least preferred value to its most preferred value? For example, is it more important to you to move the body design score to sedan or reduce the purchase price from $16,000 to $6,000?

Judy: Lower the purchase price!

Dave: Is lowering the purchase price to $6,000 the most important improvement in the whole list?

Judy: I think so. If I spend too much, I might not have enough left for gas and insurance. Then there wouldn't even be a wagon, let alone a sedan.

Dave: OK, so purchase price is number one. What would you rank second?

Judy: I think fuel economy is important, because I don't want to have to spend a lot on gas. So I'll rank that second, and body design can be third.

Judy and Dave continue this way until all of the measures have been ranked.

Q2. Which measure would you have ranked first?

Q3. Which would you have ranked second?

Points for Range

Once Judy and Dave have decided for which measure the difference between the least and most preferred scores is most important to Judy, they assign that top measure 100 points. Then, for each of the remaining measures, they must compare the importance of the difference between the least and most preferred values of this measure to that of the highest ranked measure. This comparison is captured by assigning an appropriate score between 0 and 100 in turn to each of the other measures.

The interview between Judy and Dave continues:

Dave: Start by assigning 100 points to the purchase price range, which you ranked first. Now, you ranked fuel economy second. How important is this rating, from worst to best, compared to reducing the purchase price from $16,000 to $6,000? If it's close, you should use a number close to 100.

Judy: I think purchase price is slightly more important than fuel economy, so let's use 95 points for fuel economy and then 75 for body design.

Judy and Dave continue in this manner until points have been assigned to each of the measures.

Q4. Would you have assigned the points for your top two measures as closely as Judy has done?

Calculate the Weight

Finally, the weight for each item is calculated by dividing the points allotted the item by the total number of points for all items. In essence, this amounts to rescaling the point allotments then adding up the weights.

Q5. What would it mean if the sum of the weights were 0.99?

Q6. Why might this happen?

Judy and Dave's interview concludes.

Dave: Judy, what did you get for the total number of points for all your measures? Once you have the point total, you'll need to divide the points for each measure by this total to get the weight.

Judy: I got 500 total points, and I recorded the weights in my table.

Dave: Judy, what is the total weight for each criterion?

Judy: I get a total of 0.39 for total cost, 0.17 for accessories, 0.27 for condition, and 0.17 for aesthetics.

Dave: Which criterion has the greatest weight assigned to it?

Judy: It looks like total cost, with 0.39, is way higher than the rest.

Dave: Is that the criterion you feel is the most important for choosing which used car you want to buy?

Judy: Yes, but I didn't realize I placed so much importance on it.

Dave: Do you want to change anything?

Judy: No, I really think total cost is the most important, by far.

Table 1.3.7 shows Judy's rank ordering of her measures, point assignment, and weights for each measure.

Criterion	Measures	Least preferred	Most preferred	Rank order	Points (0–100)	Weight
Total cost	Purchase price	$16,000	$6,000	1	100	0.200
	Fuel economy	20 mpg	50 mpg	2	95	0.190
Condition	Odometer reading	100,000 mi	50,000 mi	5	60	0.120
	Body condition	1 (fair)	3 (excellent)	3	75	0.150
Accessories	A/C and heater	1 (neither works)	3 (both work)	7	35	0.070
	Sound system	1 (none)	4 (radio, CD, MP3)	6	50	0.100
Aesthetics	Color	1 (blue)	5 (black)	8	10	0.020
	Body design	1 (wagon)	4 (sedan)	3	75	0.150
				Sum =	500	1.000

Table 1.3.7: Judy's rank ordering, point assignment, and weight calculation for her measures

Table 1.3.8 shows the weights for each measure and the common unit for each measure for each car.

		Common Units (Data rescaled to between 0 and 1)			
Measure	**Weight**	**Honda Civic Hybrid**	**Toyota Prius**	**Ford Focus**	**Nissan Versa**
Purchase price	0.200	0.100	0.050	0.830	0.500
Fuel economy	0.190	0.767	0.867	0.167	0.433
Odometer reading	0.120	0.300	0.400	0.100	0.700
Body condition	0.150	0.500	0.500	0.500	1.000
A/C and heater	0.070	1.000	1.000	1.000	1.000
Sound system	0.100	0.750	0.750	1.000	0.750
Color	0.020	0.500	1.000	0.750	0.000
Body design	0.150	1.000	1.000	0.000	0.500

Table 1.3.8: Judy's weights and common units for each measure

Calculate Total Scores
Table 1.3.8 provides the weights for each measure and the common units for each car with regard to each measure. Judy and Dave must calculate an overall score for her used cars. To do this, the common unit value assigned to each measure is multiplied by the weight allotted to that measure.

For example, purchase price has a weight of 0.20, and the Honda Civic Hybrid common unit value for purchase price is 0.1. So, the score for purchase price assigned to the Honda Civic Hybrid is (0.20)(0.1) = 0.02. The total score for each used car is the sum of these products. The highest total score represents Judy's preferred choice. Table 1.3.9 shows these computations for our fictitious student, Judy. A graphing calculator is helpful because it relieves most of the computational load. Using spreadsheet software is even easier and is demonstrated in the next section.

	Weight * Common Unit			
Measure	**Honda Civic Hybrid**	**Toyota Prius**	**Ford Focus**	**Nissan Versa**
Purchase price	0.0200	0.0100	0.1660	0.1000
Fuel economy	0.1463	0.1653	0.0323	0.0817
Odometer reading	0.0636	0.0720	0.0480	0.0960
Body condition	0.0750	0.0750	0.0750	0.1500
A/C and heater	0.0700	0.0700	0.0700	0.0700
Sound system	0.0750	0.0750	0.1000	0.0750
Color	0.0100	0.0200	0.0150	0.0000
Body design	0.0750	0.1500	0.0000	0.0495
Total Score	0.5349	0.6373	0.5063	0.6222

Table 1.3.9: Judy's calculation of the total score for each used car

1.3.2 Interpret the Results

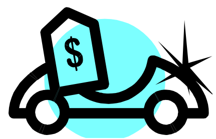
Judy's top two choices are very close.

Q7. What significant advantages does the Toyota Prius have over the Nissan Versa?

Q8. What significant advantages does the Nissan Versa have over the Toyota Prius?

Q9. Based on the results of the multiple-criteria-decision-making process Judy has used, which used car should she purchase? Why?

When the numeric values are this close, the ultimate answer may be that the decision maker will be equally satisfied with either choice. There may not be a need to dig deeper and determine to the fourth decimal place which is really and truly preferred.



1.3.3 Using Excel to Calculate Total Scores

An Excel Spreadsheet has an advantage over a calculator. It is able to display the entire matrix of information. The data in Table 1.3.8 were input into an Excel spreadsheet as presented in Figure 1.3.1 Column B in the spreadsheet has all of the weights. These are stored in cells B4 through B11. The common units for the Honda Civic Hybrid are stored in cells C4 through C11. The first calculation involves multiplying the common units by the weights. We multiplied the value in cell B4 by the value in cell C4. This product of two cells, C4*$B4, was stored in cell C18 as displayed in Figure 1.3.2. Notice the actual formula is C4*$B4 and we explain the reason for the $ sign later.

	A	B	C	D	E	F
1			Data Rescaled to Common Units (Between 0 and 1)			
2			Honda	Toyota	Ford	Nissan
3	Measure	Weights	Civic Hybrid	Prius	Focus	Versa
4	Purchase price	0.2	0.1000	0.0500	0.8300	0.5000
5	Miles per gallon	0.19	0.7670	0.8670	0.1670	0.4330
6	Odometer reading	0.12	0.3000	0.4000	0.1000	0.7000
7	Body condition	0.15	0.5000	0.5000	0.5000	1.0000
8	Air conditioner and heater	0.07	1.0000	1.0000	1.0000	1.0000
9	Sound system	0.1	0.7500	0.7500	1.0000	0.7500
10	Color	0.02	0.5000	1.0000	0.7500	0.0000
11	Body design	0.15	1.0000	1.0000	0.0000	0.5000

Figure 1.3.1: The data from Table 1.3.8 input into a spreadsheet

Excel has an important capability. Once we have written the formula, we can copy and paste the same formula into other cells. Excel has simple rules for updating the formula. For example if we copy cell C18 to C19, the formula in C19 will be C5*$B5. We have copied the cell to the next row. Excel simply updates the formula from C4*$B4 to C5*$B5. In that way we can complete the calculations for C20 through C25.

The total score for each vehicle is just the sum of the column values. The Sum command in Excel does this calculation. In Cell C26 we placed the following statement.

=Sum(C18:C25)

This tells the spreadhseet to sum the values in all of the cells from C18 through C25. The colon means include all of the cells in between. Now consider calculating the scores for the Toyota Prius and placing the values in column D. If we were to copy and paste cell C18 into cell D18 without using a $ sign, the resultant formula would be D4*C4. Excel would update both cell references by one letter in the alphabet. However, in all of our calculations, we always want to use the same column B. It has the weights and they apply to all vehicles. By placing a $ sign before the B, Excel knows not to change column letter as we copy and paste. Now, when we copy and paste cell C19 into cell D19, the result is D4*$B4. We can then copy and paste C18 into all of the cells of the table from D18 through F25. We can also copy and paste the summation formula stored in cell C26 into the adjacent cells D26 through F26. The results are displayed in Figure 1.3.2. This is exactly what was displayed in Table 1.3.9

C18 =C4*$B4

	A	B	C	D	E	F
15			**Weight * Common Unit**			
16			**Honda**	**Toyota**	**Ford**	**Nissan**
17	**Measure**		**Civic Hybrid**	**Prius**	**Focus**	**Versa**
18	Purchase price		0.0200	0.0100	0.1660	0.1000
19	Miles per gallon		0.1457	0.1647	0.0317	0.0823
20	Odometer reading		0.0360	0.0480	0.0120	0.0840
21	Body condition		0.0750	0.0750	0.0750	0.1500
22	Air conditioner and heater		0.0700	0.0700	0.0700	0.0700
23	Sound system		0.0750	0.0750	0.1000	0.0750
24	Color		0.0100	0.0200	0.0150	0.0000
25	Body design		0.1500	0.1500	0.0000	0.0750
26	**Total Score**		**0.5817**	**0.6127**	**0.4697**	**0.6363**

Figure 1.3.2: Judy's spreadsheet calculation of the scores for each used car

In Figure 1.3.2, we see all of the components of the total score. Each cell contains the product, Weight*Common Unit. If all we wanted was the final score, Excel has a simple command. This command is the SUMPRODUCT. In Figure 1.3.3 we placed in cell C12 the command =SUMPRODUCT(C4:C11,$B4:$B11). This command performs the following calculation. It multiplies C4 by B4, then C5 by B5,… until C11 by B11 and sums these products. This is done in just one command. We used a $ sign so we can copy the contents of C12 into D12 through F12. The form of the command involves stating the range of values in Column C as C4:C11. This range is followed by a comma before giving the second range $B4:$B11. There must be exactly the same number of cells in each range for the command to work.

C12 =SUMPRODUCT(C4:C11,$B4:$B11)

	A	B	C	D	E	F
1			Data Rescaled to Common Units (Between 0 and 1)			
2			**Honda**	**Toyota**	**Ford**	**Nissan**
3	**Measure**	**Weights**	**Civic Hybrid**	**Prius**	**Focus**	**Versa**
4	Purchase price	0.2	0.1000	0.0500	0.8300	0.5000
5	Miles per gallon	0.19	0.7670	0.8670	0.1670	0.4330
6	Odometer reading	0.12	0.3000	0.4000	0.1000	0.7000
7	Body condition	0.15	0.5000	0.5000	0.5000	1.0000
8	Air conditioner and heater	0.07	1.0000	1.0000	1.0000	1.0000
9	Sound system	0.1	0.7500	0.7500	1.0000	0.7500
10	Color	0.02	0.5000	1.0000	0.7500	0.0000
11	Body design	0.15	1.0000	1.0000	0.0000	0.5000
12	**Total Score**	**1**	**0.5817**	**0.6127**	**0.4697**	**0.6363**

Figure 1.3.3: Using the SUMPRODUCT(C4:C11,$B4:$B11) to calculate total score

Section 1.4: Chapter 1 (MCDM) Homework Questions

1. Olivia wants to pursue a career in medicine, but she is not sure which profession would be best for her. After some preliminary research, she narrows her choices to physician, nurse, and pharmacist. Olivia decides to consider four criteria to help structure her decision: professional preparation, personal fulfillment, financial compensation, and lifestyle. The table below shows these criteria and the measures she has decided to use for each.

Criterion	Measure	Type of Scale	Type of Data
Professional Preparation	Schooling		
	Internship		
	Difficulty		
Personal Fulfillment	Job satisfaction		
	Personal interest		
Financial Compensation	Initial salary		
	Median salary		
Lifestyle	Likely schedule		
	Maternity leave		
	Prestige		

a. Decide which type of scale would be appropriate for each measure, either *continuous-natural* or *categorical-constructed*.

b. Determine which of the data will have to be collected through *research* and what will be based on personal *opinion*.

c. The table below shows some of the data Olivia has collected for the professional preparation criterion. Based on the scale ranges, determine what you would consider most preferred and least preferred for each measure.

Criterion	Measure	Scale Range	Physician (M.D.)	Nurse (R.N.)	Pharmacist (Pharm.D.)
Professional Preparation	Schooling (years)	2-8	8	4	6
	Internship (years)	0-4	3	0	1
	Difficulty (rank)	1-3	1	3	2

d. What else must be done before obtaining common unit values?

e. Fill in the following table with scores scaled to common units.

Criterion	Measure	Physician (M.D.)	Nurse (R.N.)	Pharmacist (Pharm.D.)

Professional Preparation	Schooling			
	Internship			
	Difficulty			

f. Suppose Olivia weights Schooling at 0.109, Internship at 0.091, and Difficulty at 0.073. Complete the following table with the weighted scores.

Criterion	**Measure**	**Physician (M.D.)**	**Nurse (R.N.)**	**Pharmacist (Pharm.D.)**
Professional Preparation	Schooling			
	Internship			
	Difficulty			

2. Rana is trying to decide what type of engineering to study in college. She obtained the following data after an initial search, but now she wants to take a more organized and methodical approach.

	Mechanical	**Electrical**	**Chemical**	**Industrial**
Starting salary ($)	54,128	55,292	59,361	55,067
Median salary ($)	75,130	82,090	84,240	73,490
Job Satisfaction (%)	83.0	79.9	83.8	80.5

a. Create a list of four criteria she could use to help make this decision. Try to find criteria that would use these data.

b. Identify two measures for each criterion.

3. Give an example of a measure that uses a continuous scale, but might not be converted to common units proportionally. Explain your answer.

4. Give an example of a measure that uses a categorical scale, but might not be converted to common units proportionally. Explain your answer.

5. In problem 1.1 of the chapter, Isabelle Nueva is helping her mother and father decide on the best cell phone plan for her family.
a. What additional measures do you think should be considered?

b. Add and describe a categorical measure for the problem and create 3 categories for this new measure.

c. Add and describe a numerical measure for the problem.

6. In problem 1.2 of the chapter, Enrique Ramirez is selecting a college to attend.
a. What additional measures do you think should be considered?

b. Add and describe a categorical measure for the problem and create 3 categories for this new measure.

c. Add and describe a numerical measure for the problem.

7. In problem 1.3 of the chapter, Judy is choosing which used car to purchase from among four possibilities.
   a. What additional measures do you think should be considered?

   b. Add and describe a categorical measure for the problem and create 3 categories for this new measure.

   c. Add and describe a numerical measure for the problem.

8. A high school student wants to buy a digital camera. Checking the experts' recommendations, she creates a list of important features and ranks them as follows. She ranks Price as the most important measure and, therefore, assigns 100 points it. Brand name is slightly less important than price. It is ranked 2^{nd} and she assigns 90 points to it. She thinks that having an Anti-shake system is much less important than brand name and assigns 60 points to it. Size of view screen is ranked below anti-shake system and has a little bit less importance, thus she assigned it 55 points. Finally, ease of use is the least important factor with 40 points. Calculate the weight assigned to each measure.

Measure	Rank	Points	Weight
Size of view screen	4	55	
Price	1	100	
Brand name	2	90	
Anti-shake system	3	60	
Easy to use	5	40	
	Total		

9. Suppose you are looking to buy a digital camera for yourself.
   a. Suggest and add a relevant categorical measure in the table below. Describe the new measure.

   b. Suggest and add a relevant numerical measure in the table below. Describe the new measure.

   c. Use your personal preferences and rank the measures. Then, assign points to each measure and calculate the weight of each measure.

	Assign		**Calculate**
Measure	**Rank**	**Points**	**Weight**
Size of view screen			
Price			
Brand name			
Anti-shake system			
Easy to use			
New categorical measure:			

New numerical measure:			
Total			

10. Kim is interested in purchasing a desktop computer for her office. After reviewing the specification of different models, she ended up with the following measures. Classify each measure as numerical or categorical.

Measure	**Type: Numeric or Categorical**
Computational power	
Monitor size	
Years of warranty	
Operating system	
Price	

11. A high school has selected one of its students to be the chair of a committee planning a class trip. One of her first responsibilities is to pick a co-chair for planning the trip. Suggest two measures for each criterion. Specify the type of each measure.

Criterion	**Measure**	**Type: Numerical or Categorical**
Knowledge		
Reliability		
Personality		

12. Jay is a movie fan and is considering two companies that offer DVD rental membership, Netco and DVDco. Netco is a completely web-based company but DVDco has branch stores in addition to their web site.

a. Having studied the web sites of the two main companies, Jay summarized the measures and data as follows:

Measure	**Netco**	**DVDco**
Number of available movies	100,000	75,000
Price	10	8
Web streaming	yes	no
Number of DVDs that can be out at the same time	1	2
Availability of a recommendation system	no	yes
In-store return	no	yes

Specify the range for each measure and then determine the common unit for each of them. Insert the common units in the following table.

Measure	Netco	DVDco
Number of available movies		
Price		
Web streaming		
Number of DVDs that can be out at the same time		
Availability of a recommendation system		
In-store return		

After considering the measures, Jay ranked the measures and assigned the points as indicated below. Calculate the weight of each measure based on the points.

Measure	Rank	Points	Weight
Number of available movies	4	75	
Price	1	100	
Web streaming	2	90	
Number of DVDs that can be out at the same time	5	60	
Availability of a recommendation system	6	50	
In-store return	3	85	

b. For each alternative, calculate the product of the weight and the corresponding common unit for each measure. Determine the total score for each alternative.

Measure	Netco	DVDco
Number of available movies		
Price		
Web streaming		
Number of DVDs that can be out at the same time		
Availability of a recommendation system		
In-store return		
Total Score		

c. Which alternative is ranked 1st, and what measures contribute the most to it being ranked 1st?

13. Sam and his wife were just married and are looking for an apartment in a safe area close to Sam's school. After discussing their preferences, they came up with the following measures that are very important to them.

Measure	Description
Spaciousness	Size and design
Price	Monthly rental
Condition	Freshly painted, floors, age of appliances
Apartment building rating	Based on rating of previous tenants' rating in www.apartmentrating.com

a. After searching in a 10-mile radius around his school, they ended up with the following three apartments they like. Sam summarized the data as follows:

Measure	Ap1	Ap2	Ap3
Spaciousness	Good	Medium	Poor
Price ($/month)	700	650	550
Condition (0-1)	0.6	0.9	0.7
Apartment building rating (between 1 and 5)	4	4.5	3.8

b. Specify the range for each measure and then determine the common unit for each of them. Insert the common units in the following table.

Measure	Ap1	Ap2	Ap3
Spaciousness			
Price			
Condition(0-1)			
Apartment building rating			

14. After considering the measures, Sam and his wife ranked the measures as in the following table. Use the assigned points to calculate the weights.

Measure	Rank	Points	Weight
Spaciousness	3	70	
Price	1	100	
Condition(0-1)	2	90	
Apartment Building Rating	4	50	

15. For each alternative, calculate the product of the weight and the corresponding common unit for each measure. Determine the total score for each alternative.

Measure	Ap1	Ap2	Ap3
Spaciousness			
Price			
Condition			
Apartment Building Rating			
Total Score			

16. Which alternative is ranked 1st and what measures contribute the most to it being ranked 1st?

17. James and George are seeking a team member for their capstone project. It is a very demanding project that requires a wide range of skills. To help evaluate potential teammates, they created the following list of measures.

Measure
Writing Skills
GPA of Math courses
Total GPA
Reliability and commitment
Communication skills

a. After considering all their classmates who were not yet assigned to any project, they ended up with following three people. They summarized the data for these three as follows:

Measure	**Ed**	**Ken**	**Thad**
Writing skills	Excellent	Acceptable	Good
GPA in math courses	3.8	3.9	3.5
Total GPA	3.6	3.8	3.7
Reliability and commitment	Acceptable	Good	Good
Communication skills	Good	Excellent	Excellent

b. Specify the range for each measure and then determine the common unit for each of them. Insert the common units in the following table.

Measure	**Ed**	**Ken**	**Thad**
Writing skills			
GPA in math courses			
Total GPA			
Reliability and commitment			
Communication skills			

c. They are not sure how to rank the measures. Based on your personal preferences, rank the measures and fill out the rest of table.

Measure	**Rank**	**Points**	**Weight**
Writing skills			
GPA in math courses			
Total GPA			
Reliability and commitment			
Communication skills			
	Total		

d. For each alternative, calculate the product of the weight and the corresponding common unit for each measure. Determine total score for each alternative.

Measure	**Ed**	**Ken**	**Thad**
Writing skills			
GPA in math courses			
Total GPA			
Reliability and commitment			
Communication skills			
Total Score			

18. Which alternative is ranked 1st and what measures contribute the most to him being ranked 1st?

19. Neil is trying to find a location in Michigan to open a convenience store. Location is very important for convenience stores. Thus, he wants to be very precise in this process. After talking to some consultants and other store managers, he plans to use the following measures.

Measure	Description
Traffic through intersection	Daily number of the cars passing the intersection
Population within 2 mile	Total population over the age of 15
Distance to the nearest competitor	Miles to nearest convenience store
Cost of the property	Purchase price of property

a. After considering all available properties in the area, he ends up with the following three locations. The data for these three locations is summarized below.

Measure	L1	L2	L3
Traffic through intersection (vehicles)	16,000	15,000	19,000
Population within 2 miles	50,000	45,000	55,000
Distance to the nearest competitor (miles)	1.5	2	0.5
Cost of the property ($)	210,000	180,000	250,000

b. Specify the range for each measure and then determine the common unit for each of them. Insert the common units in the following table.

Measure	L1	L2	L3
Traffic through intersection			
Population within 2 mile			
Distance to the nearest competitor			
Cost of the property			

20. After considering the measures, he ranks the measures as in the following table. Use assigned points to calculate the weights.

Measure	Rank	Point	Weight
Traffic through intersection	2	85	
Population within 2 mile	3	80	
Distance to the nearest competitor	4	70	
Cost of the property	1	100	
	Total		

21. For each alternative, calculate the product of the weight and the corresponding common unit for each measure. Determine total score for each alternative.

Measure	L1	L2	L3
Traffic through intersection			
Population within 2 mile			
Distance to the nearest competitor			
Cost of the property			
Total Score			

22. Which alternative is ranked 1st and what measures contribute the most to it being ranked 1st?

Chapter 1 Summary

What have we learned?

We have learned that the multi-criteria decision making process provides a framework for making a subjective decision when considering several alternatives, each of which has advantages and disadvantages. As the person making the decision, you must structure the decision. What criteria or objectives will be considered? What measures of your criteria will be included? How will you rank and weight these measures to help make a decision that is best for your values and priorities?

This process allows for direct comparison and evaluation of complex alternatives. The steps are as follows:

1. Identify Criteria and Measures
2. Collect Data
3. Find the Range of Each Measure
4. Rescale Each Measure to a Common Unit
5. Conduct an Interview to Calculate Weights
6. Calculate Total Scores
7. Interpret Results

Terms

Categorical Measure	A measure whose scores are classifications
Common Unit	A value that varies from 0 to 1, where 0 always represents the worst value, 1 the best value, and intermediate values are found using a proportional scale
Continuous Measure	A measure whose scores are numeric values that can take on any value in a certain range
Criteria	Objectives or aspects of the alternatives that you wish to either maximize or minimize
Measure	A trait that will quantify an aspect of a criterion
Proportional Scale	The rescaled score for intermediate values of continuous measures (calculated by dividing the difference between the particular score and the least preferred score by the scale range)
Scale Range	The range of possible values for each measure.
Total score	For each alternative, multiply the rescaled score by the weight for each measure. The sum of all these weighted, rescaled scores is the total score.
Weighted Scores	The rescaled score for each measure, weighted according to its importance (calculated by multiplying each scaled score by the corresponding weight of the measure)

Chapter 1 (MCDM) Objectives

You should be able to:

- List the sequence of steps in the multi-criteria decision making process
- Explain the purpose of each step in the process
- Identify criteria you will use to choose between several alternatives
- Select measure(s) for each criterion
- Distinguish between categorical and continuous measures
- Determine scale types and ranges for measures
- Scale scores
- Rescale scores to common units
- Weight scores for each measure
- Calculate a total score for each alternative
- Evaluate the results of the multi-criteria decision making process by comparing the strengths and weaknesses of the top two alternatives

Chapter 1 Study Guide

1. Explain why the Multi-Criteria Decision Making (MCDM) process is useful.
2. Discuss the differences between a *criterion* and a *measure*.
3. When choosing between the same alternatives, why might you and a classmate, both using MCDM, come to a different decision?
4. Compare and contrast *continuous* and *categorical* measures.
5. Give an example of a scale range in which one end is most preferable for you, but the other end may be preferable to a classmate. Explain.
6. Why do we scale all scores between zero and one?
7. Describe how scores are scaled differently for continuous and categorical measures.
8. Describe how scaled scores are rescaled to common units differently for continuous and categorical measures.
9. Identify which steps in MCDM involve you inserting your own preferences and priorities into the process and describe how this occurs?
10. What role do the weights of the measures play in determining which alternative is the best?
11. Describe the process that occurs from collecting raw data for measures to obtaining a total score for an alternative.
12. Should you always choose the alternative with the highest total score?

References

Lenhart, A. (2010). *Teens, cell phones and texting*. Pew Internet & American Life Project: Pew Research Center.

Section 2.0: Mathematical Programming

The next five chapters in the text focus on mathematical programming. The father of mathematical programming is George Dantzig. Between 1947 and 1949, Dantzig developed the basic concepts used for framing and solving linear programming problems. During WWII, he worked on developing various plans which the military called "programs." After the war he was challenged to find an efficient way to develop and solve these programs.

Dantzig recognized that these programs could be formulated as a system of linear inequalities. Next, he introduced the concept of a goal. At that time, goals usually meant rules of thumb for carrying out a goal. For example, a navy admiral might have said, "Our goal is to win the war, and we can do that by building more battleships." Dantzig was the first to express the selection of a plan to reach a goal as a mathematical function. Today it is called the *objective function*.

All of this work would not have had much practical value without a way to solve the problem. Dantzig found an efficient method called the simplex method. This mathematical technique finds the optimal solution to a set of linear inequalities that maximizes (profit) or minimizes (cost) an objective function.

Economists were excited by these developments. Several attended an early conference on linear programming and the simplex method called "Activity Analysis of Production and Allocation." Some of them later won Nobel prizes in economics for their work. They were able to model fundamental economic principles using linear programming.

The first problem Dantzig solved was a minimum cost diet problem. The problem involved the solution of nine inequalities (nutrition requirements) with seventy-seven decision variables (sources of nutrition). The National Bureau of Standards supervised the solution process. It took the equivalent of one man working 120 days using a hand-operated desk calculator to solve the problem. Nowadays, a standard personal computer could solve this problem in less than one second. Excel spreadsheet software includes a standard "add-in" called "solver", a tool for solving linear programming problems.

Mainframe computers became available in the 1950s and grew more and more powerful. This allowed many industries, such as the petroleum and chemical industries, to use the simplex method to solve practical problems. The field of linear programming grew very fast. This led to the development of non-linear programming, in which inequalities and/or the objective function are not linear functions. Another extension is called integer programming, in which the variables can only have integer values. Together, linear, non-linear and integer programming are called **mathematical programming**.

2.0.1 An Introductory Problem

In order to get a feel for mathematical programming, this chapter begins with a problem that has a concrete model. This model can be built from Lego pieces. When a mathematical model of a real world situation is constructed in symbolic form, it is often helpful to construct a physical or visual model at the same time. The role of the latter model is to help the model builder to understand the real-world situation as well as its mathematical model.

The Problem
A certain furniture company makes only two products: tables and chairs. The manufacturing of tables and chairs can be modeled using Lego pieces. To make a table requires two large and two small pieces, and a chair requires one large and two small pieces. Figure 2.0.1 shows a table and a chair made from Legos.

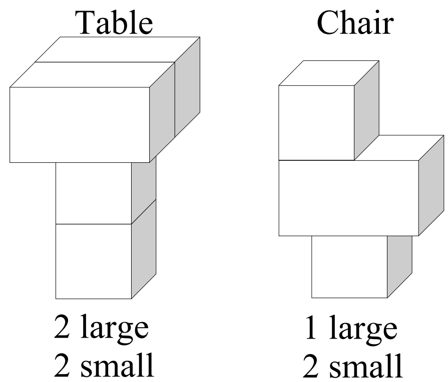


Figure 2.0.1: A Lego table and chair

If the resources needed to build tables and chairs were unlimited, the company would just manufacture as many of each as it thought it could sell. In the real world, however, resources are not unlimited. Suppose that the company can only obtain six large and eight small pieces per day. Figure 2.0.2 shows these limited resources.

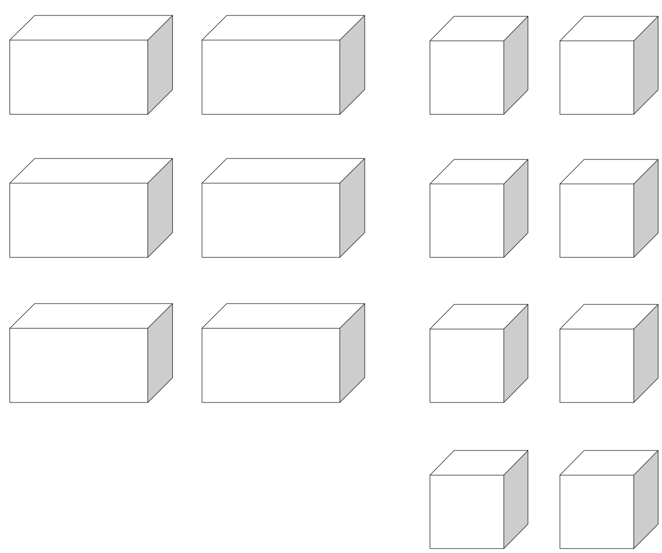


Figure 2.0.2: The furniture company's limited resources

The profit from each table is $16, and the profit from each chair is $10. The production manager wants to find the rate of production of tables and chairs per day that earns the most profit. **Production rate** refers to the number of tables and chairs this company can produce per day.

Q1. What do you think the production rates should be in order to generate the most profit?

Q2. Does the number of table and chairs produced each day have to be an integer value?

Q3. Using only eight small and six large Legos, build a physical model of this problem. If Legos are unavailable, draw pictures to explore some possibilities. Create several combinations of tables and chairs this company could make using your model.

Solving the Problem

There are many possible product mixes this company could make. A **product mix** is a combination of each product being manufactured. The various product mixes could be explored using the Lego model.

First, the company could begin by making as many tables as possible since the profit from a table is much greater than the profit from a chair. Each table requires two large pieces and two small pieces. There are only six large and eight small pieces available. Therefore, only three tables can be built. This generates 3($16) = $48 profit. There are two small pieces left over, but nothing can be built from them. Thus, $48 is the total profit if three tables (and no chairs) are built.

Three tables and zero chairs was one possible product mix. There could be other production rates that generate more profit.

No more than three tables could be made due to the limited resources available, and making three tables yielded a profit of $48. Now, suppose two tables are made. Manufacturing two tables uses four large and four small pieces. Now there are two large and four small pieces left over. These are just enough resources to build two chairs. The profit on two tables and two chairs would be 2($16) + 2($10) = $52. This is more profit than building three tables. However, the production manager wonders, "Is $52 the greatest profit possible? Is there another product mix that could generate more profit?"

Q4. In a Table 2.0.1, record other combinations of tables and chairs the company could produce. For each combination, write the production rate of tables, the production rate of chairs, and the profit for each possibility.

Production Rate of Tables	Production Rate of Chairs	Total Profit

Table 2.0.1: Exploring the total profit for each combination of tables and chairs

Q5. Which production rates generate the most profit?

Q6. Did any product mix yield a profit greater than $52?

It is impossible to find the total profit for every product mix because there are infinitely many possibilities. However, most likely no one in the class found a profit greater than $52. In the next section, you will learn how to know for certain you found the product mix with the greatest profit.

Notice that in Table 2.0.1 you used a set of similar equations to compute the profit for each possibility. These equations are the basis for the **objective function**. The two production rates *varied* across each possible product mix, and exploring these variations allows a *decision* about production to be made. Therefore, the production rates for tables and chairs are known as the **decision variables** for this problem.

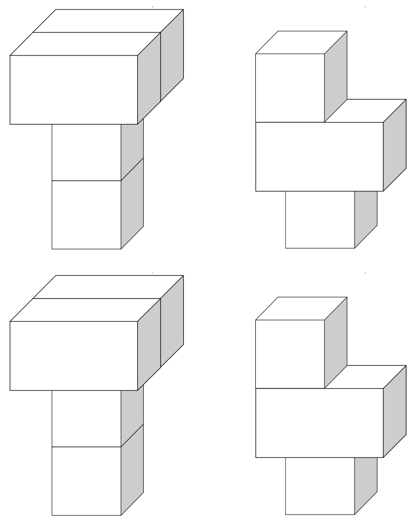


Figure 2.0.3: Two tables and two chairs yield the most profit

Because the profit has been optimized, the solution in Figure 2.0.3 is called the **optimal solution**. Besides the optimal solution, there are many other possible solutions. Although they are not optimal, each possible solution is still a **feasible** solution. Building four tables is an example of an **infeasible** solution.

Q7. Why is building four tables an example of an infeasible solution?

Q8. Give another example of an infeasible solution.

Stepping Beyond the Solution

Operations researchers understand that there is more to their work than merely finding solutions to problems. Once a solution is found, it must be interpreted. One sort of interpretation is called **sensitivity analysis**. Sensitivity analysis involves exploring how sensitive the solution is to changes in the parameters of the problem. For example, in the Lego problem above, one of the parameters of the problem is the availability of large pieces.

Q9. Would it make a difference if *seven* large pieces were available instead of six (there are still eight small pieces)? If so, what is the new optimal solution, and how much profit does it generate?

Q10. Would it make a difference if *nine* small pieces were available instead of eight (there are still six large pieces)? If so, what is the new optimal solution, and how much profit does it generate?

Q11. Would it make a difference if *seven* large pieces and *nine* small pieces were available? If so, what is the new optimal solution, and how much profit does it generate?

Growing the Problem

Suppose now that the furniture company has decided to dramatically expand production. Now it is able to obtain 27 small and 18 large Lego pieces per day. The profit on tables and chairs remains the same.

Q12. What should the daily production rates be in order to maximize profit?

Q13. Would it make a difference if 19 large pieces were available instead of 18 (there are still 27 small pieces)? If so, what is the new optimal solution, and how much profit does it generate?

Q14. Would it make a difference if 28 small pieces were available instead of 27 (there are still 18 large pieces)? If so, what is the new optimal solution, and how much profit does it generate?

Q15. Was this new problem easier or more difficult to solve than the original? Why?

Section 2.1: Computer Flips, a Junior Achievement Company

Junior Achievement (JA) is an educational program available worldwide. JA uses hands-on experiences to help young people understand the economics of life. In partnership with businesses and educators, JA brings the real world to students. The JA Company Program provides basic economic education for high school students by using support and guidance of volunteer consultants from the local business community. By organizing and operating an actual business, students learn how businesses function. They also learn about the structure of the free enterprise system and the benefits it provides.

Gates Williams is the production manager for Computer Flips, a Junior Achievement company. Computer Flips purchases a basic computer at wholesale prices and then adds a display, extra memory cards, extra USB ports, or a CD-ROM or DVD-ROM drive. The company also purchases these extra components at wholesale prices. The computers, with the added features, are then resold at retail prices.

Computer Flips produces two models: Simplex and Omniplex. The profit on each Simplex is $200, and on each Omniplex, the profit is $300. The Simplex model has fewer add-ons, so it requires only 60 minutes of installation time. The Omniplex has more add-ons and requires 120 minutes of installation time. Five JA students do all of the installation work. Each of them works 8 hours per week. Gates Williams must decide the rate of production per week of each computer model in order to maximize the company's weekly profit.

To make decisions such as the one Gates Williams faces, operations researchers use a technique known as **linear programming**. Answering the following questions will help you understand this technique.

2.1.1 Exploring the Problem

One way to approach the problem is to make some guesses and test the profit generated by each guess. For example, suppose Gates Williams decides the company should make 20 of each model.

Q1. How much profit would be generated?

Q2. Is there enough installation time available to make that number of each model?

Q3. Answer the same two questions if Gates Williams decides to make:
   a. 10 Simplex computers and 30 Omniplex computers
   b. 30 Simplex and 10 Omniplex

Q4. Can you find a product mix for which there is enough installation time?

Q5. How much profit do the production rates you found generate for the company?

2.1.2 Generalizing the Problem

Sometimes it is helpful to visualize things. The numbers, variables, and their relationships in a problem can be represented by a graph. Before graphing the Computer Flips problem, you must translate the information in the problem into mathematical statements—equations or inequalities.

First, let

x_1 represent the weekly production rate of Simplex computers and
x_2 represent the weekly production rate of Omniplex computers

The variables x_1 and x_2 are called decision variables because Gates Williams uses them to help make his decision. Mr. Williams's goal is to make as much money as possible. He does this by selling as many computers as he is able. Therefore, Mr. Williams can calculate his weekly profit (z) as a function of x_1 and x_2. Because the objective is to maximize profit, the profit function is called the objective function.

Q6. Write an equation for the profit (z) the company would earn in a week. [Hint: Look back at Section 2.1.1 and see how you calculated profit for 20 Simplex and 20 Omniplex computers.]

Q7. Write a mathematical statement in terms of x_1 and x_2 that describes the relationship between the installation time required to produce x_1 Simplex and x_2 Omniplex computers each week and the amount of available installation time each week. [Hint: Look back at Section 2.1.1 and see how you determined if there was enough installation time to produce 20 Simplex and 20 Omniplex computers.]

Q8. Can Computer Flips produce a negative number of either model?

Q9. Write two mathematical statements that describe your answer to the previous question.

The mathematical statements created in this section will be used to find the optimal solution in the following sections.

2.1.3 A Visual Approach

At this point, it should be clear that Gates Williams cannot decide to make any number of each model he chooses because there is only a certain amount of installation time available each week. That is, the available installation time *constrains* the number of Simplex and Omniplex computers that can be made each week. The inequality that captures this relationship (from Q7) is called a **constraint**. The other two inequalities (from Q9) express the fact that the decision variables in this problem cannot be negative. Thus, they are called **non-negativity constraints**.

These constraints can be graphed on a coordinate place. This graph gives a visual representation of the possible production rates for each computer model.

Q10. On the same coordinate axes, graph each of the three inequalities you wrote in the previous section (one from Q7 and one from Q9). For uniformity, place x_1 on the horizontal axis and x_2 on the vertical axis.

Q11. Give one point that satisfies all three inequalities.

Q12. Where are all of the points that satisfy all three inequalities?

Q13. What is the connection between the points identified in the previous question and the Simplex and Omniplex computers?

The points that satisfy each of the constraint inequalities represent a mix of Simplex and Omniplex computers that could be produced each week. Recall that this region of the coordinate system is called the feasible region, because those points represent feasible production mixes.

Q14. Choose any point in the feasible region, and compute the weekly profit that would be generated by producing that mix of Simplex and Omniplex computers.

Q15. Choose a second point in the feasible region that generates the same weekly profit as the first point.
- a. Draw a line through the two points.
- b. Write the equation for this line in terms of x_1 and x_2.

Every point on the line you have drawn generates the same weekly profit. For this reason, such a line is called a **line of constant profit**.

Q16. Suppose Computer Flips generates $6,000 of profit each week.
- a. Write an equation to represent this situation.
- b. Graph this equation.

Note that the points (0, 20), (15, 10), and (30, 0) are on the line you drew, and each coordinate pair generates a profit of $6,000 when substituted into the objective function.

Q17. For each of the following profits, write an equation and then graph that equation (on the same coordinate plane).
- a. $0 profit
- b. $3600 profit
- c. $4800 profit
- d. $7200 profit

Q18. What do you notice about the three lines you have drawn?

Q19. Which of the lines generates the largest weekly profit?

Q20. If you were to continue drawing lines in this way, where does the line that generates the largest weekly profit intersect the feasible region? What is the profit at that point?

2.1.4 Solving the Problem

Hopefully, the previous line of investigation has suggested that the point or points representing the largest possible weekly profit are close to the boundary of the feasible region. That is, in order to maximize profits, Computer Flips' production rates should be as large as possible, while still keeping within the available installation time.

Q21. Choose a point on the boundary of the feasible region, but not at a corner (vertex), and evaluate the profit there.

Q22. Continue to choose points on the boundary, but try to increase the amount of profit each time.

Q23. Finally, evaluate the profit at each of the corner points of the feasible region.

Q24. What is the relationship between the corner points and the feasible region?

Q25. At which of these points is the profit the greatest? How would you describe this point? Recall: The point at which the profit is maximized is called the optimal solution.

Notice that as the amount of constant profit increases, the lines are higher and further right in the first quadrant. Try to visualize a single line moving upward or to the right while its slope remains constant. The last point(s) in the feasible region that such a moving line touches will be optimal, because the profit is the greatest of any feasible points.

Q26. There is not always only one optimal solution.
Draw an example of a feasible region that could have more than one optimal solution.

2.1.5 Complicating the Problem

After several weeks of operation, one of the students in the sales department of Computer Flips does some market research. Based on this research, she decides that the company cannot sell more than 20 Simplex computers in any given week.

Q27. Write an inequality that expresses this market constraint.

Q28. Graph the new system of constraint inequalities.

Q29. What do you notice about the optimal solution you found earlier?

Q30. What is the optimal solution after adding the market constraint?

Now the students in the sales department of Computer Flips decide to extend the market research to the Omniplex model. On the basis of their research, they decide that Computer Flips cannot sell more than 16 Omniplex computers in any given week.

Q31. Write an inequality for this new market constraint.

Q32. Graph the new feasible region.

Q33. Does the previous optimal solution lie in the new feasible region?

Q34. What is the optimal solution after the addition of the second market constraint?

The students at Computer Flips notice that they are getting a lot of returns. Every computer that was returned had a problem with one of the add-ons. They realize that they need to test their finished products before shipping them. They decide to assign the task of testing the computers to only one of the student installers. To accommodate this change, the other four student installers agree to work 10 hours per week, so that the total available installation time remains 40 hours per week. The student who will do the testing also works 10 hours per week. It takes her 20 minutes to test a Simplex and 24 minutes to test an Omniplex.

Q35. Write an inequality for the testing constraint based on the information in the previous paragraph.

Q36. Graph the new feasible region.

Q37. Using the new feasible region, what is the optimal solution?

Q38. Why is it possible to have a non-integer solution?

2.1.6 Success Breeds—An Even More Complicated Problem

Computer Flips has some initial success, so the students are considering producing two additional models: Multiplex and Megaplex. Multiplex will have more add-ons than Simplex, but not as many as Omniplex. Each Multiplex will generate $250 profit. Megaplex, as the name implies, will have more add-ons than any of the other models. Each Megaplex will generate $400 profit.

Q39. What are the decision variables in the new problem? What do they represent?

Q40. Write an equation for the profit (z) the company would earn in a week.

The installation and testing times for each computer appear in Table 2.1.1. In addition, market research indicates that the *combined* sales of Simplex and Multiplex cannot exceed 20 computers per week, and the *combined* sales of Omniplex and Megaplex cannot exceed 16 computers per week.

	Simplex	Omniplex	Multiplex	Megaplex
Installation Time	60 min.	120 min.	90 min.	150 min.
Testing Time	20 min.	24 min.	24 min.	30 min.

Table 2.1.1: Installation and Testing times for all four computer models

Q41. Using the information above, formulate the constraints after the Multiplex and Megaplex models have been added to the product mix.

Q42. Is it possible to solve this problem by graphing? Why or why not?

In the next section, you will see another way to solve linear programming problems. In particular, the following section explores solving problems without graphing. You may wonder why this graphing approach cannot be used to solve every linear programming problem. If a problem contain three decision variables, it would be difficult for many people to visualize the graph. If a problem contains four or more decision variables, a graph is not even possible.

Section 2.2: SK8MAN, Inc.

SK8MAN, Inc. manufactures and sells skateboards. A skateboard is made of a deck, two trucks that hold the wheels (see Figure 2.2.1), four wheels, and a piece of grip tape. SK8MAN, Inc. manufactures the decks of skateboards in its own factory and purchases the rest of the components.

Figure 2.2.1: A skateboard truck

To produce a skateboard deck, the wood must be glued and pressed, then shaped. After a deck has been produced, the trucks and wheels are added to the deck to complete a skateboard. Skateboard decks are made of either North American maple or Chinese maple. A large piece of maple wood is peeled into very thin layers called veneers. A total of seven veneers are glued at a gluing machine and then placed in a hydraulic press for a period of time (see Figure 2.2.2). After the glued veneers are removed from the press, eight holes are drilled for the truck mounts. Then the new deck goes into a series of shaping, sanding, and painting processes. Figure 2.2.3 shows a deck during the shaping process.

Figure 2.2.2: Maple veneers in a hydraulic press

Figure 2.2.3: Shaping a deck

Currently, SK8MAN, Inc. manufactures two types of skateboards: Sporty (Figure 2.2.4, top) and Fancy (Figure 2.2.4, bottom).

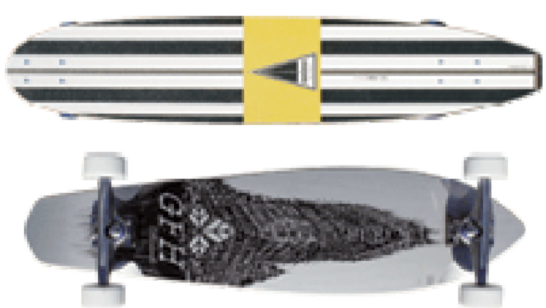


Figure 2.2.4: Sporty and Fancy skateboards

G. F. Hurley, the production manager at SK8MAN, Inc., needs to decide the production rate for each type of skateboard in order to make the most profit. Each Sporty board earns $15 profit, and each Fancy board earns $35 profit. However, Mr. Hurley might not be able to produce as many boards of either style as he would like, because some of the necessary resources, such as the North American maple and Chinese maple, are limited. That is, the production rates are *constrained* by the availability of the resources.

The Sporty board is a less expensive product, because its quality is not as good as the Fancy board. Chinese maple is used in the manufacture of Sporty decks. North American maple is used for Fancy decks. Because Chinese maple is soft, it is easier to shape. On average, it takes a worker 5 minutes to shape a Sporty board. However, a Fancy board requires 15 minutes to shape. G. F. Hurley needs to determine the production rates of Sporty and Fancy boards that will yield the maximum profit.

Q1. Develop a table to organize the information about Sporty and Fancy boards.

2.2.1 Problem Formulation

To find how to maximize profit, G. F. Hurley uses linear programming. The first step in the formulation of a linear programming problem is to define the decision variables in the problem. Let:

x_1 represent the weekly production rate of Sporty boards and
x_2 represent the weekly production rate of Fancy boards.

The decision variables are then used to define the objective function. This function captures the goal in the problem, which, in this case, is to *maximize* the company's profits per week. Therefore, the objective function should represent the weekly profit from the sale of the two different styles of skateboards. The variable z is used to represent the amount of profit SK8MAN, Inc. earns per week.

Now, since the profit for each style of skateboard is known ($15 and $35, respectively), G. F. Hurley writes the objective function by expressing the profit (z) in terms of the decision variables (x_1 and x_2):

Maximize: $z = 15x_1 + 35x_2$.

The last step in the formulation of the problem is to represent any constraints in terms of the decision variables. G. F. Hurley cannot just decide to make as many boards as he wants, because the number made is *constrained* by the available shaping time. Therefore, shaping time will be a constraint.

Suppose SK8MAN, Inc. is open for 8 hours a day, 5 days a week, which is a 40-hour workweek. However, since the information about shaping time is expressed in minutes, 40 hours is converted to 2,400 minutes. If SK8MAN, Inc. makes x_1 Sporty boards and x_2 Fancy boards per week, they use $5x_1 + 15x_2$ minutes of shaping time.

For example, making 100 Sporty boards and 150 Fancy boards would take 5(100) + 15(150) = 2,750 minutes. Note that since 2,750 minutes is greater than 2,400 minutes, this production mix is not feasible.

Thus, the shaping time constraint is:

$$5x_1 + 15x_2 \leq 2400$$

There are also two not-so-obvious but completely logical constraints. G. F. Hurley knows the production rate cannot be a negative number for either type of skateboard, so he writes the non-negativity constraints: $x_1 \geq 0$ and $x_2 \geq 0$.

The complete linear programming formulation looks like this:

Decision Variables

Let: x_1 = the weekly production rate of Sporty boards
x_2 = the weekly production rate of Fancy boards
z = the amount of profit SK8MAN, Inc. earns per week

Objective Function

Maximize: $z = 15x_1 + 35x_2$

Constraints

Subject to:

Shaping Time: $5x_1 + 15x_2 \leq 2400$
Non-Negativity: $x_1 \geq 0$ and $x_2 \geq 0$

This formulated linear programming problem can now be solved graphically. To do so, G. F. Hurley sets up a coordinate plane with x_1 as the horizontal axis and x_2 as the vertical axis. Then, he graphs the constraints, as shown in Figure 2.2.5.

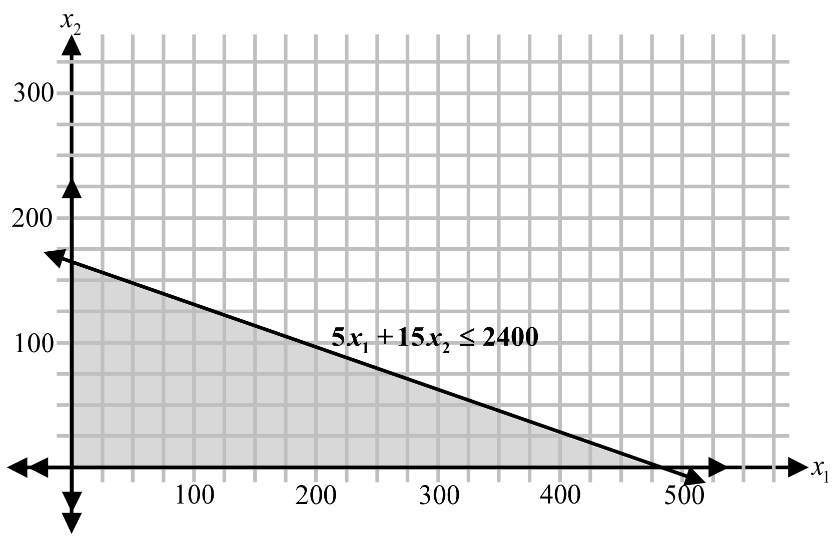


Figure 2.2.5: Graph of the system of constraints

G. F. Hurley recalls that every point in the shaded region satisfies all three constraints and is thus called the feasible region. Each ordered pair in the feasible region represents a combination of Fancy and Sporty boards that SK8MAN, Inc. could produce without violating any of the constraints. There are an infinite number of points in the feasible region, and the solution to the problem of maximizing profit is the one point that generates the most profit.

Rather than try to test an infinite number of points in the objective function, the optimal solution can be found by testing only a few points. This is due to the Corner Point Principle.

Corner Point Principle

If a linear programming problem has a unique optimal solution, it must occur at a corner point of the feasible region.

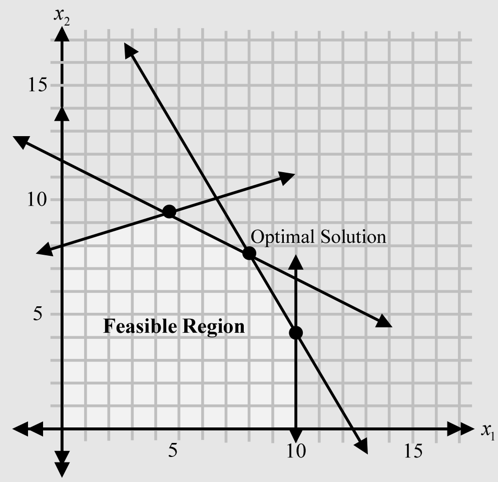


Figure 2.2.6: Example of an optimal solution at a corner point of the feasible region

The Corner Point Principle allows us to simply evaluate the objective function at each corner point of the feasible region. Instead of there being an infinite number of possibilities for the optimal solution, there are only as many possibilities as there are corners of the feasible region.

Therefore, G.F. Hurley tests only the corner points of the feasible region in the objective function, as shown in Table 2.2.1.

Point	Profit
(0, 0)	\$15(0) + \$35(0) = \$0
(480, 0)	\$15(480) + \$35(0) = \$7,200
(0, 160)	\$15(0) + \$35(160) = \$5,600

Table 2.2.1: Corner points and their profits

Based on this information, SK8MAN, Inc. should produce 480 Sporty boards and 0 Fancy boards each week. This product mix will generate a weekly profit of $7,200.

2.2.2 Adding a New Constraint

G. F. Hurley just found out that the company that supplies the trucks for SK8MAN Inc.'s boards can provide at most 2,800 trucks per month. To make the problem easier, G. F. Hurley considers a month to be four weeks, and therefore there are 700 trucks available per week. Since each skateboard needs two trucks, this new information represents another constraint.

The new complete linear programming formulation is as follows:

Decision Variables

Let: x_1 = the weekly production rate of Sporty boards
x_2 = the weekly production rate of Fancy boards
z = the amount of profit SK8MAN, Inc. earns per week

Objective Function

Maximize: $z = 15x_1 + 35x_2$

Subject to:

Constraints

Shaping Time:	$5x_1 + 15x_2 \leq 2400$
Trucks:	$2x_1 + 2x_2 \leq 700$
Non-Negativity:	$x_1 \geq 0$ and $x_2 \geq 0$

Again, G. F. Hurley graphs the constraints, as shown in Figure 2.2.6.

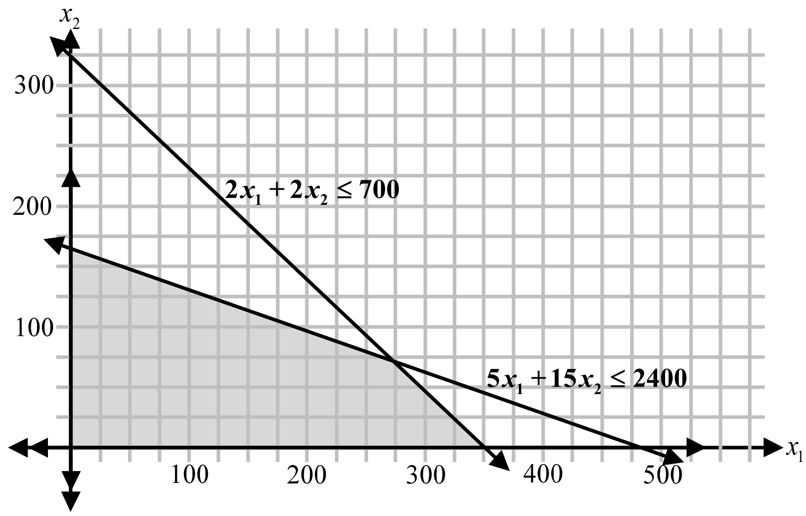


Figure 2.2.6: The feasible region after adding the truck constraint

The new constraint changes the feasible region. The previous optimal solution is no longer included. To find the new optimal solution, G. F. Hurley evaluates all the new corner points in the objective function, as seen in Table 2.2.2.

Point	Profit
(0, 0)	\$15(0) + \$35(0) = \$0
(350, 0)	\$15(350) + \$35(0) = \$5,250
(285, 65)	\$15(285) + \$35(65) = \$6,550
(0, 160)	\$15(0) + \$35(160) = \$5,600

Table 2.2.2: New corner points and their profits

Now G. F. Hurley can easily see that the maximum weekly profit SK8MAN, Inc. can earn is \$6,550, and the company does so by manufacturing 285 Sporty skateboards and 65 Fancy skateboards each week.

2.2.3 Adding a Third Constraint

The U.S. Congress recently enacted legislation regulating the consumption of North American maple by U.S. manufacturers. As a consequence, SK8MAN, Inc.'s supplier told the company that it can provide no more than 840 veneers per week. The law leads to a new constraint. Recall that to make a skateboard, seven veneers are glued together and then placed in a hydraulic press (see Figure 2.2.2). Also recall that North American maple is used only for Fancy decks (Sporty decks are made from Chinese maple).

G. F. Hurley develops the new complete linear programming formulation:

<u>Decision Variables</u>

Let: x_1 = the weekly production rate of Sporty boards
x_2 = the weekly production rate of Fancy boards
z = the amount of profit SK8MAN, Inc. earns per week

<u>Objective Function</u>

Maximize: $z = 15x_1 + 35x_2$

Subject to:

<u>Constraints</u>

Shaping Time:	$5x_1 + 15x_2 \le 2400$
Trucks:	$2x_1 + 2x_2 \le 700$
North American Maple:	$7x_2 \le 840$
Non-Negativity:	$x_1 \ge 0$ and $x_2 \ge 0$

Again, G. F. Hurley graphs the constraints, as shown in Figure 2.2.7.

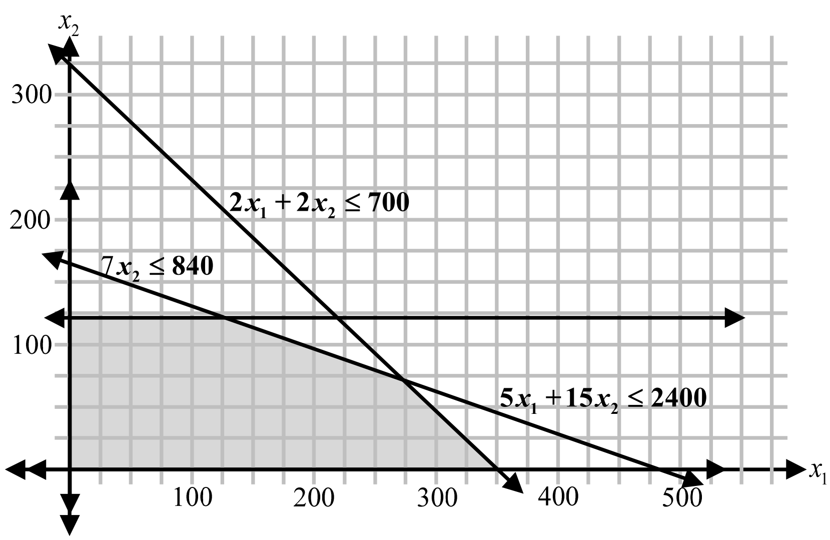


Figure 2.2.7: The feasible region after adding the North American maple constraint

When the new constraint is graphed, the feasible region changes, but the previous optimal solution (285 Sporty boards and 65 Fancy boards) is still included. Applying the corner point principle confirms that the maximum profit is unchanged because the optimal solution without the North American maple constraint remains in the feasible region after the North American maple constraint is added to the formulation. Table 2.2.3 shows the corner point calculations with the new constraint added to the formulation.

Point	Profit
(0, 0)	\$15(0) + \$35(0) = \$0
(350, 0)	\$15(350) + \$35(0) = \$5,250
(285, 65)	\$15(285) + \$35(65) = \$6,550
(120, 120)	\$15(120) + \$35(120) = \$6,000
(0, 120)	\$15(0) + \$35(160) = \$5,600

Table 2.2.3: Evaluating the objective function at each corner point of the new feasible region

Therefore, the optimal solution remains at 285 Sport boards and 65 Fancy boards. Since the North American maple constraint has no effect on the optimal product mix, it is called a **non-binding** constraint. The optimal product mix uses only 65(7) = 455 of the 840 available North American maple veneers (because the Sporty boards do not use North American maple, and the Fancy boards use 7 North American maple veneers per board). Not all of the available resource is expended in producing the optimal solution; thus there is a **slack** of 840 – 455 = 385. The ideas of non-binding constraints and slack will be explored throughout the chapter.

2.2.4 A Fourth Constraint

Finally, SK8MAN, Inc.'s Chinese maple supplier has decided to limit its exports and will deliver a maximum of 1,470 veneers per week. Now G. F. Hurley needs to determine the new mix of products that will maximize weekly profit. As before, this information leads to a new constraint, but the decision variables, objective function, and previous constraints remain the same. G. F. Hurley develops the new complete linear programming formulation:

Decision Variables

Let: x_1 = the weekly production rate of Sporty boards
x_2 = the weekly production rate of Fancy boards
z = the amount of profit SK8MAN, Inc. earns per week

Objective Function

Maximize: $z = 15x_1 + 35x_2$

Subject to:

Constraints

Shaping Time:	$5x_1 + 15x_2 \le 2400$
Trucks:	$2x_1 + 2x_2 \le 700$
North American Maple:	$7x_2 \le 840$
Chinese Maple:	$7x_1 \le 1470$
Non-Negativity:	$x_1 \ge 0$ and $x_2 \ge 0$

Figure 2.2.8 shows the new graph of the constraints.

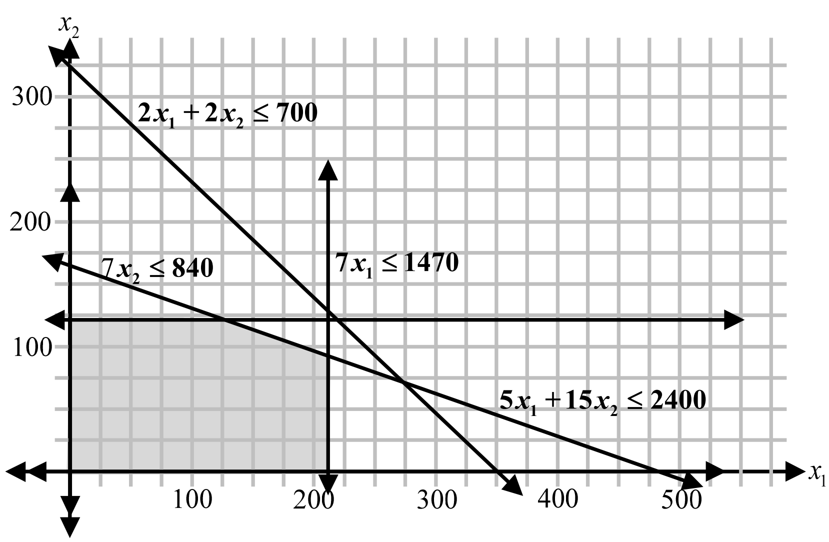


Figure 2.2.8: The feasible region after adding the Chinese maple constraint

As the graph in Figure 2.2.8 shows, the feasible region changes again. The previous optimal solution, (285, 65), is no longer feasible, so each corner point must be tested. Notice that the corner points created by the boundary of the Chinese maple constraint are new.

Point	Profit
(0, 0)	\$15(0) + \$35(0) = \$0
(210, 0)	\$15(210) + \$35(0) = \$3,150
(210, 90)	\$15(210) + \$35(90) = \$6,300
(120, 120)	\$15(120) + \$35(120) = \$6,000
(0, 120)	\$15(0) + \$35(160) = \$5,600

Table 2.2.3: Evaluating the objective function after the last constraint is added

Now SK8MAN, Inc.'s maximum profit is \$6,300 per week. The product mix that achieves that profit is 210 Sporty skateboards and 90 Fancy skateboards. Notice the tendency for maximum profit to decrease as the number of constraints increases.

2.2.5 Adding a Third Decision Variable

SK8MAN Inc. is introducing a new product—the Pool-Runner skateboard—which is made from Chinese maple. It is wider and shorter than the Sporty board so that it will be easy to use in a pool. It takes four minutes to shape a Pool-Runner board, and SK8MAN, Inc. earns \$20 for each one sold. G. F. Hurley needs to determine the new constraints and the optimal product mix. He begins by developing the new complete linear programming formulation:

Q2. Develop a table to organize the information about Sporty, Fancy, and Pool-Runner boards.

Decision Variables

Let: x_1 = the weekly production rate of Sporty boards
x_2 = the weekly production rate of Fancy boards
x_3 = the weekly production rate of Pool-Runner boards
z = the amount of profit SK8MAN, Inc. earns per week

Objective Function

Maximize: $z = 15x_1 + 35x_2 + 20x_3$

Subject to:

Constraints

Shaping Time: $5x_1 + 15x_2 + 4x_3 \leq 2400$
Trucks: $2x_1 + 2x_2 + 2x_3 \leq 700$
North American Maple: $7x_2 \leq 840$
Chinese Maple: $7x_1 + 7x_3 \leq 1470$
Non-Negativity: $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$

Adding the third decision variable makes solving this problem graphically very difficult. Graphing the feasible region with three decision variables would require three dimensions. While it is possible to do so, visualizing such a feasible region is very difficult for most people. There are two other possible ways to solve linear programming problems involving three or more decision variables. The first way is to apply a paper-and-pencil technique called the Simplex Method. This method will not be described here. Instead, the use of a spreadsheet solver will be explored. A spreadsheet solver applies a computer procedure to solve linear programming problems. The following directions will walk you through the steps needed to use the Solver function in Microsoft Excel to solve this problem.

2.2.6 Using Excel Solver

The following steps are given in terms of Microsoft Office 2010. For information regarding earlier versions, see Appendix A.

Step 0: Add in Solver
Open Excel and go to "Data" menu. The Solver option should be at the top right of the menu (see Figure 2.2.9). If it is not available, it needs to be added. To add Solver, go to "Options" under the "File" menu (see Figure 2.2.10) and click on "Add-Ins" (see Figure 2.2.11). Next, choose the "Solver Add-in" and click "Go." Finally, check "Solver Add-in" and click "OK" (See Figure 2.2.12).

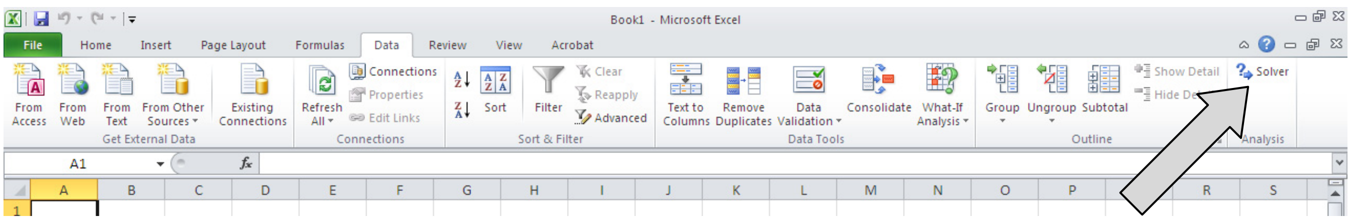


Figure 2.2.9: Location of Solver in Microsoft Excel 2010

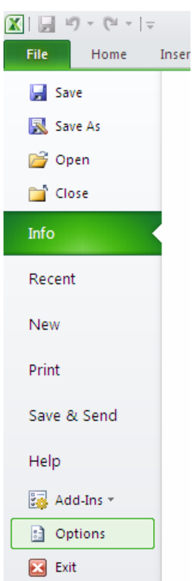


Figure 2.2.10: Choose "Options" under the "File" menu

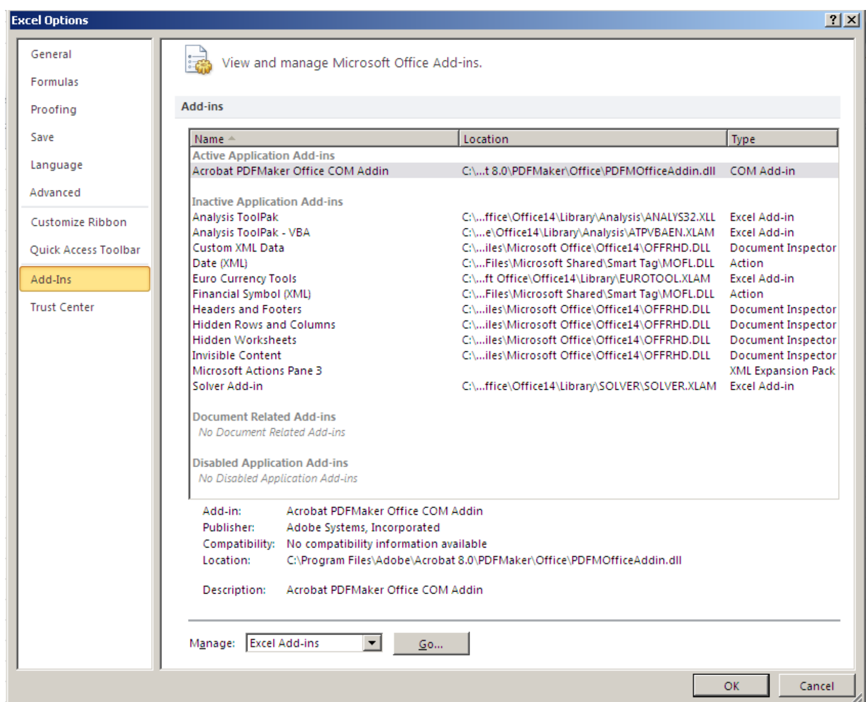


Figure 2.2.11: The Add-Ins menu in Microsoft Excel 2010

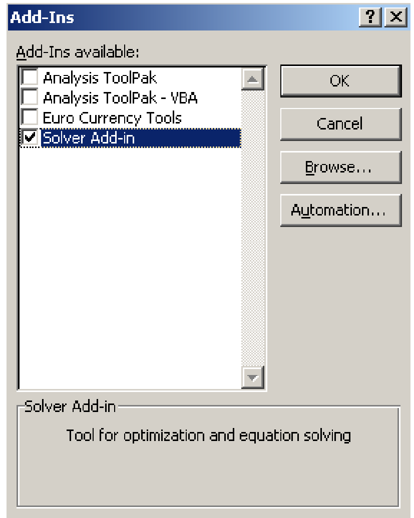


Figure 2.2.12: Choose "Solver Add-in" and click "OK"

Step 1: Set Up Spreadsheet
To set up the spreadsheet, keep in mind the complete linear programming formulation:

Decision Variables

Let: x_1 = the weekly production rate of Sporty boards
x_2 = the weekly production rate of Fancy boards
x_3 = the weekly production rate of Pool-Runner boards
z = the amount of profit SK8MAN, Inc. earns per week

Objective Function

Maximize: $z = 15x_1 + 35x_2 + 20x_3$

Subject to:

Constraints

Shaping Time:	$5x_1 + 15x_2 + 4x_3 \leq 2400$
Trucks:	$2x_1 + 2x_2 + 2x_3 \leq 700$
North American Maple:	$7x_2 \leq 840$
Chinese Maple:	$7x_1 + 7x_3 \leq 1470$
Non-Negativity:	$x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$

Begin by putting a title in column A, such as Chapter, Section, and type of problem (see Figure 2.2.13). This title will make things easier when coming back to this spreadsheet later.

Next, label cell A5 "Decision Variables" and cell A6 "Decision Values." It is helpful to type in the description of the decision values as seen in Figure 2.2.13. The decision variables in this problem were the types of skateboards being produced at SK8MAN Inc. (Sporty, Fancy, and Pool-Runner). The empty cells for the decision values are treated as zero values. These cells are where Solver will put the values it computes for the decision variables once Solver is run. Because the values in these cells are continually changing, Solver calls them "changing cells."

After the decision variables and values, the objective function will be written. It is helpful to write a short description of the objective function. In this example, G. F. Hurley wants to maximize profit, so the objective is to find profit.

Finally, the constraints are listed. The non-negativity constraints do not need to be written in the spreadsheet because Solver has an option that makes all decision values non-negative.

A2 | f_x | 2.2 SK8MAN, Inc.

	A	B	C	D
1	Chapter 2: LP Maximization			
2	2.2 SK8MAN, Inc.			
3	Profit Maximization Problem			
4				
5	Decision Variable	**Sporty** (x1)	**Fancy** (x2)	**Pool-Runner** (x3)
6	Decision Values [# to make per week]			
7				
8	Objective Function [Profit ($)]			
9				
10	**Constraints**			
11	Shaping Time (minutes)			
12	Truck Availability			
13	North American Maple Veneers			
14	Chinese Maple Veneers			

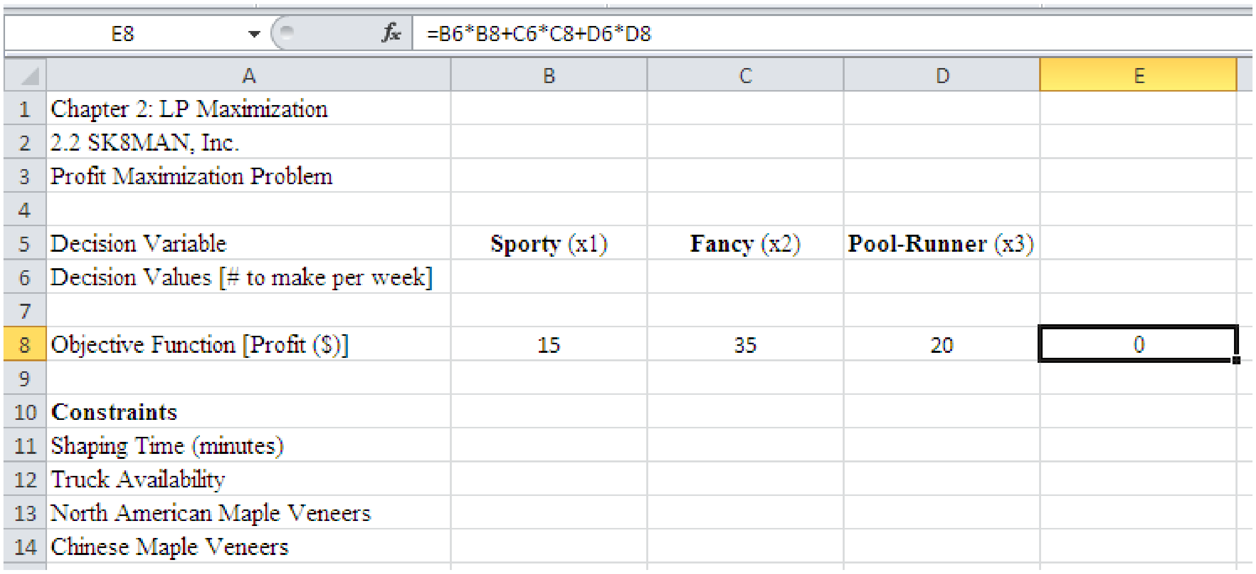
Figure 2.2.13: Setting up the problem formulation in an Excel spreadsheet

Step 2: Develop Formula for Objective Function
The objective function and each of the constraints need to be defined mathematically so that Solver will know what to compute.

First, recall that the objective function is $z = 15x_1 + 35x_2 + 20x_3$. That is, 15 is multiplied by the weekly production rate of Sporty boards, 35 is multiplied by the weekly production rate of Fancy boards, and 20 is multiplied by the weekly production rate of Pool-Runner boards.

Place the coefficients of the objective function in row 8 under the respective decision variable (see Figure 2.2.14).

The coefficients are multiplied by the cells containing the weekly production rate of each board, namely B6, C6, and D6, and then added together. This formula will be typed in cell E8. See Figure 2.2.14 for the formula for the objective function.



E8 | f_x =B6*B8+C6*C8+D6*D8

	A	B	C	D	E
1	Chapter 2: LP Maximization				
2	2.2 SK8MAN, Inc.				
3	Profit Maximization Problem				
4					
5	Decision Variable	**Sporty** (x1)	**Fancy** (x2)	**Pool-Runner** (x3)	
6	Decision Values [# to make per week]				
7					
8	Objective Function [Profit ($)]	15	35	20	0
9					
10	**Constraints**				
11	Shaping Time (minutes)				
12	Truck Availability				
13	North American Maple Veneers				
14	Chinese Maple Veneers				

Figure 2.2.14: The formula for the objective function

Q3. Compare the formula for the objective function in Excel with the objective function written algebraically (i.e., $z = 15x_1 + 35x_2 + 20x_3$).

Q4. Why did a zero appear in cell E8?

Step 3: Develop Formulas for Left-Hand Side of Constraints
The formulas for the constraints are written in the same way as the formula for the objective function. First consider the shaping time constraint: $5x_1 + 15x_2 + 4x_3 \leq 2400$. In this step, only consider the left hand side of the inequality.

Place the coefficients of the shaping time constraint inequality in row 10 under the respective decision variable (see Figure 2.2.15). Again, the coefficients are multiplied by the cells containing the weekly production rate of each board, namely B6, C6, and D6, and then added together. This formula will be typed in cell E10. Note that this expression computes the sum total of the shaping time that is consumed by a particular set of values of the three decision variables. See Figure 2.2.15 for the formula for the shaping time constraint.

E11 =B6*B11+C6*C11+D6*D11

	A	B	C	D	E
1	Chapter 2: LP Maximization				
2	2.2 SK8MAN, Inc.				
3	Profit Maximization Problem				
4					
5	Decision Variable	**Sporty** (x1)	**Fancy** (x2)	**Pool-Runner** (x3)	
6	Decision Values [# to make per week]				
7					
8	Objective Function [Profit ($)]	15	35	20	0
9					
10	**Constraints**				
11	Shaping Time (minutes)	5	15	4	0
12	Truck Availability				
13	North American Maple Veneers				
14	Chinese Maple Veneers				

Figure 2.2.15: The formula for the shaping time constraint

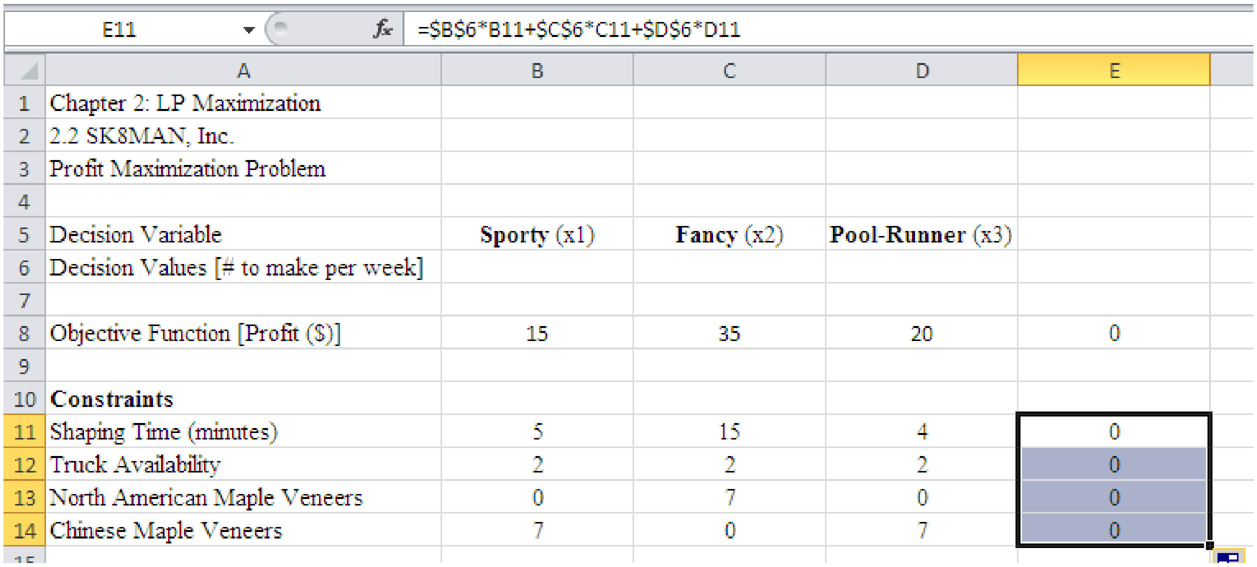
The formulas for the remaining constraints will look very similar to the shaping time constraint. Therefore, rather than type in each formula by hand, the *fill handle* will be used (see Chapter 1 for a review of the *fill handle*). Recall that in order to make some cells unchanging in a formula, dollar signs ($) must be used. Thus, the shaping time constraint should be revised to make the decision values unchanging. It should look as follows:

=$B$6 * B11 + $C$6 * C11 + $D$6 * D11

To obtain the left-hand side of each of the formulas for the remaining constraints, first type in the coefficients into the appropriate rows and then drag the fill handle into cells E12 through E14. The formulas for each of the remaining constraints look as follows:

Trucks:	**=$B$6 * B12 + $C$6 * C12 + $D$6 * D12**
North American Maple:	**=$B$6 * B13 + $C$6 * C13 + $D$6 * D13**
Chinese Maple:	**=$B$6 * B14 + $C$6 * C14 + $D$6 * D14**

See Figure 2.2.16 for the remaining constraint formulas.



E11 =$B$6*B11+$C$6*C11+$D$6*D11

	A	B	C	D	E
1	Chapter 2: LP Maximization				
2	2.2 SK8MAN, Inc.				
3	Profit Maximization Problem				
4					
5	Decision Variable	**Sporty** (x1)	**Fancy** (x2)	**Pool-Runner** (x3)	
6	Decision Values [# to make per week]				
7					
8	Objective Function [Profit ($)]	15	35	20	0
9					
10	**Constraints**				
11	Shaping Time (minutes)	5	15	4	0
12	Truck Availability	2	2	2	0
13	North American Maple Veneers	0	7	0	0
14	Chinese Maple Veneers	7	0	7	0

Figure 2.2.16: The spreadsheet with formulas for all four constraints added

Step 4: Type in Values for Right-Hand Side of Constraints

Next, the right-hand side of each constraint, which is the amount of each resource that is available, needs to be added. In addition, labels for the inequality sign should be added for convenience. This will help when setting up Solver.

For example, recall that the inequality for the shaping time constraint is: $5x_1 + 15x_2 + 4x_3 \leq 2400$. In the previous step, $5x_1 + 15x_2 + 4x_3$ (the left-hand side) was written in cell E11. This expression is less than or equal to 2400 because there are 2400 minutes available for shaping. Therefore, the symbol for "less than or equal to" should be placed in cell F11 (for ease, type "<=" rather than "≤"), and the value 2400 (the right-hand side) should be placed in cell G11.

Thus, the following should be added to cells F11 through F14 and cells G11 through G14 of the spreadsheet.

F11: <=	G11: **2400**
F12: <=	G12: **700**
F13: <=	G13: **840**
F14: <=	G14: **1470**

The spreadsheet should now look like the one in Figure 2.2.17. Notice that the amount of each resource that is consumed for a particular set of values of the decision variables will be displayed in column E, and those amounts must be less than or equal to the amount of each resource available that appears in column G.

G14 f_x 1470

	A	B	C	D	E	F	G
1	Chapter 2: LP Maximization						
2	2.2 SK8MAN, Inc.						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x1)	**Fancy** (x2)	**Pool-Runner** (x3)			
6	Decision Values [# to make per week]						
7							
8	Objective Function [Profit ($)]	15	35	20	0		
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	0	<=	2400
12	Truck Availability	2	2	2	0	<=	700
13	North American Maple Veneers	0	7	0	0	<=	840
14	Chinese Maple Veneers	7	0	7	0	<=	1470

Figure 2.2.17: The spreadsheet with the SK8MAN, Inc. problem completely formulated

Step 5: Open Solver

Before the problem can be solved, parameters need to be set up in the Solver program. These parameters include all of the parts of the problem formulation: the decision variables, the objective function, and the constraints. Solver needs to be told where in the spreadsheet each of these parameters is located.

First, click on the cell containing the objective function (cell E8).

Second, go to the Data menu and choose Solver (see Figure 2.2.9). A Solver Parameters window should come up with the target cell being $E$8 (see Figure 2.2.18). Notice that the target cell is the cell in which the objective function is defined. The dollar signs merely indicate that specific cell. When the Solver Parameters window is opened, if cell E8, containing the value of the objective function, is not already selected, select it now.

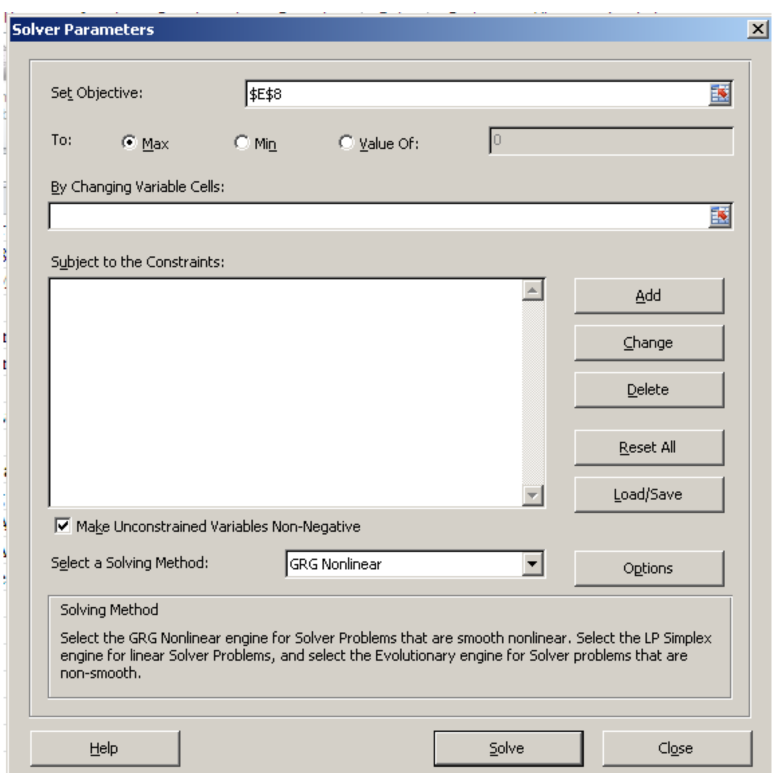


Figure 2.2.18: Beginning the Solver Parameters setup

Step 6: Choose Type of Linear Programing Problem
Recall that the objective for this problem is for SK8MAN, Inc. to *maximize* profit. Therefore, in the Solver Parameters window, make sure the "Max" circle is filled in.

Step 7: Choose Decision Variable Cells
Next, look at the "By Changing Variable Cells" title. Solver needs to be told that the decision variable cells are B6, C6, and D6. To do so, type in B6, C6, and D6 or use the shortcut B6:D6. Solver will add dollar signs in the cell names, and you can leave them as they are.

Step 8: Setting Up Constraints
Click inside the box labeled "Subject to the Constraints." The constraints will be added, one at a time. Click "Add." As shown in Figure 2.2.19, another window will appear titled "Add Constraint."

The value of the formula in cell E11 should be less than or equal to the value in cell G11 (because the amount of shaping time used needs to be less than or equal to 2400 minutes). To do this, type E11 into "Cell Reference" and G11 into "Constraint." The inequality symbol can be changed by using the drop-down menu. In this case it should be left as is (<=). Click "Add" and then continue to the next constraint.

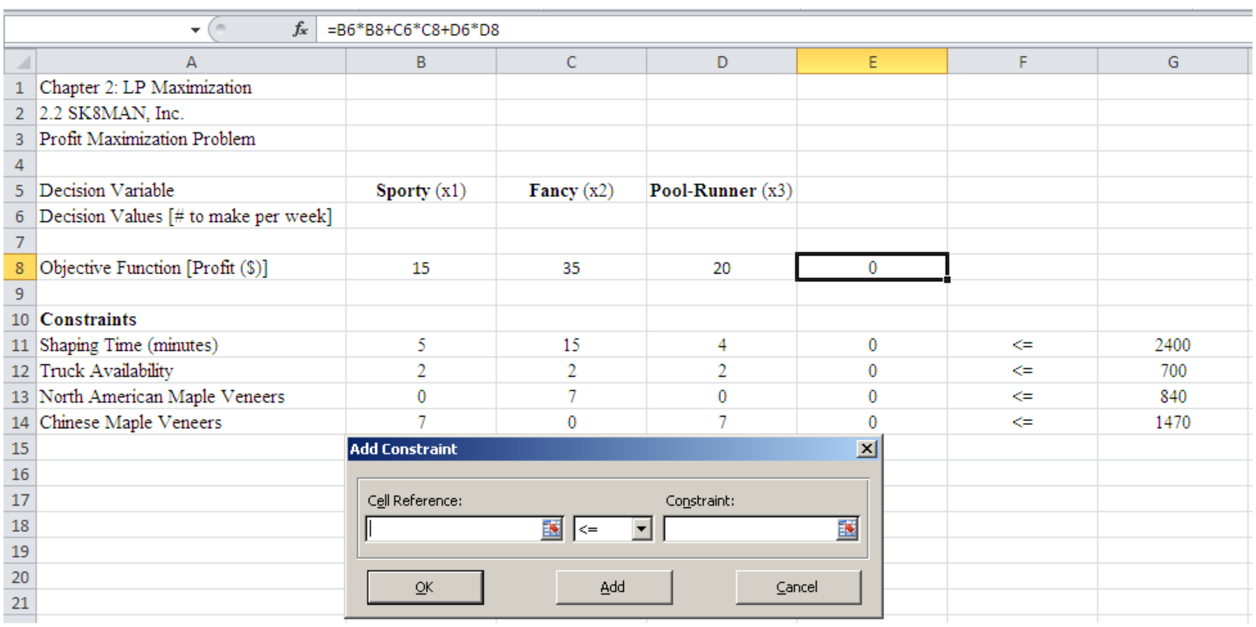


Figure 2.2.19: The "Add Constraint" window

For the Truck availability constraint, E12 should be less than or equal to G12, so type E12 into "Cell Reference" and G12 into "Constraint" and then click "Add." Continue in the fashion for the North American Maple Veneers constraint and the Chinese Maple Veneers constraint. After completing the Chinese Maple Veneers constraint, click "OK." When finished, the constraints should be listed in the "Subject to the Constraints" box in the Solver Parameters window (see Figure 2.2.20).

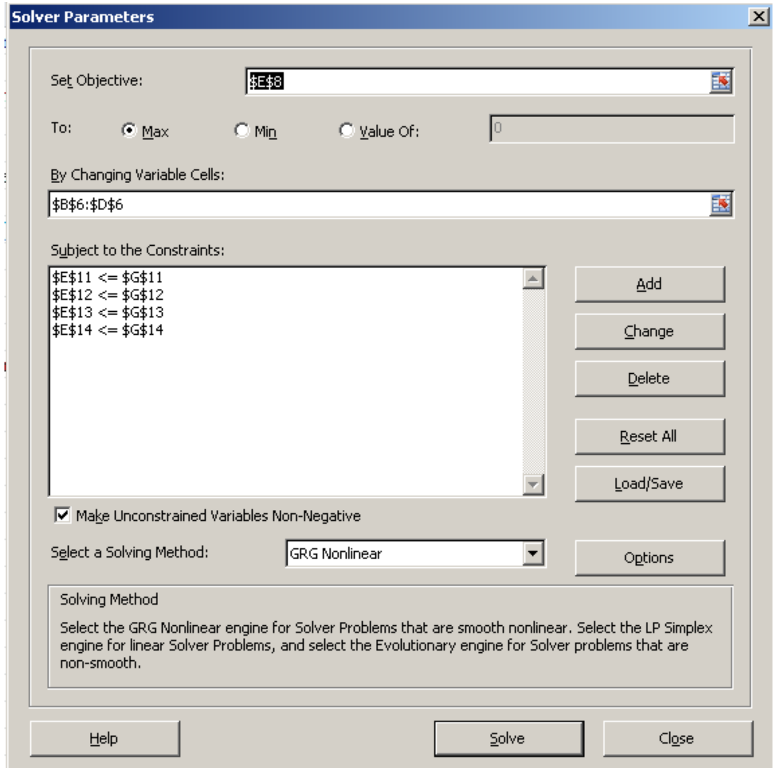


Figure 2.2.20: Continuing to set problem parameters in Solver

Step 9: Verify Non-Negativity Constraint
Notice that the non-negative constraints were not included because Solver has a shortcut for doing so. Under the "Subjects to the Constraints" box, make sure "Make Unconstrained Variables Non-Negative" is checked.

Step 10: Select Solving Method
To solve this linear programming problem, Solver must use the Simplex method. Therefore, in the "Select a Solving Method" drop-down menu, select "Simplex LP."

Step 11: Set Up Options Menu
Click on the "Options" button in the Solver Parameters window. In the "All Methods" tab, check the box that says "Use Automatic Scaling." Also check the box that says "Ignore Integer Constraints." Make sure the "Constraint Precision" is at most 0.000000001 (8 zeros after the decimal point). Note: if you find that your solution is not precise enough, you may need to make the constraint precision smaller. Finally, set the "Solving Limits." Set "Max Time (Seconds)" to 100 and "Iterations" to 100. Then click "OK." Figure 2.2.21 shows the appropriate settings for this linear programming problem.

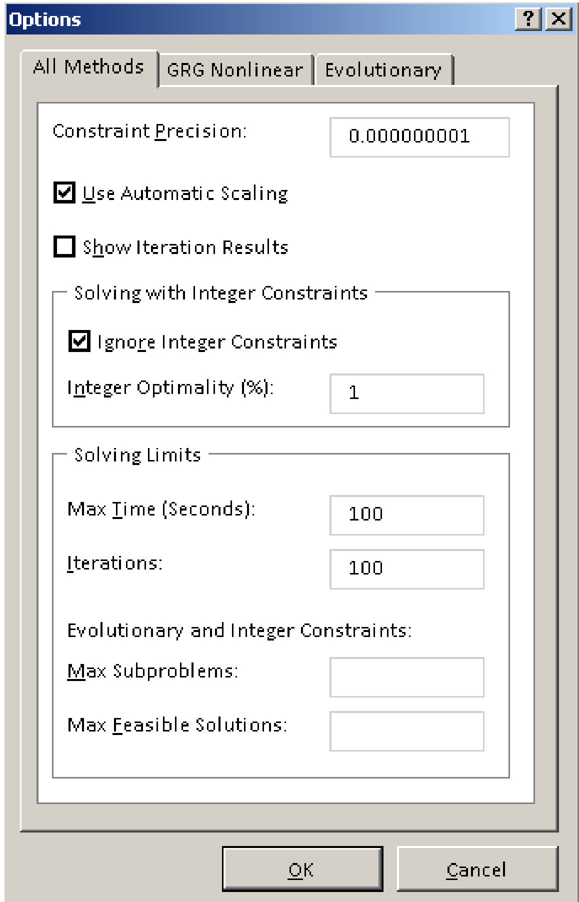


Figure 2.2.21: Adding the Solver Options

Step 12: Solve and Print Appropriate Reports
Finally, click "Solve." The Solver Results window appears, and the results can be seen in the spreadsheet, as shown in Figure 2.2.22.

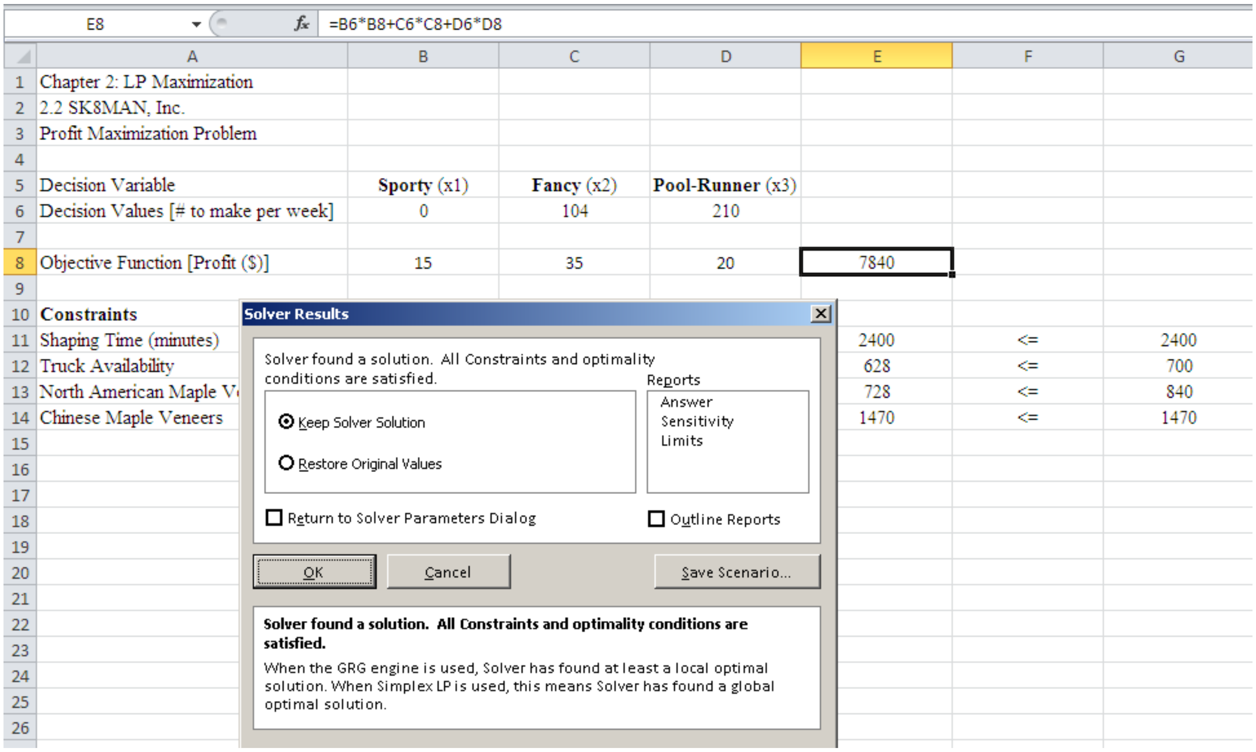


Figure 2.2.22: Solver has solved the three-decision-variable SK8MAN, Inc. problem

The Solver Results window shows various options to choose. For now, click "OK." Some of these options will be explored later.

Notice that x_1 has a final value of 0, x_2 has a final value of 104, and x_3 has a final value of 210. Thus, the optimal product mix calls for producing 0 Sporty skateboards, 104 Fancy skateboards, and 210 Pool-Runner skateboards per week. With this product mix, SK8MAN, Inc. will make a weekly profit of 15(0) + 35(104) + 20(210) = $7840.

Furthermore, notice that the workers at SK8MAN, Inc. will spend 5(0) + 15(104) + 4(210) = 2400 minutes shaping the three skateboards. Since there were only 2400 minutes available for shaping and they use all this time, this constraint is **binding**.

Q5. What other constraint is binding? How do you know?

Q6. What constraints are non-binding? How do you know?

Recall that the slack of each constraint can be found by subtracting the resources used from the available resources. For example, since 2400 minutes were available for shaping and 2400 minutes were used, the slack for the shaping time constraint is 2400 – 2400 = 0. Therefore, there is no time left over for shaping skateboards.

Q7. Calculate the slack for each of the remaining constraints. Interpret each value in terms of the context of the problem.

Q8. What is the relationship between slack and whether a constraint is binding?

To review, the steps for solving a maximization linear programming problem using Excel Solver are given in Table 2.2.4.

Step	Description
0	Add in Solver (skip this step once Solver has been added).
1	Set up the spreadsheet using the linear programming formulation.
2	Develop the formula for the objective function.
3	Develop the formulas for left-hand side of the constraints.
4	Type in the values for the right-hand side of the constraints.
5	Click on objective formula cell and choose Solver from the Data menu.
6	Verify that that "Max" circle is filled in.
7	Fill in the decision variable cells into the "By Changing Variable Cells" section.
8	Add constraints into the "Subject to the Constraints" section.
9	Verify that "Make Unconstrained Variables Non-Negative" is checked.
10	Choose "Simplex LP" from the "Select a Solving Method" drop-down menu.
11	Choose all appropriate options in the "All Methods" tab of the "Options" menu (see Figure 2.2.21).
12	Click "Solve." Interpret and analyze the results. Examine the Answer and Sensitivity Reports when desired.

Table 2.2.4: Steps for solving a maximization linear programming problem using Excel Solver

Q9. Based on these results, how many of each type of skateboard should SK8MAN, Inc. produce each week?

Q10. Under what conditions may G. F. Hurley decide to make a different product mix?

Q11. Summarize the main ideas of linear programming.

Section 2.3: The Pallas Sport Shoe Company

The Pallas Sport Shoe Company manufactures six different lines of sport shoes: High Rise, Max-Riser, Stuff It, Zoom, Sprint, and Rocket. Table 2.3.1 displays the amount of profit generated by each pair of shoes for each of these six lines. The production manager of the company would like to determine the daily production rates for each line of shoes that will maximize profit.

Product	High Rise	Max-Riser	Stuff It	Zoom	Sprint	Rocket
Profit	$18	$23	$22	$20	$18	$19

Table 2.3.1: Profit per pair for six lines of sport shoes

There are six main steps in the production of a pair of sport shoes at Pallas. Some of these steps can be seen in Figure 2.3.1.

1. **Stamping**: The parts that go together to form the upper portion of the shoe are cut using patterns on a large stamping machine. This process resembles cutting dough with a cookie cutter.
2. **Upper Finishing**: These parts are stitched or cemented together to form an upper, and holes for the laces are punched.
3. **Insole Stitching**: An insole is stitched to the sides of the upper.
4. **Molding**: The completed upper is then placed on a plastic mold, called a *last*, to form the final shape of the shoe.
5. **Sole-to-Upper Joining**: After the upper has been molded, it is cemented to the bottom sole using heat and pressure.
6. **Inspecting**: Finally, the shoe is inspected, and any excess cement is removed.

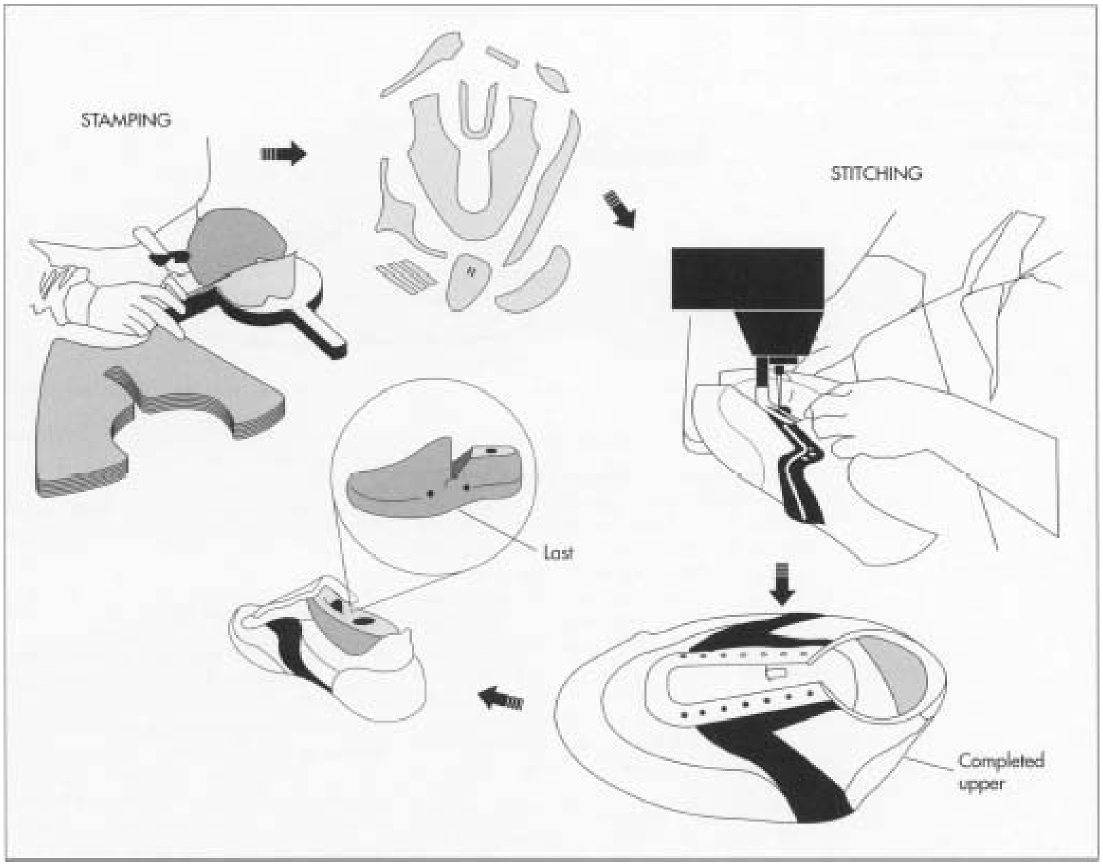


Figure 2.3.1: The steps in manufacturing a sport shoe

The time it takes to complete each of these steps differs across the six lines, and the total time available for each process constrains the daily production rates. Table 2.3.2 shows the time, in minutes, required for each of the six production steps for each of the six lines of sport shoe produced by Pallas as well as the total number of minutes available per day for each step.

	High Rise	Max-Riser	Stuff It	Zoom	Sprint	Rocket	Total Time Available
Stamping	1.25	2	1.5	1.75	1	1.25	420
Upper Finishing	3.5	3.75	5	3	4	4.25	1,260
Insole Stitching	2	3.25	2.75	2.25	3	2.5	840
Molding	5.5	6	7	6.5	8	5	2,100
Sole-to-Upper Joining	7.5	7.25	6	7	6.75	6.5	2,100
Inspecting	2	3	2	3	2	3	840

Table 2.3.2: Time, in minutes, per production step for each line of shoes and total time available

Q1. Based on the information in Tables 2.3.1 and 2.3.2, predict what the optimal solution will be for this problem. Explain your reasoning.

2.3.1 Problem Formulation

The linear programming formulation of the Pallas Sport Shoe Company problem appears below.

<u>Decision Variables</u>

Let: x_1 = the daily production rate of High Rise
x_2 = the daily production rate of Max-Riser
x_3 = the daily production rate of Stuff It
x_4 = the daily production rate of Zoom
x_5 = the daily production rate of Sprint
x_6 = the daily production rate of Rocket
z = the amount of profit Pallas Sport Shoe Company earns per day

Objective Function

Maximize: $z = 18x_1 + 23x_2 + 22x_3 + 20x_4 + 18x_5 + 19x_6$

Subject to:

Constraints

Stamping Time: $1.25x_1 + 2x_2 + 1.5x_3 + 1.75x_4 + x_5 + 1.25x_6 \leq 420$
Upper Finishing Time: $3.5x_1 + 3.75x_2 + 5x_3 + 3x_4 + 4x_5 + 4.25x_6 \leq 1{,}260$
Insole Stitching Time: $2x_1 + 3.25x_2 + 2.75x_3 + 2.25x_4 + 3x_5 + 2.5x_6 \leq 840$
Molding Time: $5.5x_1 + 6x_2 + 7x_3 + 6.5x_4 + 8x_5 + 5x_6 \leq 2{,}100$
Sole-to-Upper Joining Time: $7.5x_1 + 7.25x_2 + 6x_3 + 7x_4 + 6.75x_5 + 6.5x_6 \leq 2{,}100$
Inspecting Time: $2x_1 + 3x_2 + 2x_3 + 3x_4 + 2x_5 + 3x_6 \leq 840$
Non-Negativity: $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$

Figure 2.3.2 contains this formulation in an Excel spreadsheet format for use with Solver.

	A	B	C	D	E	F	G	H	I	J
1	Chapter 2: LP Maximization									
2	2.3 Pallas Sport Show Company									
3	Profit Maximization									
4										
5	Decision Variable	**High Rise** (x_1)	**Max-Riser** (x_2)	**Stuff It** (x_3)	**Zoom** (x_4)	**Sprint** (x_5)	**Rocket** (x_6)			
6	Decision Values [# to make per day]									
7										**Total Profit**
8	Objective Function [Profit ($)]	18	23	22	20	18	19			**$0.00**
9										
10	**Constraints**							**Used**		**Available**
11	Stamping (minutes)	1.25	2	1.5	1.75	1	1.25	0	≤	420
12	Upper Finishing (minutes)	3.5	3.75	5	3	4	4.25	0	≤	1260
13	Insole Stitching (minutes)	2	3.25	2.75	2.25	3	2.5	0	≤	840
14	Molding (minutes)	5.5	6	7	6.5	8	5	0	≤	2100
15	Sole-to-Upper Joining (minutes)	7.5	7.25	6	7	6.75	6.5	0	≤	2100
16	Inspecting (minutes)	2	3	2	3	2	3	0	≤	840

Figure 2.3.2: An Excel spreadsheet formulation of the Pallas Shoe problem

2.3.2 Problem Solution

After solving this linear programming problem in Excel, an Answer Report can be generated, as shown in Figure 2.3.3. Figure 2.3.4 shows this Answer Report.

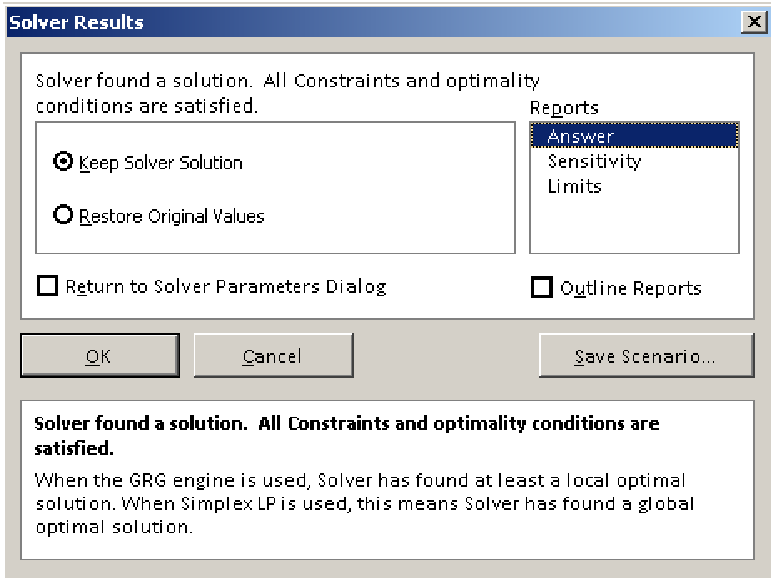


Figure 2.3.3: Generating an Answer Report in Excel Solver

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$J$8	Objective Function [Profit ($)] Total Profit	$0.00	$6,132.57

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per day] High Rise (x1)	0	0	Contin
$C$6	Decision Values [# to make per day] Max-Riser (x2)	0	4.282944345	Contin
$D$6	Decision Values [# to make per day] Stuff It (x3)	0	45.12172352	Contin
$E$6	Decision Values [# to make per day] Zoom (x4)	0	72.32746858	Contin
$F$6	Decision Values [# to make per day] Sprint (x5)	0	104.9019749	Contin
$G$6	Decision Values [# to make per day] Rocket (x6)	0	89.82118492	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$H$11	Stamping (minutes) Used	420	$H$11<=$J$11	Binding	0
$H$12	Upper Finishing (minutes) Used	1260	$H$12<=$J$12	Binding	0
$H$13	Insole Stitching (minutes) Used	840	$H$13<=$J$13	Binding	0
$H$14	Molding (minutes) Used	2100	$H$14<=$J$14	Binding	0
$H$15	Sole-to-Upper Joining (minutes) Used	2100	$H$15<=$J$15	Binding	0
$H$16	Inspecting (minutes) Used	799.3421903	$H$16<=$J$16	Not Binding	40.65780969

Figure 2.3.4: Answer Report for Pallas Sport Shoe Company

Notice that the Answer Report is split into three sections: (1) Objective Cell (Max), (2) Variable Cells, and (3) Constraints.

Q2. Use the "Objective Cell (Max)" section of the Answer Report in Figure 2.3.4 to complete the following.

a. Why is “$J$8” listed under “Cell?”
b. Why do you think “Objective Function [Profit ($)] Total Profit” is listed under “Name?”
c. Why do you think the “Original Value” is 0?
d. What is the “Final Value” referring to?

Q3. Use the “Variable Cells” section of the Answer Report in Figure 2.3.4 to complete the following.
a. What are the “Original Value” and “Final Value” columns referring to?
b. Interpret the information given in the “Final Value” column in terms of the problem context.
c. How do you think the production manager should handle the decimal values that appear in the “Final Value” column?

Q4. Use the “Constraints” section of the Answer Report in Figure 2.3.4 to complete the following.
a. Interpret the information given in the “Cell Value” column in terms of the problem context.
b. The first five constraints are binding and the sixth one is not. What does this mean in terms of the problem context?
c. How are the values in the “Slack” column calculated?
d. If you were only given the slack value for a constraint, how could you determine whether that constraint is binding?

Q5. Which of the six sport shoe lines should be produced, and at what daily rates, in order to maximize profit? (Approximate to two decimal places.)

Suppose Pallas Sport Shoe Company considers adding another five minutes of cutting time each day. Therefore, the cutting time constraint is changed from 420 to 425. The Answer Report for this new scenario is shown in Figure 2.3.5.

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$J$8	Objective Function [Profit ($)] Total Profit	6132.57307	6160.473968

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per day] High Rise (x1)	0	0	Contin
$C$6	Decision Values [# to make per day] Max-Riser (x2)	4.282944345	6.700179533	Contin
$D$6	Decision Values [# to make per day] Stuff It (x3)	45.12172352	48.89766607	Contin
$E$6	Decision Values [# to make per day] Zoom (x4)	72.32746858	74.55655296	Contin
$F$6	Decision Values [# to make per day] Sprint (x5)	104.9019749	100.4452424	Contin
$G$6	Decision Values [# to make per day] Rocket (x6)	89.82118492	85.86714542	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$H$11	Cutting Time (minutes) Used	425	$H$11<=$J$11	Binding	0
$H$12	Upper Finishing Time (minutes) Used	1260	$H$12<=$J$12	Binding	0
$H$13	Insole Stitching Time (minutes) Used	840	$H$13<=$J$13	Binding	0
$H$14	Molding Time (minutes) Used	2100	$H$14<=$J$14	Binding	0
$H$15	Sole-to-Upper Joining Time (minutes) Used	2100	$H$15<=$J$15	Binding	0
$H$16	Inspecting Time (minutes) Used	800.0574506	$H$16<=$J$16	Not Binding	39.94254937

Figure 2.3.5: Answer Report for Pallas Sport Shoe Company with 425-minute Cutting Time constraint

Q6. How does the Answer Report in Figure 2.3.5 differ from the one in Figure 2.3.4?

Q7. Is the first constraint still binding? Do you think Pallas Sport Shoe Company should add this extra five minutes of cutting time each day? Explain your reasoning.

Next, suppose Pallas Sport Shoe Company considers subtracting (rather than adding) five minutes of cutting time each day. Therefore, the cutting time constraint is changed from 420 to 415. The Answer Report for this new scenario is shown in Figure 2.3.6.

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$J$8	Objective Function [Profit ($)] Total Profit	6132.57307	6104.672172

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per day] High Rise (x1)	0	0	Contin
$C$6	Decision Values [# to make per day] Max-Riser (x2)	4.282944345	1.865709156	Contin
$D$6	Decision Values [# to make per day] Stuff It (x3)	45.12172352	41.34578097	Contin
$E$6	Decision Values [# to make per day] Zoom (x4)	72.32746858	70.0983842	Contin
$F$6	Decision Values [# to make per day] Sprint (x5)	104.9019749	109.3587074	Contin
$G$6	Decision Values [# to make per day] Rocket (x6)	89.82118492	93.77522442	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$H$11	Cutting Time (minutes) Used	415	$H$11<=$J$11	Binding	0
$H$12	Upper Finishing Time (minutes) Used	1260	$H$12<=$J$12	Binding	0
$H$13	Insole Stitching Time (minutes) Used	840	$H$13<=$J$13	Binding	0
$H$14	Molding Time (minutes) Used	2100	$H$14<=$J$14	Binding	0
$H$15	Sole-to-Upper Joining Time (minutes) Used	2100	$H$15<=$J$15	Binding	0
$H$16	Inspecting Time (minutes) Used	798.62693	$H$16<=$J$16	Not Binding	41.37307002

Figure 2.3.6: Answer Report for Pallas Sport Shoe Company with 415-minute Cutting Time constraint

Q8. How does the Answer Report in Figure 2.3.6 differ from the one in Figure 2.3.4?

Q9. Is the first constraint still binding? Do you think Pallas Sport Shoe Company should subtract this five minutes of cutting time each day? Explain your reasoning.

Now, suppose Pallas Sport Shoe Company considers adding another 130 minutes of cutting time each day. Therefore, the cutting time constraint is changed from 420 to 550. The Answer Report for this new scenario is shown in Figure 2.3.7.

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$J$8	Objective Function [Profit ($)] Total Profit	6132.57307	6781.570681

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per day] High Rise (x1)	0	0	Contin
$C$6	Decision Values [# to make per day] Max-Riser (x2)	4.282944345	61.57068063	Contin
$D$6	Decision Values [# to make per day] Stuff It (x3)	45.12172352	131.9371728	Contin
$E$6	Decision Values [# to make per day] Zoom (x4)	72.32746858	123.1413613	Contin
$F$6	Decision Values [# to make per day] Sprint (x5)	104.9019749	0	Contin
$G$6	Decision Values [# to make per day] Rocket (x6)	89.82118492	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$H$11	Cutting Time (minutes) Used	536.5445026	$H$11<=$J$11	Not Binding	13.45549738
$H$12	Upper Finishing Time (minutes) Used	1260	$H$12<=$J$12	Binding	0
$H$13	Insole Stitching Time (minutes) Used	840	$H$13<=$J$13	Binding	0
$H$14	Molding Time (minutes) Used	2093.403141	$H$14<=$J$14	Not Binding	6.596858639
$H$15	Sole-to-Upper Joining Time (minutes) Used	2100	$H$15<=$J$15	Binding	0
$H$16	Inspecting Time (minutes) Used	818.0104712	$H$16<=$J$16	Not Binding	21.9895288

Figure 2.3.7: Answer Report for Pallas Sport Shoe Company with 550-minute Cutting Time constraint

Q10. How does the Answer Report in Figure 2.3.7 differ from the one in Figure 2.3.4?

Q11. Do you think Pallas Sport Shoe Company should add this extra 130 minutes of cutting time each day? Explain your reasoning.

Recall that the decision variable x_1 is not in the optimal solution. A logical question to ask is whether increasing the profitability of x_1 could allow it to enter the optimal solution and, if so, how much of an increase would be necessary. Return again to the spreadsheet in Figure 2.3.2, reset the cutting time constraint to its original value of 420 minutes, and change the value of the objective function coefficient of x_1 from 18 to 19. The Answer Report for this new scenario is shown in Figure 2.3.8.

Q12. In terms of the problem, what does this change represent?

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$J$8	Objective Function [Profit ($)] Total Profit	6132.57307	6216.898396

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per day] High Rise (x1)	0	170.6951872	Contin
$C$6	Decision Values [# to make per day] Max-Riser (x2)	4.282944345	8.983957219	Contin
$D$6	Decision Values [# to make per day] Stuff It (x3)	45.12172352	125.7754011	Contin
$E$6	Decision Values [# to make per day] Zoom (x4)	72.32746858	0	Contin
$F$6	Decision Values [# to make per day] Sprint (x5)	104.9019749	0	Contin
$G$6	Decision Values [# to make per day] Rocket (x6)	89.82118492	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$H$11	Cutting Time (minutes) Used	420	$H$11<=$J$11	Binding	0
$H$12	Upper Finishing Time (minutes) Used	1260	$H$12<=$J$12	Binding	0
$H$13	Insole Stitching Time (minutes) Used	716.4705882	$H$13<=$J$13	Not Binding	123.5294118
$H$14	Molding Time (minutes) Used	1873.15508	$H$14<=$J$14	Not Binding	226.8449198
$H$15	Sole-to-Upper Joining Time (minutes) Used	2100	$H$15<=$J$15	Binding	0
$H$16	Inspecting Time (minutes) Used	619.8930481	$H$16<=$J$16	Not Binding	220.1069519

Figure 2.3.8: Answer Report for Pallas Sport Shoe Company with $19 High Rise shoe profit

Q13. How does the Answer Report in Figure 2.3.8 differ from the one in Figure 2.3.4?

Q14. Why may Pallas Sport Shoe Company not want this new optimal solution?

Q15. Explain, in your own words, the usefulness of Answer Reports.

Section 2.4: Chapter 2 (LP Maximization) Homework Questions

1. Refer back to Question 17 in Section 2.1: Computer Flips, a Junior Achievement Company to explore what happens if the lines of constant profit are rotated. Recall that these lines are determined by the objective function.
   a.) Consider the graph in Figure 2.4.1. What changes in the objective function would result in such a rotation?

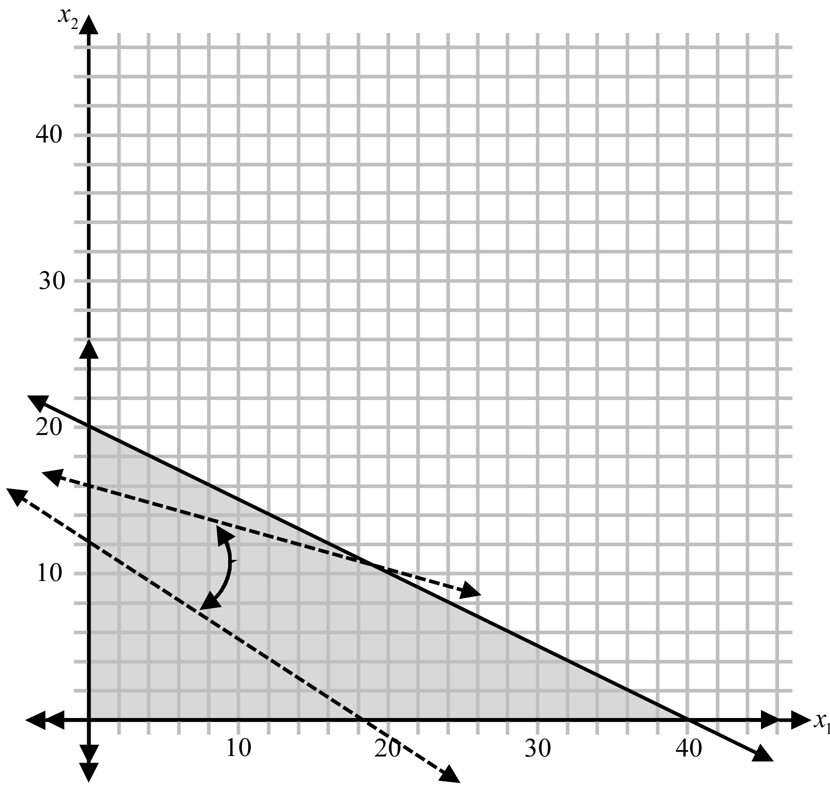


Figure 2.4.1: Feasible region with rotated lines of constant profit

   b.) Now, consider what would happen if the slope of the rotated lines of constant profit matched the slope of one of the constraints (see Figure 2.4.2). How many points of intersection between the rotated lines of constant profit and the feasible region would there be in that case?

   c.) How would this effect the optimal solution?

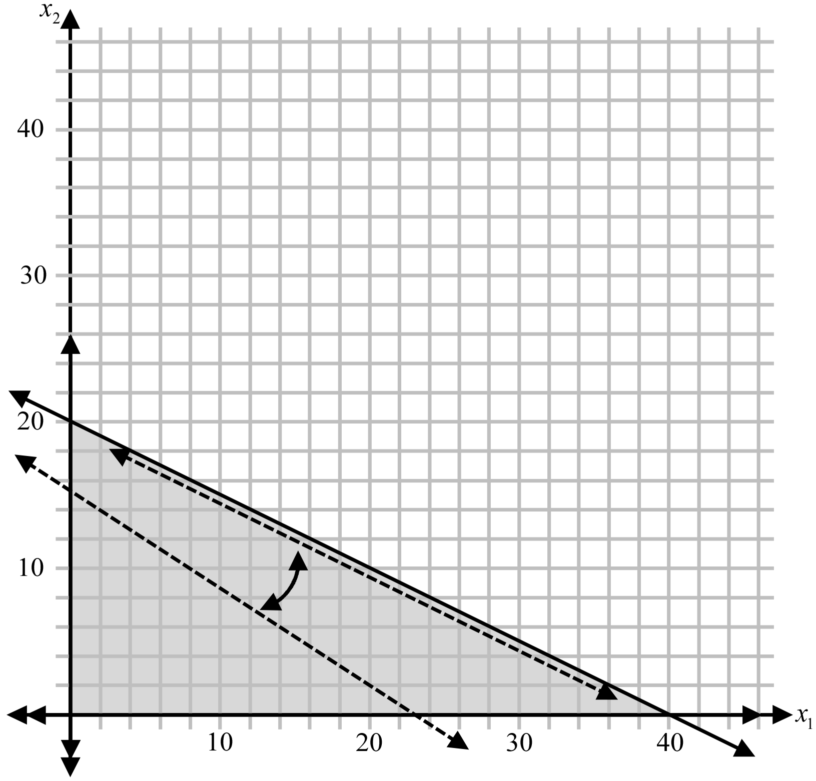


Figure 2.4.2: Feasible region where slope of rotated lines of constant profit matches slope of constraint

2. Suppose SK8MAN, Inc. wants to increase its product variety with different colors and decorations. SK8MAN, Inc. decides to produce three more products in addition to Sporty, Fancy, and Pool-Runner skateboards. The new products will be named Pool-Beauty, Recluse, and Ringer skateboards. Each Pool-Beauty skateboard will earn $22 profit, is manufactured using North American maple, and requires 10 minutes of shaping time. Each Recluse skateboard will earn $17 profit, is made of Chinese maple, and requires 7 minutes of shaping time. Each Ringer skateboard will earn $19 profit, is made of Chinese maple, and uses 4 minutes of shaping time. Since SK8MAN only purchases trucks from a single supplier, the three new products are also subject to the truck availability constraint.

Since the SK8MAN, Inc. products will differ from one another in color and decoration, they have added a worker to do screen-printing. The time required for screen-printing each deck as well as the available time for doing so must be taken into account. The information for this problem appears in Table 2.4.1.

Recall that SK8MAN, Inc. is open for 8 hours a day, 5 days a week, for a total of 2400 minutes per week. However, SK8MAN is now providing all of its workers with two 15-minute breaks each day, so the total time available for both the screen-printing and shaping operations must be

$$2400 \text{ minutes} - 2(15 \text{ minutes})(5 \text{ days}) = 2250 \text{ minutes}.$$

	Sporty	**Fancy**	**Pool-Runner**	**Pool-Beauty**	**Recluse**	**Ringer**
Profit	$15	$35	$20	$22	$17	$19
Type of Maple	Chinese	North American	Chinese	North American	Chinese	Chinese

Number of Veneers	7	7	7	7	7	7
Shaping Time	5 minutes	15 minutes	4 minutes	10 minutes	7 minutes	4 minutes
Screen-Printing Time	7 minutes	10 minutes	8 minutes	10 minutes	5 minutes	6 minutes

Table 2.4.1: Information for each skateboard

a.) Identify the six decision variables for this problem.

b.) Develop the complete linear programming formulation.

Since this problem has six decision variables, a graphical solution cannot be visualized. Therefore, Solver must be employed to find a solution. Use Excel Solver to find the optimal product mix for SK8MAN, Inc.

c.) What is the optimal product mix?

d.) What is the profit for this product mix?

e.) Compare this profit with the profit found in the three-variable problem in Section 2.2. What characteristics of this six-decision-variable problem do you think caused a lower profit?

f.) Open up the Answer Report. Identify the constraints the constraints that are binding and those that are non-binding. Interpret the meaning of these binding and non-binding constraints in terms of the problem context.

g.) Find the slack for each constraint. Interpret the meaning of the slack in terms of the problem context.

3. Figure 2.4.3 shows a sensitivity report for the SK8MAN, Inc. problem with four constraints. The sensitivity report helps to determine how “sensitive” the optimal solution is to changes in data values. This includes analyzing changes in the objective function’s coefficients and the right-hand side (RHS) values of the constraints.

a.) Recall that x_1 represents the production rate of Sporty skateboards. If the profit on each Sporty skateboard increased from \$15 to \$22.50, would that be a large enough increase to add Sporty boards to the optimal product mix?

b.) If the RHS of the shaping time constraint is changed to 2500, what effect would that have on the optimal solution? Explain.

c.) If the RHS of the Chinese Maple constraint were changed to 1100, what would be the effect on the optimal solution? Explain.

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [2010_10-26_SK8MAN.xlsx]Sheet1
Report Created: 11/2/2010 7:31:04 PM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Values [# to make per week] Sporty (x1)	0	-7.333333333	15	7.333333333	1E+30
$C$6	Decision Values [# to make per week] Fancy (x2)	104	0	35	40	35
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	210	0	20	1E+30	7.333333333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$11	Shaping Time (minutes)	2400	2.333333333	2400	240	1560
$E$12	Truck Availability	628	0	700	1E+30	72
$E$13	North American Maple Veneers	728	0	840	1E+30	112
$E$14	Chinese Maple Veneers	1470	1.523809524	1470	343.6363636	420

Figure 2.4.3: Sensitivity report for the SK8MAN, Inc. problem

4. At the end of Section 2.1, you were asked to formulate the Computer Flips problem after the addition of two new products. Your formulation should have used four decision variables and should have included eight constraints, including four non-negativity constraints. In an Excel spreadsheet, set up this formulation, and use solver to obtain a solution. Then create an Answer Report and a Sensitivity Report.

    a.) What is the optimal solution?

    b.) What does this optimal product mix imply about the planned addition of new products?

    c.) What research do you think the Sales Department should conduct before implementing the optimal product mix?

    d.) Market research indicates that Computer Flips could increase the price of a Megaplex computer so as to increase the profit on each by $50 without affecting the number that could be sold per week. Would doing so affect the optimal solution?

    e.) Assuming that a price increase would not affect the number that could be sold each week, by how much could Computer Flips increase the profit on a Simplex computer without affecting the optimal solution?

    f.) Suppose that production costs decreased the profit on a Simplex computer by $30. Would that decrease affect the optimal solution?

    g.) New market research indicates that the combined number of Simplex and Multiplex computers sold per week cannot exceed 15, and the combined number of Optiplex and Megaplex computers sold per week cannot exceed 12. What is the effect on the optimal solution?

John Farmer (Questions 5-8)

5. John Farmer is studying operations research in school. He is curious about applying what he learns in class to actual problems. John's father owns a farm in Missouri and has 640 acres under cultivation. John would like to help his father find the mix of corn and soybeans to plant that would maximize his father's profit. Table 2.4.2 contains some data that John collected about corn and soybean production in Missouri.

	Corn	**Soybeans**
Price/Bushel	\$2.90-3.50	\$7.85-8.85
Yield/Acre	155.8 bu.	41.4 bu.
Seed Cost/Acre	\$45.50	\$34.00
Fertilizer cost/acre	\$88.10	\$37.80
Fuel Cost/Acre	\$24.40	\$10.00
Worker Cost/Acre	\$13.00	\$9.50

Table 2.4.2: 2007-08 USDA corn and soybean estimates for Missouri.

a.) In order to formulate the problem, John must know the price per bushel at which corn and soybeans can be sold. However, the USDA data contains a range of values for both of these crops. What value do you think John should use for each? Why?

b.) What is the largest gross revenue before considering expenses that Mr. Farmer can make? What crop mix produces this gross revenue?

c.) What is the largest net revenue after expenses that Mr. Farmer can make. What crop mix produces this net revenue?

d.) Suppose Mr. Farmer has budgeted only \$60,000 to cover all of the expenses of his crop production. What is the largest net revenue he can earn under this constraint?

6. Corn and soybeans are used in the production of biofuels. Biofuel consumption is important for the environment, because greenhouse gas emissions are reduced 12% by ethanol combustion and 41% by biodiesel combustion. The total corn and soybean production in the United States can meet only 12% of the demand for gasoline and 6% of the demand for diesel fuel. In order to encourage corn production, the USDA pays a subsidy to increase the price per bushel to \$3.50. However, to get the subsidy, the farmer must produce at least 40,000 bushels of corn.

a.) Assuming that the price per bushel that Mr. Farmer can sell his corn for without the subsidy is \$2.90, should Mr. Farmer accept the constraint of producing at least 40,000 bushels of corn in order to earn the subsidy?

b.) If accepting the subsidy is more profitable, what is the new net revenue? If accepting the subsidy is not more profitable, what would the subsidy have to be per bushel in order to make accepting it more profitable?

7. After doing some research, John learned that soybean followed by corn a year after, increases corn yield by 7.5% and saves 25% of the soybean residue nitrogen, which will reduce the fertilizer use in corn production by \$2.50 per acre. The reduction in fertilizer use will also reduce its harmful effect to the environment. In the news, he heard that nitrate leaching causes surface and ground water to degrade. This will harm the living things that use water for drinking or

swimming. For all of these reasons, John wants to convince his father, who planted 200 acres of land with soybean last year, to begin to rotate corn and soybeans. Should John's father begin to rotate his corn and soybean crops?

8. John gathered some data related to wheat production (Table 2.4.3). Is it profitable to produce some wheat under the constraints in question 5?

	Corn	**Soybean**	**Wheat**
Price/bushel	$2.90-3.50	$7.85-8.85	$3.75-4.15
Yield/acre	155.8 bushels	41.4 bushels	60 bushels
Seed cost/acre	$45.50	$34	$24
Fertilizer cost/acre	$88.10	$37.80	$69.25
Fuel cost/acre	$24.40	$10	$40
Worker cost/acre	$13	$9.50	$9

Table 2.4.3: 2007-2008 Corn, soybean, and wheat estimates for Missouri

Elegant Fragrances (Questions 9-14)

9. Elegant Fragrances, Ltd. decides to produce two new perfumes, *L'Arbre d'Amour* and *Evening Rose.* The factory management asked the industrial engineering department to develop a mathematical model for maximizing the profit obtained from producing these products. There are also some limitations in resources, budget and the capacity of the factory that should be considered in the model. At the beginning, the industrial engineering team studied the problem and gathered information from which a model can be developed. The team collected the following information:

- Each perfume is made of two main components. A fragrant perfume oil and a solvent such as a combination of ethanol and water, which is necessary to reduce the allergic reactions of skin to the perfume oil.
- A fragrant oil for *L'Arbre d'Amour* is obtained from Mango Pulp, Tea Leaves, and Juniper Berry, and a fragrant oil for *Evening Rose* is obtained from Mango Pulp, Tea Leaves, and White Rose.
- There are two main processes in the production of these perfumes: Extraction and blending. In the extraction stage, physical and chemical processes change the raw materials and the perfume oil is extracted. In the blending stage, the perfume oil is blended with the solvent. However, Elegant Fragrances does not work directly with the fragrance raw materials, but instead purchases fragrance essences from suppliers and only blends them.
- The annual budget for producing the two new perfumes is limited to $1,000,000.
- The capacity of blending regardless of the perfume type is 50,000 pounds per year.

The other information needed for formulating this problem is summarized in Tables 2.4.4 and 2.4.5:

<table>
<tr><th colspan="3"></th><th>L'Arbre d'Amou</th><th>Evening Rose</th></tr>
<tr><td colspan="3">Income (per pound)</td><td>$408.1</td><td>$209.9</td></tr>
<tr><td rowspan="5">Percentage of raw materials</td><td rowspan="4">Fragrance Oil (Pure)</td><td>Mango</td><td>3</td><td>5</td></tr>
<tr><td>Tea Leaves</td><td>5</td><td>7</td></tr>
<tr><td>Juniper Berry</td><td>7</td><td>0</td></tr>
<tr><td>White Rose</td><td>0</td><td>8</td></tr>
<tr><td>Solvent</td><td>Ethanol and Water</td><td>85</td><td>80</td></tr>
<tr><td colspan="2">Process Costs (Per Pound)</td><td>Blending</td><td>$12</td><td>$12</td></tr>
</table>

Table 2.4.4: Some information about the perfumes

	Availability (pounds per year)	**Cost ($ per pounds)**
Mango	3500	14.4
Tea Leaves	2920	10.8
Juniper Berry	1530	384
White Rose	2000	56
Ethanol and Water	40000	7

Table 2.4.5: Ingredient information

a. Define the decision variables as the amount of annual production of each perfume and help the industrial engineering team formulate this problem for maximizing the profit within the given constraints.

b. Use Solver to obtain the optimal solution to the Elegant Fragrances problem.

10. The management at Elegant Fragrances decides to produce two more new perfumes, *Evergreen* and *Embrasser du Soir.* The production capacity remains the same, but $500,000 more will be allocated for production. The income and costs, as well as the key ingredients and their proportions for the new perfumes are given in Table 2.4.6 below. If the availability of resources is unchanged, what production rates for each of the four new perfumes should the industrial engineering team recommend in order to maximize profit?

			Evergreen	***Embrasser du Soir***
Income (per pound)			$479.4	$218.8
Percentage of raw materials	**Fragrance Oil (Pure)**	**Mango**	8	7
		Tea Leaves	7	5
		Juniper Berry	9	0
		White Rose	0	10
	Solvent	**Ethanol and Water**	76	78
Process Costs (Per Pound)		**Blending**	$12	$12

Table 2.4.6: New Perfumes Information

11. *Fleur de Lis*, a company that plants and trades lily flowers, decides to develop a linear programming model in order to maximize its annual profit. There are two types of lilies the company plans to produce: Asiatic and Stargazer. The company purchases lily bulbs from Holland and, after planting and harvesting, sells the flowers. The purchasing cost, the approximate production cost, and the selling price for each lily type are given in Table 2.4.7.

Lily Type	**Purchasing Cost**	**Production Cost**	**Selling Price**
Asiatic	$1.59/bulb	$1/bulb	$2.74/stem
Stargazer	$2.33/bulb	$1.5/bulb	$3.06/stem

Table 2.4.7: Information about two types of lily

If the annual budget is $1 million and annual capacity is enough to grow lilies from 350,000 bulbs, what is the optimal production plan if the lilies are sold in stems, and each bulb yields two stems?

12. *Fleur de Lis* realizes that their model is not realistic because lilies are typically not sold in stems, but in bunches or pots. Accordingly, it must change the model. The total budget is increased to $1.5 million and the annual capacity is allocated to the production of cut lilies from 350,000 bulbs and potted lilies from 50,000 bulbs.

    One bulb is inserted in a pot to produce a potted Asiatic or Stargazer lily. Asiatic and Stargazer lily cuts are sold in bunches; there are 10 stems in a bunch of an Asiatic lily and 7 stems in a bunch of Stargazer lily. The cost of pots used for Asiatic and Stargazer lilies are $2 and $1, respectively. Use this information and the information in Table 2.4.8 to reformulate the problem. What is the optimal product mix?

Category Type	**Selling Price for each Lily Type**	
	Asiatic	**Stargazer**
Potted Lilies	$8.48/pot	$9.15/pot
Cut Lily Bunches	$21.66/bunch	$19.89/bunch

Table 2.4.8: Selling prices of Potted Lilies and Cut Lilies

13. *Fleur de Lis*'s production is further constrained when it learns that the company can obtain no more than 100,000 Stargazer bulbs a year from Holland. Add this constraint to the constraints of question 11. What, if any, is the effect on the optimal solution?

14. There is a customer group that begins to buy potted Asiatic lilies in quantity, since They are cheaper than potted Stargazer lilies and have more blooms in one pot. *Fleur de Lis*'s sales records show that demand for potted Asiatic lilies is now more than 40,000 a year. What effect does this have on the problem formulation? What effect, if any, does is have on the optimal solution?

15. Suppose that SK8MAN, Inc., wants to increase its product variety with different colors and decorations. SK8MAN decides to produce three more products in addition to Sporty, Fancy, and Pool-Runner. The new products will be named Pool-Beauty, Recluse, and Ringer. Each Pool-Beauty will earn $22 profit, is manufactured using North American maple, and requires 10 minutes of shaping time. Each Recluse will earn $17 profit, is made of Chinese maple, and requires 7 minutes of shaping time. Each Ringer will earn $19 profit, is made of Chinese maple, and uses 4 minutes of shaping time. Since SK8MAN only purchases trucks from a single supplier, the three new products are subject to the same truck availability constraint.

    Since the SK8MAN products will differ from one another in color and decoration, they have added a worker to do screen-printing, and they must take into account the time required for screen-printing each deck as well as the available time for doing so. The screen-printing times appear in Table 2.4.9, and we know that SK8MAN is open for 8 hours a day, 5 days a week. However, SK8MAN is now providing all of its workers with two 15-minute breaks each day, so the total time available for both the screen-printing and shaping operations must be adjusted accordingly.

Product	**Sporty**	**Fancy**	**Pool-Runner**	**Pool-Beauty**	**Recluse**	**Ringer**
Time (min.)	7	10	8	10	5	6

Table 2.4.9: Screen-printing time for each skateboard

a.) Formulate this larger SK8MAN problem.

b.) Use a spreadsheet solver to find the optimal solution.

Chapter 2 Summary

What have we learned?

Linear programming is a process of taking a real world situation, modeling it with inequalities, and finding the best or optimal solution.

- Modeling the situation
    - We start by finding the decision variables – what things can you choose? Typically this is how much to make of a particular product or how much to invest in a particular option. We assign a variable for each of these choices.
    - Next we write an objective function that captures the goal of the problem. – What will determine when you have found the optimal solution? This is the equation that we want to maximize.
    - Finally we define the constraints - those things that limit our choices. These are typically the amount of money, time, people, or resources available.

- Once we have defined our problem, we use a spreadsheet program such as Microsoft Excel to find the optimal solution. After entering the inequalities we set up the solver parameters and run solver. This gives us an answer and sensitivity report.

- The answer report will show:
    - The objective function's final value
    - The value for each of the decision variables
    - The amount of each constraint that is used

- Last we perform sensitivity analysis to determine the effect of different changes.
    - What if the situation changes? Will the optimal solution change and if so by how much?
    - How much would the situation have to change before we would need to re-run solver?

Terms

Adjustable Cells	A column in the Sensitivity Report that shows the increase/decrease of an objective function coefficient without changing the final values and only applies to one objective function coefficient at a time (all other coefficients must remain constant)
Allowable Decrease	A column in the Sensitivity Report that tells how much you can *decrease* the objective coefficient without changing the final values; this decrease will cause the optimal total profit to decrease by an unknown amount
Allowable Increase	A column in the Sensitivity Report that tells how much you can *increase* the objective coefficient without changing the final values; this increase will cause the optimal total profit to increase by an unknown amount
Answer Report	A report that details the optimal solution, lists whether constraints are binding or non-binding, and gives the slack for each constraint
Binding Constraint	A constraint that is satisfied as a strict equality in the optimal solution; all of the available constraint is used
Constraint	A condition that must be satisfied, represented by equations or inequalities
Decision Variable	A quantity that the decision-maker controls
Feasible Solution	A solution that satisfies all the constraints
Final Value	A column in the Answer and Sensitivity Reports that refers to the number in the decision variable cells *after* you use Solver (i.e., the decision variable and objective function values for the optimal solution)
Line of Constant Profit	A line representing the objective function, where every point on the line generates the same profit
Linear Programming	A mathematical technique for finding the optimal value of a linear objective function subject to linear constraints when the decision variables can take on fractional values.
Mathematical Programming	A mathematical approach to allocating limited resources among options in an optimal manner (includes linear programming, integer programming, and binary programming)
Non-Binding Constraint	A constraint for which all available resources are not used
Objective Function	The function that is to be optimized

Optimal Solution	The feasible solution with the best value for the objective function
Original Value	A column in the Answer Report that refers to the number in the decision variable cells *before* you use Solver
Product Mix	The composition of all goods and/or services being produced
Production Rate	The number of products made in a given period of time
Reduced Cost	A column in the Sensitivity Report that shows how much an objective function's coefficient would have to change in order to change the optimal mix; in other words, reduced cost is the per-unit amount that the product contributes to profits, minus the shadow price (note: if reduced cost is negative, then the product is not profitable to make)
Sensitivity Report	A report that shows how changes to the situation will affect the optimal solution
Shadow Price	A column in the Sensitivity Report that gives the amount the objective function would change if there is a *one unit* change in the right-hand side of a constraint
Slack	The amount of a resource that is not used in the optimal solution

Chapter 2 (LP Maximization) Objectives

You should be able to:

- Identify the decision variables
- Define the objective function by finding the goal to be solved for the situation
- Identify the constraints and write inequalities to model them.
- Enter each of these into Microsoft Excel
- Use solver to find the Optimal Solution and generate Answer and Sensitivity Reports
- Analyze the Answer Report
- Analyze the Sensitivity Report

Troubleshooting

What can go wrong?

Troubleshooting is a valuable skill when using Excel and Solver. Quite often a problem is developed and solved and the answer will not make sense. As you work through this book try to remember the mistakes you make so you can avoid them in the future.

- One thing that students may do when working through the problems with the text is to look ahead. They realize that the answers to the questions are on the next page and will copy them onto their spreadsheet. This can lead to problems running solver. When setting up the constraints, the right side must be a value – how much of the resource is available. The left side must be a formula that multiplies the decision variables by the amount used for each one. This formula does not appear on the screen, only the value appears. Typing values printed in the book rather than the formula will cause a problem.
- Another common mistake is to forget to change the direction of the inequality in solver. The default in solver is a less than or equal (<=) constraint. For example, if you have up to 40 hours to produce something, you use the <= constraint. Some situations require something to be at least a certain value. For example if you make tables and chairs you have to make at least 4 chairs for each table. This requires you to use the greater than or equal (>=) constraint.
- When a model changes, don't forget to make the changes in solver. Recall that when we develop a spreadsheet, we start by entering our decision variables, then write our objective function, and finally a row for each constraint. We then start solver and enter the decision variables, objective function and constraints. If you revise a problem, either by adding new decision variables or adding new constraints, you follow the same process. First change the decision variables and/or constraints in excel but then you must make the same changes in solver. Otherwise the computer will not recognize that the situation has changed.

Chapter 2 Study Guide

1. What are decision variables? Where do they come from in the word problem?

2. What is the objective function? Where does it come from in the word problem?

3. What are constraints? Where do they come from in the word problem?

4. What information have we used from the Answer Reports?

5. Write a definition for each in your own words.
   a. Binding constraint-

   b. Non-binding constraint-

6. What is the "Final Value" on an Answer Report?

7. What is slack? What does the cell value tell us?

8. What information have we used from the Sensitivity Reports?

9. For the decision variables, what do the allowable increase and decrease tell us about the variables?

10. What does the reduced cost tell us?

11. Complete each statement describing what happens to the objective function:
    a. If the shadow price for a constraint is 0, then

    b. If the shadow price for a constraint is 100, then

12. How does "Constraint R. H. Side" relate to the word problem?

13. What information is listed in the final value column?

14. Which report would you use to find the final value of the objective function?

15. In 5 or more COMPLETE, GRAMMATICALLY CORRECT sentences compare and contrast the answer report and sensitivity report. Tell how they can be used to analyze problems. **Use at least one example** of how we have used them with the problems done in class.

References

Kolecki, C. (2010). How running shoe is made - material, manufacture, used, parts, components, machine, Raw Materials, Design, The Manufacturing Process of running shoe, Quality Control. Retrieved November 8, 2010, from http://www.madehow.com/Volume-1/Running-Shoe.html

Skateboard - Wikipedia, the free encyclopedia. (2011, April 28). Retrieved April 28, 2011, from https://secure.wikimedia.org/wikipedia/en/wiki/Skateboard

Sk8 Factory. (2010). Retrieved April 28, 2011, from http://www.sk8factory.com/

Appendix A: Using Excel Solver in Microsoft Office 2003 and 2007

The majority of the steps for using Excel Solver are the same in all versions of Microsoft Office. However, there a few differences.

Microsoft Office 2003

To add in Solver in Microsoft Office 2003, go to Add-Ins under the Tools menu and click on it. The Add-Ins window will appear. Then, check the Solver Add-In box and then click OK.

Next, set up the spreadsheet in the same way as detailed in Steps 1-4 of Section 2.2.6.

Once the spreadsheet is properly set up, click on the cell containing the objective function and choose Solver from the Tools menu. A Solver Parameters window will open, as shown in Figure 2.A.1.

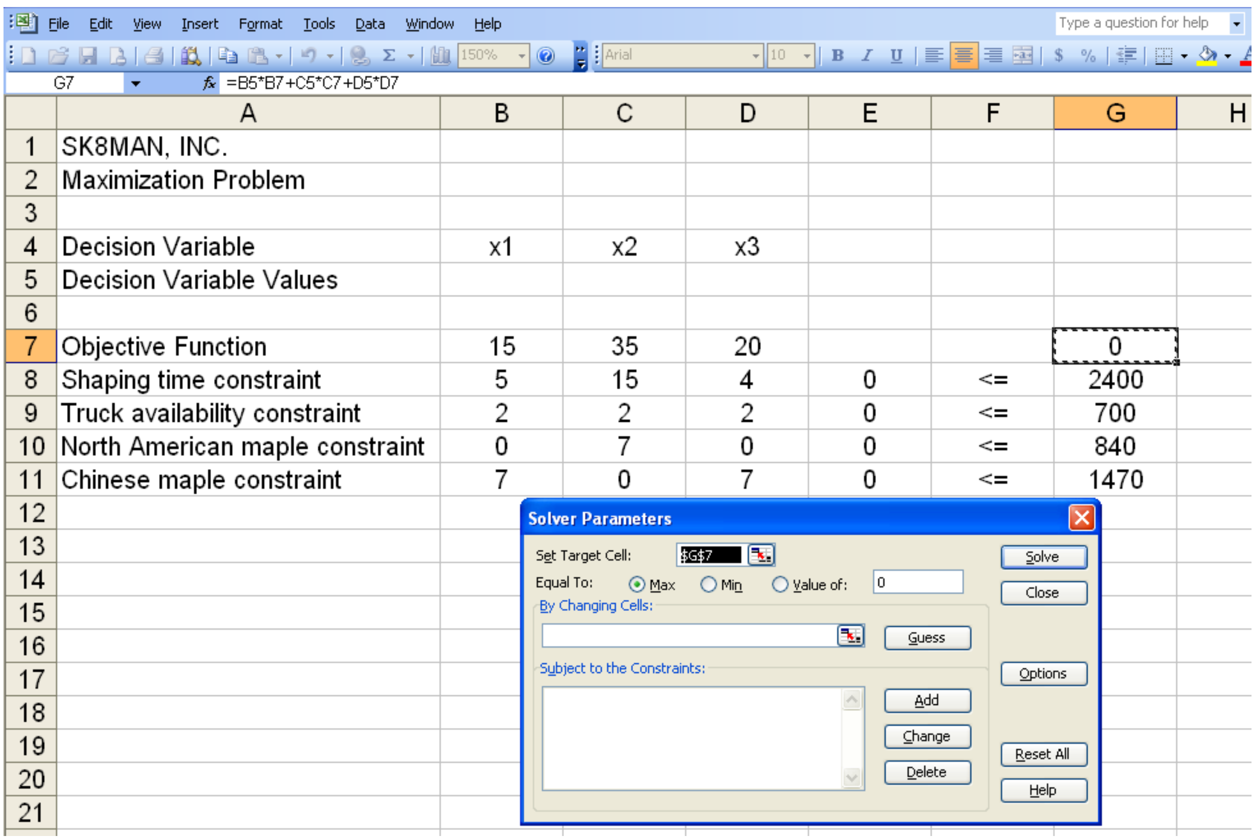


Figure 2.A.1: Solver Parameters window in Microsoft Office 2003

Verify that the “Max” circle is filled in. Then fill in the “By Changing Cells” and “Subject to the Constraints” windows as described in Steps 7-8 in Section 2.2.6

To include the non-negativity constraint, choose Options and check the box that says “Assume Non-Negative.” Also, check the box that says “Assume Linear Model.” See Figure 2.A.2 for an illustration of this.

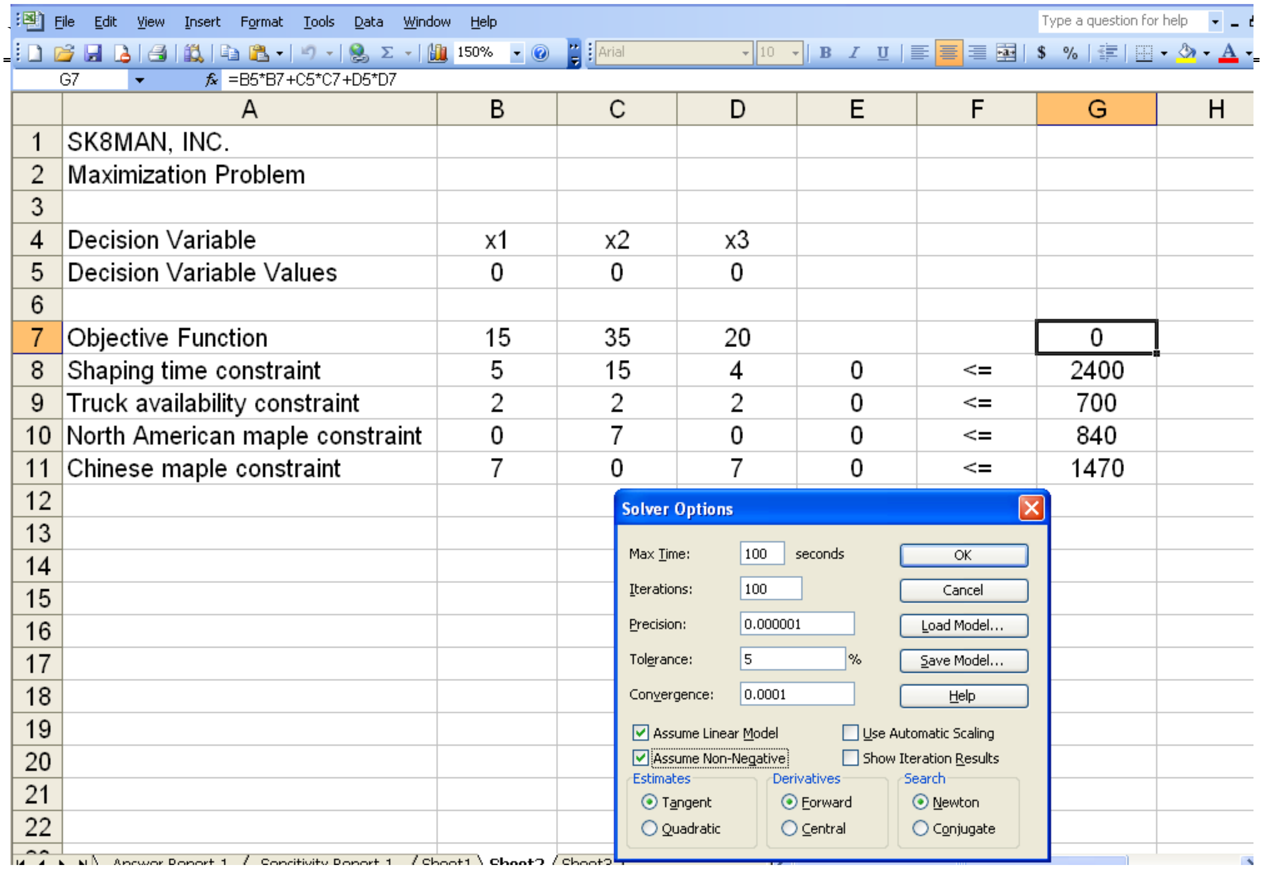


Figure 2.A.2: Solver Options window in Microsoft 2003

Finally, click "Solve" and choose the desired reports, as shown in Figure 2.A.3.

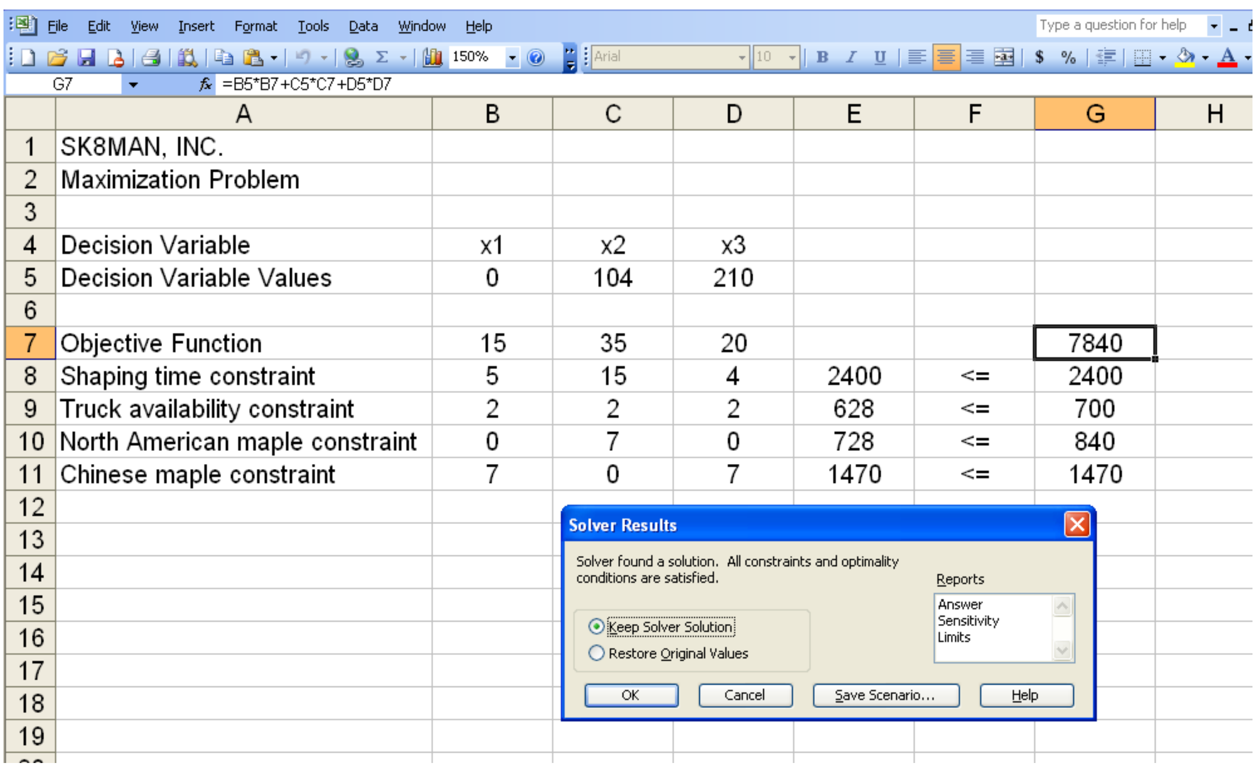


Figure 2.A.3: Solver results window

Microsoft Office 2007

To add in Solver in Microsoft Office 2007, click the Microsoft Office Button and choose “Excel Options.” Click “Add-Ins.” Under the “Manage” box, select “Excel Add-ins” and click “Go.” Check the “Solver Add-in” box and click “OK.”

Once Solver has been added, the Solver command is in the Analysis group on the Data tab, as shown in Figure 2.2.9.

All other steps for using Excel Solver in Microsoft Office 2007 are the same as for Microsoft Office 2003, given above.

Section 3.0: Introduction

In addition to solving problems, operations researchers are often interested in learning how sensitive their solutions are to changes in the parameters of the problem. Consider the Computer Flips problem in the previous chapter. How sensitive is the solution to changes in the amount of profit that is made on each type of computer? What would be the effect of increasing the amount of available installation time or testing time? Questions such as these are part of what is called **sensitivity analysis**.

Section 3.1: Computer Flips, a Junior Achievement Company

Recall from Chapter 2 that Computer Flips is a Junior Achievement Company that begins producing two computer models: Simplex and Omniplex. The pertinent data from the Computer Flips problem appear in Table 3.1.1.

	Simplex	Omniplex
Profit per Computer	\$200	\$300
Installation Time per Computer	60 min.	120 min.

Table 3.1.1: Computer Flips information for two computer types

In addition, Computer Flips has 2,400 min of installation time available per week (five students, each working eight hours per week). They are also under two market restrictions. They estimate that they cannot sell more than 20 Simplex computers or 16 Omniplex computers per week. Gates Williams, the production manager for Computer Flips, wants to find the production rate per week for each type of computer that will maximize total profit.

3.1.1 Problem Formulation

Gates Williams writes the complete linear programming formulation for this problem. He begins with the definition of the decision variables, then he writes the objective function, and finally he lists the constraints.

Decision Variables

Let: x_1 = the weekly production rate of Simplex computers
x_2 = the weekly production rate of Omniplex computers
z = the amount of profit Computer Flips earns per week

Objective Function

Maximize: $z = 200x_1 + 300x_2$

Subject to:

Constraints

Installation Time: $60x_1 + 120x_2 \leq 2400$
Simplex Market: $x_1 \leq 20$
Omniplex Market: $x_2 \leq 16$
Non-Negativity: $x_1 \geq 0$ and $x_2 \geq 0$

Q1. Using Excel Solver, find the optimal solution to the Computer Flips problem.
a. How many Simplex and Omniplex computers should be produced per week?
b. How much profit will Computer Flips earn per week?

3.1.2 Solver Answer Report

To begin to analyze this solution, Gates Williams has Solver generate the Answer and Sensitivity Reports, as shown in Figure 3.1.1. He starts by looking at the Answer Report. He notices that the optimal solution and the value of the objective function are listed on this report.

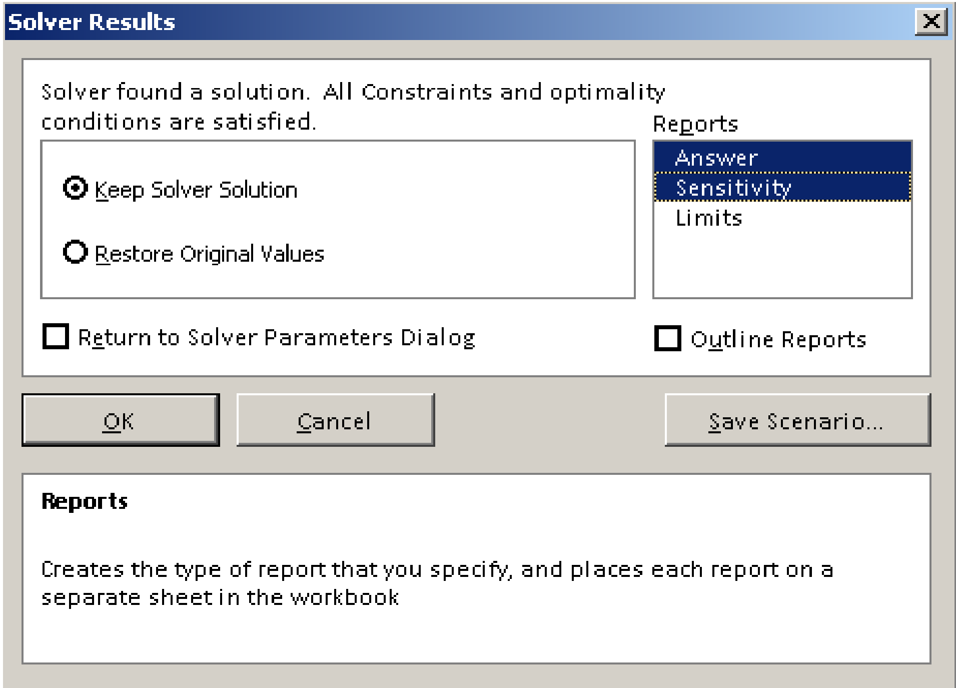


Figure 3.1.1: Choosing Answer and Sensitivity Reports

Q2. How could you determine the optimal solution and objective function value by looking *only* at the Answer Report?

Next, Gates Williams looks at the Constraints portion of the Answer Report. He notices constraints are either listed as Binding or Not Binding. He also notices that there is a column for Slack.

Q3. In the context of this problem, what does the information given in the Cell Value column mean?

Q4. Connect your response to the previous question to the amount of constraints available.
- a. Based on this information, what do you think *Binding* means?
- b. Based on this information, what do you think *Slack* means?
- c. If you were only given the slack value for a constraint, how could you determine whether that constraint is binding?

Gates Williams is considering asking each student to work an additional hour each week. This would increase the available installation time by: (5 students)(1 hour)(60 minutes) = 300. Therefore, the new total installation time available would be 2,700 minutes.

Important!

Before re-solving a problem using Excel Solver, delete the decision variable values currently in the cells. The algorithm Excel uses relies on the values in these cells. To get consistent solutions, be sure to delete these values. But be careful not to delete your formulas!

Chapter 2: LP Maximization					
2.1 Computer Flips Problem (2 variables)					
Profit Maximization					
Decision Variable	**Simplex** (x_1)	**Omniplex** (x_2)			
Decision Values [# to make per week]					
					Total Profit
Objective Function [Profit ($)]	200	300			**$0**
Constraints			**Used**		**Available**
Maximum amount of Installation Time (minutes)	60	120	**0**	≤	2,400
Maximum Marketability	1	0	**0**	≤	20
	0	1	**0**	≤	16

Q5. In your Excel worksheet, change the right hand side of the installation time constraint to 2700 minutes. Solve the problem again and open the Answer Report.
  a. What changes to do you observe?
  b. Do you think it is worth it for Gates Williams to ask the students to work this extra time each week?

Gates Williams decides that he will not ask students to work this extra hour each week. Instead, he wonders if an increase in the Omniplex marketability will increase the weekly profit.

Q6. In your Excel worksheet, change the right hand side of the installation time constraint back to 2400 minutes. Then, increase the Omniplex marketability constraint to 17. Solve the problem again and open the Answer Report.
  a. What changes to do you observe from the original problem?
  b. What happens if the Omniplex marketability constraint is 20 computers? 50 computers?
  c. Do you think Gates Williams should try to increase the marketability of the Omniplex computer? Why or why not?

3.1.3 Solver Sensitivity Report

Now, Gates Williams opens the Sensitivity Report. He notices that the report is split into two sections: Variable Cells and Constraints. (Note: in older versions of Microsoft Office, Variable Cells are referred to as Adjustable Cells.)

In general, **Variable Cells** tell Gates Williams how the objective function coefficients may change. More specifically, Variable Cells show the increase or decrease of an objective function coefficient without changing the optimal solution. These changes only apply to one objective function coefficient at a time (all other coefficients must remain constant).

Next, the **Constraint** cells tell Gates Williams how the objective function value changes. Specifically, Constraint cells give the objective function value based upon an increase or decrease of the right hand side (RHS) of a constraint. These changes only apply to one RHS constraint at a time (all other RHS constraints must remain the same).

Gates Williams feels that he only has control over how profitable each computer is. That is, he cannot change any of the constraints, but he could consider increasing the price of a computer. Therefore, he decides to only explore the Variable Cells in the Sensitivity Report.

Variable Cells
The information in the Variable Cells section of the Sensitivity Report tells how sensitive the optimal solution is to changes in the objective function coefficients of the decision variables. Solver considers changes made to one coefficient at a time. In particular, **Allowable Increase** refers to how much the objective coefficient can be increased without changing the final values. Similarly, **Allowable Decrease** tells how much the objective coefficient can be decreased without changing the final values.

For now, Gates Williams only concerns himself with the Allowable Increase column of the Variable Cells section. He considers the coefficient of x_2, which is the amount of profit generated by the sale of one Omniplex computer. Currently, that profit is $300 per computer. He sees that the Allowable Increase for the coefficient of x_2 is $100.

Gates Williams is curious about the effect of increasing the profit per Omniplex computer. He considers increasing the profit by a value below the Allowable Increase, above the Allowable Increase, and exactly at the Allowable Increase. Thus, he explores the effect of increasing the profit by $50 (below the Allowable Increase), $200 (above the Allowable Increase), and $100 (exactly the Allowable Increase). The corresponding new objective function coefficients of x_2 are $350, $500, and $400, respectively. These changes affect only the objective function in the formulation of the problem. All of the constraints remain the same.

Q7. Write the new objective functions for each of these three changes.

First, Gates Williams considers increasing the profit of Omniplex computers so that the profitability is now $350. However, he notices that this increase in profitability, $50, is less than the Allowable Increase.

Q8. In your Excel worksheet, change the objective function coefficient for Omniplex computers to $350. Solve the problem again.
   a. What changes to do you observe from the original problem?
   b. Do you think Gates Williams should try to increase the profitability of the Omniplex computer by $50? Why or why not?

Next, Gates Williams looks at the effect of increasing the profitability of Omniplex by more than the Allowable Increase. He increases the profitability by $200.

Q9. In your Excel worksheet, change the objective function coefficient for Omniplex computers to $500. Solve the problem again.
   a. What changes to do you observe from the original problem?
   b. Do you think Gates Williams should try to increase the profitability of the Omniplex computer by $200? Why or why not?

At this point, Gates Williams has looked at increasing the profitability of Omniplex by $50, which is less than the Allowable Increase, and by $200, which is more than the Allowable Increase. He wonders what would happen if the profitability of Omniplex increases by exactly $100, which is the Allowable Increase.

Q10. In your Excel worksheet, change the objective function coefficient for Omniplex computers to $400. Solve the problem again.
   a. What changes to do you observe from the original problem?
   b. Did your classmates obtain the same optimal solution as you?

c. Do you think Gates Williams should try to increase the profitability of the Omniplex computer by $100? Why or why not?

In order to gain a better understanding of the three examples explored above, Gates Williams considers the geometry of the situation.

Q11. Draw a graph of the feasible region for the original problem, including the original line of constant profit ($z = 200x_1 + 300x_2$) passing through the optimal corner point (refer to Section 2.1).

Q12. Draw another graph of the feasible region for the original problem.
- a. Draw the line of constant profit when the profitability of the Omniplex computer has increased by $50 to $350. Which corner point maximizes profit in this situation?
- b. Draw the line of constant profit when the profitability of the Omniplex computer has increased by $200 to $500. Which corner point maximizes profit in this situation?
- c. Draw the line of constant profit when the profitability of the Omniplex computer has increased by $100 to $400. Which corner point maximizes profit in this situation? How is this situation different from the previous two?

Q13. Consider the case where the profit margin on Omniplex is increased to $400.
- a. What is the profit if 8 Simplex and 16 Omniplex computers are produced?
- b. What is the profit if 20 Simplex and 10 Omniplex computers are produced?
- c. Are there are other feasible points that produce this profit? If so, where are they? If not, why not?

Q14. Based on what you saw in this section, describe what you think would happen if you considered the Allowable Decrease instead.
- a. How do you think the final values would change if Gates Williams decreased the profitability of Omniplex computers by $200?
- b. How do you think the final values would change if Gates Williams decreased the profitability of Omniplex computers by $400?
- c. How do you think the final values would change if Gates Williams decreased the profitability of Omniplex computers by $300?
- d. Put these changes into Excel to see if your predictions were correct.

Finally, Gates Williams notices that the Sensitivity Report shows an Allowable Increase of 1E+30 in the coefficient of x_1 in the objective function. The number 1E+30 is Solver's way of conveying the expression $1 \cdot 10^{30}$. This very large number is the best Solver can do to indicate an infinite Allowable Increase.

To understand why the Allowable Increase is infinite, Gates Williams first needs to think about what the coefficient of x_1 in the objective function represents. It is the profit margin on Simplex computers. Solver is showing that no matter how much the profitability of Simplex computers increases, it will not change the optimal solution. In other words, increasing the profit margin on Simplex computers is not going to change the optimal number to make. This makes sense because the optimal solution shows that to maximize profits, 20 Simplex (and 10 Omniplex) computers should be made. Twenty is the most that can be made and still satisfy the market limit on weekly sales of Simplex computers. Increasing the profit margin of Simplex computers will not change the fact that no more than 20 per week can be sold, and they're already making 20 each week.

In this section, we have explored the effects on the optimal solution of increasing or decreasing the profitability of one of the computer models. In reality, the situation is much more complicated. For example, if Computer Flips increased the price of an Omniplex computer by $100-$200 to make it more profitable, doing so might affect the Omniplex market constraint. The increased price might lower the market constraint. So, in practice, a company would try to explore all of the ramifications of making changes in the important parameters of the problem.

Section 3.2: SK8MAN, Inc.

Recall from Chapter 2 that SK8MAN, Inc. manufactures skateboards. G.F. Hurley, the production manager at SK8MAN, Inc., needed to determine the production rate for each type of skateboard in order to make the most profit. Table 3.2.1 contains the relevant data from Chapter 2.

	Sporty	Fancy	Pool Runner	Amount Available
Profit per skateboard	$15	$35	$20	
Shaping time required	5	15	4	2,400 minutes
Truck availability	2	2	2	700 trucks
North American maple veneers required	0	7	0	840 veneers
Chinese maple veneers required	7	0	7	1,470 veneers

Table 3.2.1: SK8MAN, Inc. data

3.2.1 Problem Formulation

In Chapter 2, the following problem formulation was developed:

Decision Variables

Let:
x_1 = the weekly production rate of Sporty boards
x_2 = the weekly production rate of Fancy boards
x_3 = the weekly production rate of Pool Runners boards
z = the amount of profit SK8MAN earns per week

Objective Function

Maximize: $z = 15x_1 + 35x_2 + 20x_3$

Subject to:

Constraints

Shaping Time: $5x_1 + 15x_2 + 4x_3 \leq 2{,}400$
Trucks: $2x_1 + 2x_2 + 2x_3 \leq 700$
North American Maple: $7x_2 \leq 840$
Chinese Maple: $7x_1 + 7x_3 \leq 1{,}470$
Non-Negativity: $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$

3.2.2 Solver Answer Report

A spreadsheet formulation of the problem and an Answer Report showing the optimal solution appear in Figures 3.2.1 and 3.2.2.

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	0	104	210			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	35	20			$7,840.00
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2400	≤	2400
12	Truck Availability	2	2	2	628	≤	700
13	North American Maple Veneers	0	7	0	728	≤	840
14	Chinese Maple Veneers	7	0	7	1470	≤	1470

Figure 3.2.1: Formulation for the 3-decision variable SK8MAN problem

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Profit ($)] Total Profit	$0.00	$7,840.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per week] Sporty (x1)	0	0	Contin
$C$6	Decision Values [# to make per week] Fancy (x2)	0	104	Contin
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	0	210	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Shaping Time (minutes)	2400	$E$11<=$G$11	Binding	0
$E$12	Truck Availability	628	$E$12<=$G$12	Not Binding	72
$E$13	North American Maple Veneers	728	$E$13<=$G$13	Not Binding	112
$E$14	Chinese Maple Veneers	1470	$E$14<=$G$14	Binding	0

Figure 3.2.2: Answer Report for the 3-variable SK8MAN problem

As seen in Figure 3.2.1, the optimal solution is $x_1 = 0$, $x_2 = 104$, and $x_3 = 210$; that is, G.F. Hurley should produce no Sporty boards, 104 Fancy boards, and 210 Pool-Runner boards per week. From this product mix, SK8MAN will make a total profit of $7,840 each week. This information also appears in the Answer Report shown in Figure 3.2.2, along with some information about the constraints.

Examining the Answer Report, the first section refers to the Objective Cell (Max), and the variable name is “Objective Function [Profit ($)] Total Profit.” The target cell in the spreadsheet, G8, stores the value of the objective function. Solver finds the maximum profit (because “Max” was selected during the Solver Parameters set-up) that meets all of the constraints. The maximum total profit Solver reports under the column Final Value is $7,840.00. The Original Value of $0.00 simply refers to the amount that was in the cell before Solver was run.

The second section is labeled Variable Cells and refers to the decision variables, x_1, x_2, and x_3. These are adjusted as Solver searches for the optimal solution. The Final Values of 0, 104, and 210, respectively, are the optimal solution. That is, the optimal solution is $x_1 = 0$, $x_2 = 104$, and $x_3 = 210$.

The third section of the Answer Report is labeled Constraints. The four constraints are all less than or equal to (≤) constraints. The left hand side value of each constraint represents the total amount used by the production plan. These totals are stored in cells E11, E12, E13, and E14 and are reported in the column labeled "Cell Value." The right hand side values for each of the three constraints are stored in column G in cells G11, G12, G13, and G14.

G.F. Hurley notices that of the four constraints, two of them are binding and two are not binding. But, he wonders what this means.

He notices that for the two binding constraints, there is a 0 in the column labeled Slack. He looks back to the Solver solution in Figure 3.2.1 and notices that for each of the two binding constraints, the left hand side of the constraint is equal to the right hand side.

For example, the workers at SK8MAN will use 5(0) + 15(104) + 4(210) = 2400 minutes for shaping (cell E11). They have 2400 minutes available for shaping (cell G11). In addition, they will use 7(0) + 7(210) = 1470 Chinese maple veneers (cell E14). They have 1470 Chinese maple veneers available (cell G14).

In other words, there is no slack because every bit of each of those resources is being used up by the optimal solution.

However, for the non-binding constraints in the Answer Report, the Slack values are not zero. They are listed as 72 and 112. Again returning to the Solver solution in Figure 3.1.1, G.F. Hurley notices that the left hand side of the truck availability constraint is 628, and the right hand side is 700. This time the two sides of the constraint are not equal because the optimal solution does not use up all available trucks. There is a Slack of 700 – 628 = 72. That means that SK8MAN could use 72 more trucks. They will not do that, though, because in order to use these extra trucks, they would have to make more skateboards, which is impossible due to the shaping time and the Chinese maple veneers constraints.

Similarly, G.F. Hurley notices that the left hand side of the North American maple veneers constraint is 728 and the right hand side is 840. Thus, they have an extra 840 – 728 = 112 North American maple veneers available.

3.2.3 Solver Sensitivity Report

Figure 3.2.3 contains the Sensitivity Report for the problem. Use it to answer the questions that follow.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Values [# to make per week] Sporty (x1)	0	-7.333333333	15	7.333333333	1E+30
$C$6	Decision Values [# to make per week] Fancy (x2)	104	0	35	40	35
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	210	0	20	1E+30	7.333333333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$11	Shaping Time (minutes)	2400	2.333333333	2400	240	1560
$E$12	Truck Availability	628	0	700	1E+30	72
$E$13	North American Maple Veneers	728	0	840	1E+30	112
$E$14	Chinese Maple Veneers	1470	1.523809524	1470	343.6363636	420

Figure 3.2.3: Sensitivity Report for the 3-variable SK8MAN problem

The Sensitivity Report provides information about each of the decision variables, which it calls Variable Cells. It also provides information about each of the constraints.

Variable Cells: Allowable Increase

The information provided in the Adjustable Cells section of the Sensitivity Report tells how sensitive the optimal solution is to changes in the objective function coefficients of the decision variables. Solver considers changes made to one coefficient at a time.

G.F. Hurley notices that for x_2, there is an Allowable Increase of 40. But he wonders what this refers to. What can be increased by 40? What does such an increase “allow”? Allowable Increase refers to increasing the coefficient of the decision variable in the objective function. In this case, the coefficient of x_2 is the amount of profit generated by the sale of one Fancy skateboard. Currently, that profit is $35 per skateboard.

G.F. Hurley is curious about the effect of increasing the profit per Fancy skateboard. He first explores the effect of increasing the profit by a value below the Allowable Increase. He chooses to increase the profit of Fancy boards by $25 to $60. Doing so changes the objective function to:

$$z = 15x_1 + 60x_2 + 20x_3.$$

The effect of this change is shown in Figures 3.2.4a and 3.2.4b.

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	0	104	210			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	60	20			$10,440.00
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2400	≤	2400
12	Truck Availability	2	2	2	628	≤	700
13	North American Maple Veneers	0	7	0	728	≤	840
14	Chinese Maple Veneers	7	0	7	1470	≤	1470

Figure 3.2.4a: Formulation when the profitability of Fancy boards is $60

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Profit ($)] Total Profit	$0.00	$10,440.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per week] Sporty (x1)	0	0	Contin
$C$6	Decision Values [# to make per week] Fancy (x2)	0	104	Contin
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	0	210	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Shaping Time (minutes)	2400	$E$11<=$G$11	Binding	0
$E$12	Truck Availability	628	$E$12<=$G$12	Not Binding	72
$E$13	North American Maple Veneers	728	$E$13<=$G$13	Not Binding	112
$E$14	Chinese Maple Veneers	1470	$E$14<=$G$14	Binding	0

Figure 3.2.4b: Answer Report when the profitability of Fancy boards is $60

G.F. Hurley notices that in Figure 3.2.4, the profitability of Fancy boards has been changed to $60. However, when looking at the Answer Report for that change, he sees that the optimal solution is the same: make Sporty boards at the rate of 0 per week, make Fancy boards at the rate of 104 per week, and make Pool-Runner boards at the rate of 210 per week.

Therefore, because G.F. Hurley increased the objective function coefficient by an amount *less than* the Allowable Increase, there was no change in the optimal solution. Although the optimal solution is the same, the increase in Omniplex profitability did change the weekly profit from $7,840 to $10,440. This $2,600 increase results from the $25 increase in profit on Fancy boards, and SK8MAN makes 104 per week: ($25)(104 Fancy boards) = $2600.

Next, G.F. Hurley increases the profitability of Fancy boards by an amount greater than the Allowable Increase. Figures 3.2.5a and 3.2.5b show the effect of increasing the objective function coefficient from \$35 to \$80 (an increase of \$45), so that the objective function would now be:

$$z = 15x_1 + 80x_2 + 20x_3$$

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	0	120	150			
7							**Total Profit**
8	Objective Function [Profit (\$)]	15	80	20			\$12,600.00
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2400	≤	2400
12	Truck Availability	2	2	2	540	≤	700
13	North American Maple Veneers	0	7	0	840	≤	840
14	Chinese Maple Veneers	7	0	7	1050	≤	1470

Figure 3.2.5a: Formulation when the profitability of Fancy boards is \$80

Objective Cell (Max)

Cell	Name	Original Value	Final Value
\$G\$8	Objective Function [Profit (\$)] Total Profit	\$0.00	\$12,600.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
\$B\$6	Decision Values [# to make per week] Sporty (x1)	0	0	Contin
\$C\$6	Decision Values [# to make per week] Fancy (x2)	0	120	Contin
\$D\$6	Decision Values [# to make per week] Pool-Runner (x3)	0	150	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$E\$11	Shaping Time (minutes)	2400	\$E\$11<=\$G\$11	Binding	0
\$E\$12	Truck Availability	540	\$E\$12<=\$G\$12	Not Binding	160
\$E\$13	North American Maple Veneers	840	\$E\$13<=\$G\$13	Binding	0
\$E\$14	Chinese Maple Veneers	1050	\$E\$14<=\$G\$14	Not Binding	420

Figure 3.2.5b: Answer Report when the profitability of Fancy boards is \$80

In this case, G.F. Hurley sees that changing the profitability of Fancy boards to \$80 yields a new optimal solution. This time, the optimal solution changes the production rates to 0 Sporty boards, 120 Fancy boards, and 150 Pool-Runner boards per week. Also, the amount of profit generated has now increased to \$12,600 per week.

At this point, G.F. Hurley sees that increasing the profitability of Fancy boards by an amount *less than* the Allowable Increase has no impact on the optimal solution. However, increasing the profitability by an amount *greater than* the Allowable Increase changes the optimal solution. Next, he wonders what will happen if he increases the profitability by exactly the Allowable Increase.

Figures 3.2.6a and 3.2.6b show the effect of increasing the profitability of Fancy boards by exactly $40, where the objective function is:

$$z = 15x_1 + 75x_2 + 20x_3$$

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	0	120	150			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	75	20			$12,000.00
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2400	≤	2400
12	Truck Availability	2	2	2	540	≤	700
13	North American Maple Veneers	0	7	0	840	≤	840
14	Chinese Maple Veneers	7	0	7	1050	≤	1470

Figure 3.2.6a: Formulation when the profitability of Fancy boards is $75

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Profit ($)] Total Profit	$0.00	$12,000.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per week] Sporty (x1)	0	0	Contin
$C$6	Decision Values [# to make per week] Fancy (x2)	0	120	Contin
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	0	150	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Shaping Time (minutes)	2400	$E$11<=$G$11	Binding	0
$E$12	Truck Availability	540	$E$12<=$G$12	Not Binding	160
$E$13	North American Maple Veneers	840	$E$13<=$G$13	Binding	0
$E$14	Chinese Maple Veneers	1050	$E$14<=$G$14	Not Binding	420

Figure 3.2.6b: Answer Report when the profitability of Fancy boards is $75

G.F. Hurley notices that when the profitability of Fancy boards increases by exactly $40 (to $75), the optimal solution reported by Solver is to produce 0 Sporty boards, 120 Fancy boards, and 150 Pool-

Runner boards. This solution is the same as when the profitability of Fancy boards was $80. The total profit is now $12,000:

$$15(0) + 75(120) + 20(150) = \$12,000.$$

However, he also notices that if the profitability of Fancy boards goes up to $75, the amount of weekly profit generated by the original optimal solution (0 Sporty boards, 104 Fancy boards, and 210 Pool-Runner boards) is also $12,000:

$$15(0) + 75(104) + 20(210) = \$12,000.$$

Both production plans lie on the same plane: $z = 15x_1 + 75x_2 + 20x_3$. In fact, any point that lies on this plane and is within the feasible region is an optimal solution. Therefore, there are infinitely many optimal solutions when the profitability of Fancy boards is $75 (i.e., when the coefficient of x_2 is increased exactly by the amount of the Allowable Increase).

Note: This idea was explored graphically in the previous section, where the objective function was a line, rather than a plane. Visualizing the SK8MAN problem graphically is much more difficult because it has 3 decision variables.

Finally, G.F. Hurley notices that the Sensitivity Report in Figure 3.2.3 shows an Allowable Increase of 1E+30 in the coefficient of x_3 in the objective function. The number 1E+30 is Solver's way of expressing the number 1×10^{30}. This very large number is the best Solver can do to indicate an *infinite* Allowable Increase.

To understand why the Allowable Increase is infinite, G.F. Hurley first needs to think about what the coefficient of x_3 in the objective function represents. It is the profitability of Pool-Runner boards. Solver is showing that no matter how much the profitability of Pool-Runner boards increases, it will not change the optimal solution. In other words, increasing the profitability of Pool-Runner boards is not going to change the optimal number to make.

This makes sense because the optimal solution shows that to maximize profits, 210 Pool-Runner boards (as well as 0 Sporty boards and 104 Fancy boards) should be made. Making 210 Pool-Runner boards per week consumes 1470 Chinese maple veneers, which is exactly the number available per week. Therefore, no more than 210 Pool-Runner boards can be made per week, no matter how much their profitability increases. SK8MAN is already making all of the Pool-Runners that it possibly can!

Variable Cells: Allowable Decrease

Next, G.F. Hurley returns to the original problem and considers decreasing one of the coefficients in the objective function. Referring again to Figure 3.2.3, he notices the Sensitivity Report indicates an Allowable Decrease of approximately 7.33 for the coefficient of x_3 in the objective function.

He considers decreasing the coefficient of x_3 by 5 (a value smaller than the Allowable Decrease), 15 (a value greater than the Allowable Decrease) and 7.33 (the Allowable Increase). The corresponding new profit coefficients for each case are 15, 5, and 12.67, respectively.

Figures 3.2.7, 3.2.8, and 3.2.9 show the formulation and Answer Report for each of those decreases in the profit margin of Pool-Runner boards.

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	0	104	210			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	35	15			$6,790.00
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2400	≤	2400
12	Truck Availability	2	2	2	628	≤	700
13	North American Maple Veneers	0	7	0	728	≤	840
14	Chinese Maple Veneers	7	0	7	1470	≤	1470

Figure 3.2.7a: Formulation when the profitability of Pool-Runner boards is $15

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Profit ($)] Total Profit	$0.00	$6,790.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per week] Sporty (x1)	0	0	Contin
$C$6	Decision Values [# to make per week] Fancy (x2)	0	104	Contin
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	0	210	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Shaping Time (minutes)	2400	$E$11<=$G$11	Binding	0
$E$12	Truck Availability	628	$E$12<=$G$12	Not Binding	72
$E$13	North American Maple Veneers	728	$E$13<=$G$13	Not Binding	112
$E$14	Chinese Maple Veneers	1470	$E$14<=$G$14	Binding	0

Figure 3.2.7b: Answer Report when the profitability of Pool-Runner boards is $15

In Figures 3.2.7a and 3.2.7b, the profitability of Pool-Runner boards has been decreased from $20 to $15. This is a decrease of $5, which is smaller than the Allowable Decrease of $7.33. G.F. Hurley notices that the optimal solution is still to make 0 Sporty boards, 104 Fancy boards, and 210 Pool-Runner boards. But the total weekly profit has decreased by $1,050 to $6,790. This is because the profit on each of the Pool-Runner boards made has decreased by $5, and 210 · $5 = $1,050

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	210	90	0			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	35	5			$6,300.00
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2400	≤	2400
12	Truck Availability	2	2	2	600	≤	700
13	North American Maple Veneers	0	7	0	630	≤	840
14	Chinese Maple Veneers	7	0	7	1470	≤	1470

Figure 3.2.8a: Formulation when the profitability of Pool-Runner boards is $5

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Profit ($)] Total Profit	$0.00	$6,300.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per week] Sporty (x1)	0	210	Contin
$C$6	Decision Values [# to make per week] Fancy (x2)	0	90	Contin
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Shaping Time (minutes)	2400	$E$11<=$G$11	Binding	0
$E$12	Truck Availability	600	$E$12<=$G$12	Not Binding	100
$E$13	North American Maple Veneers	630	$E$13<=$G$13	Not Binding	210
$E$14	Chinese Maple Veneers	1470	$E$14<=$G$14	Binding	0

Figure 3.2.8b: Answer Report when the profitability of Pool-Runner boards is $5

In Figures 3.2.8a and 3.2.8b, the profitability of Pool-Runner boards has been decreased from $20 to $5. This is a decrease of $15, which is more than the Allowable Decrease. This time, G.F. Hurley notices that the optimal solution has changed from 0 Sporty boards, 104 Fancy boards, and 210 Pool-Runner boards to 210 Sporty boards, 90 Fancy boards, and 0 Pool-Runner boards.

This happened because the profit margin is $15 for Sporty boards, $35 for Fancy boards, and $5 for Pool-Runner boards. In this case, making 210 Sporty boards, 90 Fancy boards, and 0 Pool-Runner boards is more profitable than making 0 Sporty boards, 104 Fancy boards, and 210 Pool-Runner boards. That is, producing 0 Sporty boards, 104 Fancy boards, and 210 Pool-Runner boards generates a profit of:

$$15(0) + 35(104) + 5(210) = \$4690.$$

On the other hand, producing 210 Sporty boards, 90 Fancy boards, and 0 Pool-Runner boards generates a profit of:

$$15(210) + 35(90) + 5(0) = \$6300.$$

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	210	90	0			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	35	12.66666667			$6,300.00
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2400	≤	2400
12	Truck Availability	2	2	2	600	≤	700
13	North American Maple Veneers	0	7	0	630	≤	840
14	Chinese Maple Veneers	7	0	7	1470	≤	1470

Figure 3.2.9a: Formulation when the profitability of Pool-Runner boards is approximately $12.67

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Profit ($)] Total Profit	$0.00	$6,300.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per week] Sporty (x1)	0	210	Contin
$C$6	Decision Values [# to make per week] Fancy (x2)	0	90	Contin
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Shaping Time (minutes)	2400	$E$11<=$G$11	Binding	0
$E$12	Truck Availability	600	$E$12<=$G$12	Not Binding	100
$E$13	North American Maple Veneers	630	$E$13<=$G$13	Not Binding	210
$E$14	Chinese Maple Veneers	1470	$E$14<=$G$14	Binding	0

Figure 3.2.9b: Answer Report when the profitability of Pool-Runner boards is approximately $12.67

In Figures 3.2.9a and 3.2.9b, the profitability of Pool-Runner boards has been decreased from $20 to $12.67. This is a decrease of $7.33, which is the Allowable Decrease. Examining the Answer Report for this case, G.F. Hurley sees that Solver reports 210 Sporty boards, 90 Fancy boards, and 0 Pool-Runner boards as the optimal solution. This yields a total profit of $6,300:

$$15(210) + 35(90) + 12.67(0) = \$6300.$$

However, the original optimal solution of 0 Sporty boards, 104 Fancy boards, and 210 Pool-Runner boards also yields a total profit of $6,300:

$$15(0) + 35(104) + 12.67(210) = \$6300.$$

This is another case where the plane representing the objective function coincides with one of the boundaries of the feasible region. Once again, there are infinitely many possible optimal solutions along that boundary.

G.F. Hurley turns his attention to the Allowable Decrease in the objective coefficient of x_2. He notices that Solver reports a value of $35. Since the profitability of Fancy boards is currently $35, an Allowable Decrease of $35 means that no matter how small the profit margin is, as long as Fancy boards generate a positive profit margin, the optimal solution will not change.

Variable Cells: Reduced Cost

Next, G.F. Hurley notices that there is a Reduced Cost of approximately -7.33 listed for Sporty boards (x_1). To see what this means, G.F. Hurley forces the production of one Sporty board by adding the constraint $x_1 = 1$. Figure 3.2.10 shows the new formulation after Solver has found the optimal solution.

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	1	103.933333	209			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	35	20			$7,832.67
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2400	≤	2400
12	Truck Availability	2	2	2	627.867	≤	700
13	North American Maple Veneers	0	7	0	727.533	≤	840
14	Chinese Maple Veneers	7	0	7	1470	≤	1470
15	Force One Sporty Board	1	0	0	1	=	1

Figure 3.2.10: Forcing the production of one Sporty board (x_1)

The new optimal solution is 1 Sporty board, approximately 103.933 Fancy boards, and 209 Pool-Runner boards. (There is nothing wrong with having a non-integer solution since the decision variables are production rates per week, not number of skateboards sold.) This production mix generates a total profit of $7,832.67 per week. This is a decrease of $7840 - $7832.67 = $7.33 per week. Therefore, **Reduced Cost** refers to the change in the Final Value of the objective function that is caused by increasing a decision variable by one unit.

Q1. SK8MAN has a regular customer who wants to special order 10 Sporty boards. If SK8MAN manufactures those boards, how will that affect profit for that week?

Alternatively, one could think of Reduced Cost as the amount by which the objective function coefficient would have to increase to before it would profitable to make that item. For example, G.F. Hurley notices that the Allowable Increase for Sporty boards is approximately 7.33 and the Reduced Cost for Sporty boards is approximately -7.33. This is not a coincidence! If G.F. Hurley increases the profitability of Sporty boards by more than $7.33, then Sporty boards would be profitable to produce.

To experiment with this idea, G.F. Hurley changes the profitability of Sporty boards to $23. The result is shown in Figure 3.2.11. In this case, it is now profitable to produce Sporty boards.

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	**210**	**90**	**0**			
7							**Total Profit**
8	Objective Function [Profit ($)]	23	35	20			**$7,980.00**
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	**2400**	≤	2400
12	Truck Availability	2	2	2	**600**	≤	700
13	North American Maple Veneers	0	7	0	**630**	≤	840
14	Chinese Maple Veneers	7	0	7	**1470**	≤	1470
15							

Figure 3.2.11: Formulation when the profitability of Sporty boards is $23

In a maximization problem, the Reduced Cost value will always be less than or equal to zero. If the Reduced Cost is less than zero, then it is not profitable to make the product. G.F. Hurley considers what it means to have a Reduced Cost value of zero.

The Sensitivity Report in Figure 3.2.3 shows that the Reduced Cost values for the other two decision variables, x_2 and x_3,are both 0. That is because both of those decision variables are already part of the optimal solution. If a decision variable is not part of the optimal solution, its final value is 0. Its Reduced Cost measures the amount the final value of the objective function would be reduced if the value of the decision variable were increased by just 1 unit.

Constraints: Shadow Price

G.F. Hurley then moves to the Constraints section of the Sensitivity Report (see Figure 3.2.12), where the most useful piece of information is the Shadow Price. The **Shadow Price** tells the effect on the value of the objective function of increasing the resource that is constraining the solution by 1 unit. In other words, the Shadow Price refers to the amount by which the objective function value changes given a 1-unit increase or decrease in one right hand side of a constraint.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Values [# to make per week] Sporty (x1)	0	-7.333333333	15	7.333333333	1E+30
$C$6	Decision Values [# to make per week] Fancy (x2)	104	0	35	40	35
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	210	0	20	1E+30	7.333333333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$11	Shaping Time (minutes)	2400	2.333333333	2400	240	1560
$E$12	Truck Availability	628	0	700	1E+30	72
$E$13	North American Maple Veneers	728	0	840	1E+30	112
$E$14	Chinese Maple Veneers	1470	1.523809524	1470	343.6363636	420

Figure 3.2.12: Sensitivity Report for the 3-variable SK8MAN problem

For example, the shaping time constraint shows a Shadow Price of approximately 2.33. That means if the amount of shaping time were increased by 1 unit, the value of the objective function would increase by 2.33 units.

In the context of the SK8MAN problem, the units of shaping time are minutes, and the units of the objective function are dollars. Suppose the workers agree to work a total of 100 minutes longer each week. Doing so would increase the available shaping time by 100 minutes. According to the Sensitivity Report, that should increase the value of the objective function for the optimal solution by approximately 100 · \$2.3333 = \$233.33.

Figures 3.2.13a and 3.2.13b show that the objective has increased to \$8,073.33. The new production plan is 0 Sporty boards, approximately 110.67 Fancy boards, and 210 Pool-Runner boards.

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	0	110.666667	210			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	35	20			$8,073.33
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2500	≤	2500
12	Truck Availability	2	2	2	641.333	≤	700
13	North American Maple Veneers	0	7	0	774.667	≤	840
14	Chinese Maple Veneers	7	0	7	1470	≤	1470

Figure 3.2.13a: Formulation when available shaping time is increased by 100 minutes

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Profit ($)] Total Profit	$0.00	$8,073.33

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per week] Sporty (x1)	0	0	Contin
$C$6	Decision Values [# to make per week] Fancy (x2)	0	110.6666667	Contin
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	0	210	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Shaping Time (minutes)	2500	$E$11<=$G$11	Binding	0
$E$12	Truck Availability	641.3333333	$E$12<=$G$12	Not Binding	58.66666667
$E$13	North American Maple Veneers	774.6666667	$E$13<=$G$13	Not Binding	65.33333333
$E$14	Chinese Maple Veneers	1470	$E$14<=$G$14	Binding	0

Figure 3.2.13b: Answer Report when available shaping time is increased by 100 minutes

On the other hand, suppose that the available shaping time is reduced by 150 minutes. That reduction should then reduce the final value of the objective function by approximately 150 · $2.333 = $350. Figures 3.2.14a and 3.2.14b demonstrate that the objective function has decreased to $7,490. The new production plan is 0 Sporty boards, 94 Fancy boards, and 210 Pool-Runner boards.

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	**0**	**94**	**210**			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	35	20			**$7,490.00**
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	**2250**	≤	2250
12	Truck Availability	2	2	2	**608**	≤	700
13	North American Maple Veneers	0	7	0	**658**	≤	840
14	Chinese Maple Veneers	7	0	7	**1470**	≤	1470

Figure 3.2.14a: Formulation when available shaping time is reduced by 150 minutes

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Profit ($)] Total Profit	$0.00	$7,490.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# to make per week] Sporty (x1)	0	0	Contin
$C$6	Decision Values [# to make per week] Fancy (x2)	0	94	Contin
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	0	210	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Shaping Time (minutes)	2250	$E$11<=$G$11	Binding	0
$E$12	Truck Availability	608	$E$12<=$G$12	Not Binding	92
$E$13	North American Maple Veneers	658	$E$13<=$G$13	Not Binding	182
$E$14	Chinese Maple Veneers	1470	$E$14<=$G$14	Binding	0

Figure 3.2.14b: Answer Report when available shaping time is reduced by 150 minutes

Thus, the Shadow Price shows how much the value of the objective function will increase or decrease for each unit of increase or decrease in the availability of one of the constraining resources.

Constraints: Allowable Increase and Allowable Decrease

Returning once again to the Sensitivity Report in Figure 3.2.12, G.F. Hurley puts his attention towards the columns for an **Allowable Increase** and **Allowable Decrease** for each of the constraints. These refer to increases or decreases in the right hand side of a constraint (i.e., increasing or decreasing the availability of one of the constraining resources, such as shaping time). If an increase or decrease falls within the range determined by the Allowable Increase and Allowable Decrease, then the Shadow Price will remain the same.

For example, from the Sensitivity Report, the Allowable Increase in shaping time is 240 minutes, and the Allowable Decrease is 1,560 minutes. So, if a change in the availability of shaping time falls in the range between an increase of 240 minutes and a decrease of 1,560 minutes, the Shadow Price will stay constant at approximately $2.33.

G.F. Hurley wonders what happens if a change in the available shaping time falls outside this range. He supposes that the shaping time increases by 241 minutes. Since 241 is greater than the Allowable Increase, there should be an effect on the Shadow Price. Figures 3.2.15a and 3.2.15b show the spreadsheet formulation and sensitivity report, respectively, for this change.

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	0	120	210			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	35	20			$8,400.00
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	2640	≤	2641
12	Truck Availability	2	2	2	660	≤	700
13	North American Maple Veneers	0	7	0	840	≤	840
14	Chinese Maple Veneers	7	0	7	1470	≤	1470

Figure 3.2.15a: Formulation when the shaping time constraint increases by 241 minutes

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Values [# to make per week] Sporty (x1)	0	-5	15	5	1E+30
$C$6	Decision Values [# to make per week] Fancy (x2)	120	0	35	1E+30	35
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	210	0	20	1E+30	5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$11	Shaping Time (minutes)	2640	0	2641	1E+30	1
$E$12	Truck Availability	660	0	700	1E+30	40
$E$13	North American Maple Veneers	840	5	840	0.466666667	840
$E$14	Chinese Maple Veneers	1470	2.857142857	1470	1.75	1470

Figure 3.2.15b: Sensitivity Report when the shaping time constraint increases by 241 minutes

G.F. Hurley notices that the Shadow Price has changed to 0. A Shadow Price of 0 means that there is no value in increasing the availability of installation time any further. Therefore, increasing the availability of a resource beyond the Allowable Increase *decreases* the Shadow Price.

He also notices that increasing the available shaping time changes the optimal solution to 0 Sporty boards, 120 Fancy boards, and 210 Pool-Runner boards per week. Furthermore, the Allowable Increase for shaping time has changed to infinity (1E+30). This means there is no value in increasing the available shaping time any more.

Next, G.F. Hurley investigates what happens if the available shaping time decreases below the Allowable Decrease of 1,560. Suppose he decreases it by 1,561 minutes. The available shaping time becomes 839 minutes. Figures 3.2.16a and 3.2.16b show the spreadsheet formulation and sensitivity report for this change.

	A	B	C	D	E	F	G
1	Chapter 3: Sensitivity Analysis						
2	3.2 SK8MAN, Inc. (3 variables)						
3	Profit Maximization Problem						
4							
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)			
6	Decision Values [# to make per week]	0	0	209.75			
7							**Total Profit**
8	Objective Function [Profit ($)]	15	35	20			$4,195.00
9							
10	**Constraints**						
11	Shaping Time (minutes)	5	15	4	839	≤	839
12	Truck Availability	2	2	2	419.5	≤	700
13	North American Maple Veneers	0	7	0	0	≤	840
14	Chinese Maple Veneers	7	0	7	1468.25	≤	1470

Figure 3.2.16a: Formulation when the shaping time constraint decreases by 1,561 minutes

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Values [# to make per week] Sporty (x1)	0	-10	15	10	1E+30
$C$6	Decision Values [# to make per week] Fancy (x2)	0	-40	35	40	1E+30
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	209.75	0	20	1E+30	8

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$11	Shaping Time (minutes)	839	5	839	1	839
$E$12	Truck Availability	419.5	0	700	1E+30	280.5
$E$13	North American Maple Veneers	0	0	840	1E+30	840
$E$14	Chinese Maple Veneers	1468.25	0	1470	1E+30	1.75

Figure 3.2.16b: Sensitivity Report when the shaping time constraint decreases by 1,561 minutes

G.F. Hurley notices that this change increases the value of the Shadow Price to $5. That is, decreasing the availability of a resource beyond the Allowable Decrease *increases* the Shadow Price. This makes economic sense, because decreasing the availability of a resource, as he did, increases the value per unit of that resource.

3.2.3 Adding a Fourth Product—Is it profitable?

The managers at SK8MAN, Inc. are now considering adding a fourth line of skateboards to their portfolio of products. The EasyRider skateboard will be made from seven North American maple veneers, require 12 minutes of shaping time, and, of course, require 2 trucks. The managers believe that each EasyRider skateboard manufactured will earn $25 profit. They are excited by the prospect of adding a new product to their line, but the key question is whether it will be profitable to do so. Figures 3.2.17 and 3.2.18 display the problem formulation with a fourth decision variable and the optimal solution, as well as the Sensitivity Report.

	A	B	C	D	E	F	G	H
1	Chapter 3: Sensitivity Analysis							
2	3.2 SK8MAN, Inc. (4 variables)							
3	Profit Maximization Problem							
4								
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)	**EasyRider** (x_4)			
6	Decision Values [# to make per week]	**0**	**104**	**210**	**0**			
7								**Total Profit**
8	Objective Function [Profit ($)]	15	35	20	25			**$7,840.00**
9								
10	**Constraints**							
11	Shaping Time (minutes)	5	15	4	12	**2400**	≤	2400
12	Truck Availability	2	2	2	2	**628**	≤	700
13	North American Maple Veneers	0	7	0	7	**728**	≤	840
14	Chinese Maple Veneers	7	0	7	0	**1470**	≤	1470

Figure 3.2.17: Formulation for the 4-decision variable SK8MAN problem

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Values [# to make per week] Sporty (x1)	0	-7.333333333	15	7.333333333	1E+30
$C$6	Decision Values [# to make per week] Fancy (x2)	104	0	35	40	3.75
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	210	0	20	1E+30	7.333333333
$E$6	Decision Values [# to make per week] EasyRider (x4)	0	-3	25	3	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$11	Shaping Time (minutes)	2400	2.333333333	2400	240	1560
$F$12	Truck Availability	628	0	700	1E+30	72
$F$13	North American Maple Veneers	728	0	840	1E+30	112
$F$14	Chinese Maple Veneers	1470	1.523809524	1470	343.6363636	420

Figure 3.2.18: Sensitivity Report for the 4-variable SK8MAN problem

Notice that the optimal solution has not changed, despite the addition of a new product. That means it is not profitable to make the new product. SK8MAN, Inc. will earn more profit by continuing to make only Fancy and Pool-Runner skateboards. Now the question is what, if anything, can be done so that making the new EasyRider boards would be part of SK8MAN's optimal production plan.

To answer that question, G.F. Hurley turns his attention to the Sensitivity Report. Considering the information on the EasyRider board (product x_4), he sees that the Allowable Increase in the objective coefficient is 3. That means that the profitability of EasyRider boards would have to increase by at least $3 (to $28) per board before they would become part of the optimal solution.

The shadow prices on the constraints help explain why the profit margin would need to be at least $28 for each EasyRider skateboard. Each EasyRider board requires 7 North American maple veneers and 2 trucks. The related resource constraints have 0 shadow prices because not all of these resources are currently being used. However, each EasyRider requires 12 minutes of installation. Each minute has a shadow price of $2.333. If G.F. Hurley multiplies 12 by $2.333, he obtains $28. Thus the resources needed to produce an EasyRider board are valued at $28 with the current optimal production plan.

Now, suppose the marketing division at SK8MAN, Inc. has just signed a contract with Allie Loop, the top female skateboarder in the world. She will endorse the new EasyRider board. Taking into consideration the cost of Allie Loop's endorsement contact, the marketing division estimates that the retail price of an EasyRider can be increased by $5. This would then increase the profitability of EasyRider to $30 per board. Since the increase in profitability is larger than the Allowable Increase, this should be enough to make it profitable to produce EasyRider boards.

Figures 3.2.19a and 3.2.19b shows the problem formulation and the Sensitivity Report after increasing the profitability of the EasyRider board (x_4) to $30 per board.

	A	B	C	D	E	F	G	H
1	Chapter 3: Sensitivity Analysis							
2	3.2 SK8MAN, Inc. (4 variables)							
3	Profit Maximization Problem							
4								
5	Decision Variable	**Sporty** (x_1)	**Fancy** (x_2)	**Pool-Runner** (x_3)	**EasyRider** (x_4)			
6	Decision Values [# to make per week]	**0**	**40**	**210**	**80**			
7								**Total Profit**
8	Objective Function [Profit ($)]	15	35	20	30			**$8,000.00**
9								
10	**Constraints**							
11	Shaping Time (minutes)	5	15	4	12	**2400**	≤	2400
12	Truck Availability	2	2	2	2	**660**	≤	700
13	North American Maple Veneers	0	7	0	7	**840**	≤	840
14	Chinese Maple Veneers	7	0	7	0	**1470**	≤	1470

Figure 3.2.19a: Formulation when the profitability of EasyRider boards is $30

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Values [# to make per week] Sporty (x1)	0	-6.666666667	15	6.666666667	1E+30
$C$6	Decision Values [# to make per week] Fancy (x2)	40	0	35	2.5	5
$D$6	Decision Values [# to make per week] Pool-Runner (x3)	210	0	20	1E+30	6.666666667
$E$6	Decision Values [# to make per week] EasyRider (x4)	80	0	30	5	2

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$11	Shaping Time (minutes)	2400	1.666666667	2400	240	120
$F$12	Truck Availability	660	0	700	1E+30	40
$F$13	North American Maple Veneers	840	1.428571429	840	70	112
$F$14	Chinese Maple Veneers	1470	1.904761905	1470	140	420

Figure 3.2.19b: Sensitivity Report when the profitability of EasyRider boards is $30

When the profitability of the EasyRider board is increased to $30 per board, it becomes profitable to produce 80 of them per week. G.F. Hurley compares this optimal production plan with the optimal production plan before SK8MAN got the Allie Loop endorsement (see Figure 3.2.17). He notices that 210 Pool-Runner boards (x_3) will still be produced, but only 40 Fancy boards will be produced. So, in order to produce 80 EasyRider boards, 64 *fewer* Fancy boards would have to be made.

G.F. Hurley wonders why this is more profitable to produce 64 fewer Fancy boards while producing 80 more EasyRider boards. He considers the profit margins on each of the boards. Making 64 fewer Fancy boards would decrease the total profit by (64)($35) = $2,240. At the same time, making 80 EasyRider boards that were not being made before would increase the total profit by (80)($30) = $2,400. Thus, the total profit is being increased by $2,400 – $2,240 = $160 per week.

Section 3.3: The Pallas Sport Shoe Company

Recall from Chapter 2 that the Pallas Sport Shoe Company manufactures six different lines of sport shoes: High Rise, Max-Riser, Stuff It, Zoom, Sprint, and Rocket. Table 3.3.1 displays the amount of daily profit generated by each pair of shoes for each of these six products. It also lists the amount of time each line of shoes requires for the six steps of production. The last line of the table shows the total amount of time per day available for each of the six production steps. Sue Painter, the production manager of the company would like to determine the daily production rates for each line of shoes that will maximize profit.

	High Rise	Max-Riser	Stuff It	Zoom	Sprint	Rocket	Total Time Available (minutes per day)
Profit	\$18	\$23	\$22	\$20	\$18	\$19	
Stamping	1.25	2	1.5	1.75	1	1.25	420
Upper Finishing	3.5	3.75	5	3	4	4.25	1,260
Insole Stitching	2	3.25	2.75	2.25	3	2.5	840
Molding	5.5	6	7	6.5	8	5	2,100
Sole-to-Upper Joining	7.5	7.25	6	7	6.75	6.5	2,100
Inspecting	2	3	2	3	2	3	840

Table 3.3.1: Profit and production detail per pair for six lines of sport shoes

3.3.1 Problem Formulation

The formulation of the problem is given below.

Decision Variables

Let:
x_1 = the daily production rate of High Rise
x_2 = the daily production rate of Max-Riser
x_3 = the daily production rate of Stuff It
x_4 = the daily production rate of Zoom
x_5 = the daily production rate of Sprint
x_6 = the daily production rate of Rocket
z = the amount of profit Pallas Sport Shoe Company earns per day

Objective Function

Maximize: $z = 18x_1 + 23x_2 + 22x_3 + 20x_4 + 18x_5 + 19x_6$

Subject to:

Constraints

Stamping Time: $1.25x_1 + 2x_2 + 1.5x_3 + 1.75x_4 + x_5 + 1.25x_6 \leq 420$
Upper Finishing Time: $3.5x_1 + 3.75x_2 + 5x_3 + 3x_4 + 4x_5 + 4.25x_6 \leq 1{,}260$
Insole Stitching Time: $2x_1 + 3.25x_2 + 2.75x_3 + 2.25x_4 + 3x_5 + 2.5x_6 \leq 840$
Molding Time: $5.5x_1 + 6x_2 + 7x_3 + 6.5x_4 + 8x_5 + 5x_6 \leq 2{,}100$
Sole-to-Upper Joining Time: $7.5x_1 + 7.25x_2 + 6x_3 + 7x_4 + 6.75x_5 + 6.5x_6 \leq 2{,}100$
Inspecting Time: $2x_1 + 3x_2 + 2x_3 + 3x_4 + 2x_5 + 3x_6 \leq 840$
Non-Negativity: $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$

This formulation as it appears in a spreadsheet is presented in Figure 3.3.1. Solver has been run, and the optimal solution also appears in the spreadsheet.

	A	B	C	D	E	F	G	H	I	J
1	Chapter 3: Sensitivity Analysis									
2	3.3 Pallas Sport Show Company									
3	Profit Maximization									
4										
5	Decision Variable	**High Rise** (x_1)	**Max-Riser** (x_2)	**Stuff It** (x_3)	**Zoom** (x_4)	**Sprint** (x_5)	**Rocket** (x_6)			
6	Decision Values [# to make per day]	0	4.28294434	45.12172	72.3275	104.902	89.82118			
7										**Total Profit**
8	Objective Function [Profit ($)]	18	23	22	20	18	19			$6,132.57
9										
10	**Constraints**							**Used**		**Available**
11	Stamping (minutes)	1.25	2	1.5	1.75	1	1.25	420	≤	420
12	Upper Finishing (minutes)	3.5	3.75	5	3	4	4.25	1260	≤	1260
13	Insole Stitching (minutes)	2	3.25	2.75	2.25	3	2.5	840	≤	840
14	Molding (minutes)	5.5	6	7	6.5	8	5	2100	≤	2100
15	Sole-to-Upper Joining (minutes)	7.5	7.25	6	7	6.75	6.5	2100	≤	2100
16	Inspecting (minutes)	2	3	2	3	2	3	799.34219	≤	840

Figure 3.3.1: Pallas Sport Shoes Spreadsheet Formulation and Optimal Solution

3.3.2 Interpreting the Solution

Q1. Without referring to an Answer or Sensitivity Report, which of the constraints in the spreadsheet in Figure 3.3.1 are binding and which are non-binding? How do you know?

Q2. Similarly, which one of the constraints will show a Shadow Price of zero in the Sensitivity Report, and why does that make sense?

Sue Painter has seen the Answer and Sensitivity Reports. She wonders, "How do I go about implementing this optimal solution?" In order to answer this question, the production manager must understand what the optimal solution means.

Q3. The optimal solution given in the spreadsheet from Figure 3.3.1 lists $x_1 = 0$. What does that mean? What does it mean that $x_2 \approx 4.2829$?

Recalling that the decision variables in the problem were defined as daily production rates, $x_2 \approx 4.2829$means that on most days, 4 Max-Riser shoes will be produced. Then, approximately every fourth day, 5 Max-Riser shoes will be produced. This production plan would yield 4.25 Max-Riser shoes every four days.

Similarly, a daily production rate for $x_3 \approx 45.1217$ means that on most days 45 will be produced, but on about every eighth day, 46 will be produced. This production plan would yield 45.125 Stuff It shoes every eight days.

Q4. How might the production rate of 72.3275 for product x_4 be implemented?

So, in order to implement the optimal production plan, the production manager will have to allocate production resources in such a way that the optimal production rates are achieved.

Figure 3.3.2 shows the Sensitivity Report for the optimal solution to the Pallas Sport Shoe problem. The production manager notices that it reports an Allowable Increase of about $0.0507 in the coefficient of x_1 in the objective function.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Values [# to make per day] High Rise (x1)	0	-0.05070018	18	0.05070018	1E+30
$C$6	Decision Values [# to make per day] Max-Riser (x2)	4.282944345	0	23	0.083039285	0.239247312
$D$6	Decision Values [# to make per day] Stuff It (x3)	45.12172352	0	22	0.100842737	1.120253165
$E$6	Decision Values [# to make per day] Zoom (x4)	72.32746858	0	20	0.25648415	0.055429065
$F$6	Decision Values [# to make per day] Sprint (x5)	104.9019749	0	18	6.260393168	0.529761905
$G$6	Decision Values [# to make per day] Rocket (x6)	89.82118492	0	19	0.572583906	0.040500229

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$11	Stamping (minutes) Used	420	5.580179533	420	113.5815474	8.859180036
$H$12	Upper Finishing (minutes) Used	1260	1.793895871	1260	15.4507772	111.8007117
$H$13	Insole Stitching (minutes) Used	840	0.255655296	840	72.56195965	4.008064516
$H$14	Molding (minutes) Used	2100	0.203375224	2100	29.90972919	134.3521595
$H$15	Sole-to-Upper Joining (minutes) Used	2100	0.422262118	2100	20.14864865	179.7068966
$H$16	Inspecting (minutes) Used	799.3421903	0	840	1E+30	40.65780969

Figure 3.3.2: Sensitivity Report for the Pallas Sport Shoe problem

Q5. Suppose Pallas Shoes was able to increase the profit margin on x_1 to $18.05. Would this change affect the optimal solution? Why or why not?

Q6. Suppose Pallas Shoes was able to increase the profitability of x_1 to $18.10. What would be the effect on the optimal solution of this increase?

3.3.3 Using the Sensitivity Report to Make Decisions

Pallas Shoes is considering adding an hour of overtime to one of the workers. Sue Painter must decide to which of the production tasks the overtime should go.

Q7. Using the Sensitivity Report in Figure 3.3.2 to guide the decision, to which of the six production tasks should the extra time be added? Why?

Suppose that the union contract mandates that any overtime work be paid at double the normal rate of $28 per hour.

Q8. Would it be profitable to add to one hour of overtime? If so, how much larger than the cost of the overtime would the increase in profits be? If not, at what hourly pay rate would it be profitable?

Finally, the managers at Pallas Sport Shoes are considering adding another line of shoes. The data for the new Pro-Go model is shown in Table 3.3.2.

	Pro-Go
Profit	$20
Stamping	1.5
Upper Finishing	3.9
Insole Stitching	2.6
Molding	6.3
Sole-to-Upper Joining	6.8
Inspecting	2.5

Table 3.3.2: Profit and production detail per pair of Pro-Go sport shoes

At present, there are no plans to increase the total amount of time available for each of the six steps of production. Figure 3.3.3 contains the new spreadsheet and optimal solution with the information for Pro-Go as decision variable x_7. Figure 3.3.4 shows the Sensitivity Report.

	A	B	C	D	E	F	G	H	I	J	K
1	Chapter 3: Sensitivity Analysis										
2	3.3 Pallas Sport Show Company										
3	Profit Maximization										
4											
5	Decision Variable	**High Rise** (x_1)	**Max-Riser** (x_2)	**Stuff It** (x_3)	**Zoom** (x_4)	**Sprint** (x_5)	**Rocket** (x_6)	**Pro-Go** (x_7)			
6	Decision Values [# to make per day]	0	4.2829443	45.1217	72.3275	104.902	89.8212	0			
7											**Total Profit**
8	Objective Function [Profit ($)]	18	23	22	20	18	19	20			$6,132.57
9											
10	**Constraints**								**Used**		**Available**
11	Stamping (minutes)	1.25	2	1.5	1.75	1	1.25	1.5	420	≤	420
12	Upper Finishing (minutes)	3.5	3.75	5	3	4	4.25	3.9	1260	≤	1260
13	Insole Stitching (minutes)	2	3.25	2.75	2.25	3	2.5	2.6	840	≤	840
14	Molding (minutes)	5.5	6	7	6.5	8	5	6.3	2100	≤	2100
15	Sole-to-Upper Joining (minutes)	7.5	7.25	6	7	6.75	6.5	6.8	2100	≤	2100
16	Inspecting (minutes)	2	3	2	3	2	3	2.5	799.34219	≤	840

Figure 3.3.3: Formulation with seven decision variables

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Values [# to make per day] High Rise (x1)	0	-0.05070018	18	0.05070018	1E+30
$C$6	Decision Values [# to make per day] Max-Riser (x2)	4.282944345	0	23	0.083039285	0.239247312
$D$6	Decision Values [# to make per day] Stuff It (x3)	45.12172352	0	22	0.100842737	1.120253165
$E$6	Decision Values [# to make per day] Zoom (x4)	72.32746858	0	20	0.25648415	0.055429065
$F$6	Decision Values [# to make per day] Sprint (x5)	104.9019749	0	18	6.260393168	0.529761905
$G$6	Decision Values [# to make per day] Rocket (x6)	89.82118492	0	19	0.572583906	0.040500229
$H$6	Decision Values [# to make per day] Pro-Go (x7)	0	-0.183813285	20	0.183813285	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$I$11	Stamping (minutes) Used	420	5.580179533	420	113.5815474	8.859180036
$I$12	Upper Finishing (minutes) Used	1260	1.793895871	1260	15.4507772	111.8007117
$I$13	Insole Stitching (minutes) Used	840	0.255655296	840	72.56195965	4.008064516
$I$14	Molding (minutes) Used	2100	0.203375224	2100	29.90972919	134.3521595
$I$15	Sole-to-Upper Joining (minutes) Used	2100	0.422262118	2100	20.14864865	179.7068966
$I$16	Inspecting (minutes) Used	799.3421903	0	840	1E+30	40.65780969

Figure 3.3.4: Sensitivity Report with seven decision variables

Q9. Why was it not profitable to produce the new product?

Q10. How much would its profit margin have to increase to make it profitable enough to produce?

Section 3.4: Chapter 3 (Sensitivity Analysis) Homework Questions

Chapter 5 Summary

What have we learned?

Terms

Allowable Decrease

Allowable Increase

Binding

Constraint

Final Value

Original Value

Reduced Cost

Sensitivity Analysis

Shadow Price

Slack

Variable Cells

Chapter 3 (Sensitivity Analysis) Objectives

You should be able to:

-

Chapter 3 Study Guide

1.

Section 4.0: Introduction

The last two chapters focused on maximizing profit. First, Chapter 2 explained how to explore maximization linear programming problems by hand and with Excel. Then, Chapter 3 was dedicated to interpreting and analyzing the solutions and constraints of these maximization problems. In both of these chapters, the goal was to find the largest value of the objective function, given a set of constraints.

In this chapter, minimization linear programming problems are introduced. Excel Solver is again used, but this time, the intent is to obtain the smallest value of the objective function for a given set of constraints. The chapter begins by exploring the problem of finding a food program for Malawian children. The food program needs to meet daily nutritional requirements while minimizing calories.

Next, a group of planners is trying to reduce the amount of water pollution coming into two watersheds in Wisconsin. They need to keep in mind a number of constraints as they attempt to minimize the cost of the project. Excel Solver is used to solve this problem.

Finally, the chapter ends with a linear programming problem involving a gasoline distributer. The distributor needs to determine the optimal gasoline blend while minimizing the cost of this blend. Again, Excel Solver is used to solve this problem.

In each of these problems, the goal is to minimize the value of the objective function, representing either calories or cost. The methods used to formulate and solve these problems are very similar to the methods used to solve maximization problems. The largest difference is simply the way one thinks about the set-up of the problem. There will also be a greater mix of constraint types. In the previous chapters all of the constraints were of the form, “less than or equal to” and involved a resource or market demand limit. In this chapter many of the constraints will be of the form, “greater than or equal to”.

Section 4.1: Nutrition in Malawi

Malawi is a landlocked country in southern Africa (see Figures 4.1.1 and 4.1.2). Its population of over 13,000,000 lives in an area about the size of the state of Pennsylvania. Malawi's economy is largely based on agriculture. Much of its population is impoverished. As a result, the diets of Malawian children are frequently deficient in essential nutrients.

Figure 4.1.1: Map of southern Africa with Malawi in black

Figure 4.1.2: Detailed map of Malawi

Malawi is an impoverished nation, so the financial aspect of any food program is a vital concern. For that reason, Dr. Corr, an administrator at the World Health Organization, needs to determine an optimal food program for this country. In particular, he needs to minimize the total number of calories while meeting the minimum requirements for key nutrients, using the highest nutrient concentrated food combinations. This will be the most efficient way to meet the children's minimum nutritional requirements. A common problem in poor regions is that their diet is dominated by low cost high caloric foods with little other nutritional value. All of the foods under consideration are readily available in Malawi at low cost.

The key nutrients Dr. Corr takes into account and the minimum daily requirements recommended by the World Health Organization are listed in Table 4.1.1. The foods available to the Malawians are shown in Figure 4.1.3. Notice that the units of measurement are not all the same in Table 4.1.1. For example, protein is measured in grams. Calcium and iron are measured in milligrams. Vitamins B_9 and B_{12} are measured in micrograms. Nutritional facts per gram for these foods appear in Table 4.1.2. These nutrients are all scaled based on the same units of measurement in Table 4.1.1. For example, let's look at a gram of maize flour. Each gram of maize flour contains 0.08120 grams of protein. It also contains 0.0612 milligrams of calcium and 0.03450 milligrams of iron. Each gram of maize flour contains 0.2450 micrograms of vitamin B_9 and so forth. The caloric content per gram of each of the food sources is given in Table 4.1.3.

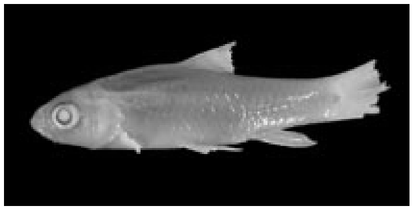
Nutrient	Minimum daily requirement
Protein	20 grams (g)
Calcium (Ca)	400 milligrams (mg)
Iron (Fe)	7 mg
Folate (Vitamin B_9)	50 micrograms (μg)
Cyanocobalamin (Vitamin B_{12})	0.5 μg
Ascorbic acid (Vitamin C)	20 mg
Thiamine (Vitamin B_1)	0.7 mg
Riboflavin (Vitamin B_2)	1.1 mg
Niacin (Vitamin B_3)	12.1 mg
Retinol (Vitamin A)	400 μg

Table 4.1.1: Nutrients and minimum daily requirements

Maize flour

Tangerines

Pigeon peas



Matemba

Potatoes

Chinese cabbage

Figure 4.1.3: Malawian foods

Food	Protein (g)	Ca (mg)	Fe (mg)	Vit. B_9 (μg)	Vit. B_{12} (μg)	Vit. C (mg)	Vit. B_1 (mg)	Vit. B_2 (mg)	Vit. B_3 (mg)	Vit. A (μg)
Maize flour	0.08120	0.0612	0.03450	0.2450	0	0	0.00385	0.00201	0.03630	0.112
Tangerines	0.00805	0.3640	0.00156	0.1560	0	0.268	0.00058	0.00036	0.00377	0.338
Pigeon peas	0.06760	0.4290	0.10000	1.1000	0	0	0.00148	0.00571	0.00781	0
Matemba	0.20100	0.1000	0.00556	0.2440	0.0158	0	0.00041	0.00063	0.03900	0
Potatoes	0.01960	0.0507	0.00350	0.0922	0	0.128	0.00105	0.00021	0.01390	0
Chinese cabbage	0.01500	1.0500	0.00800	0.6630	0	0.450	0.00040	0.00070	0.00500	2.230

Table 4.1.2: Nutritional content per gram of foods

Food	Energy content (cal/g)
Maize flour	3.620
Tangerines	0.532
Pigeon peas	1.190
Matemba	0.956
Potatoes	0.931
Chinese cabbage	0.131

Table 4.1.3: Energy content of foods

Dr. Corr uses linear programming to design a diet that meets all nutritional requirements while keeping the intake of calories at a minimum. In this case, each nutrient under consideration acts as a constraint. The total number of calories is the objective function. Note that the objective is to *minimize* the number of calories. Therefore, the method used to solve this problem must be different from the previous chapters, where the objective was maximization.

Q1. How do you think this minimization problem differs from the maximization problems in the previous two chapters?

Minimization Linear Programming Problems

Minimization linear programming problems are solved very similarly to maximization problems. However, instead of the optimal solution being the *largest* value of the objective function, it is now the *smallest*.

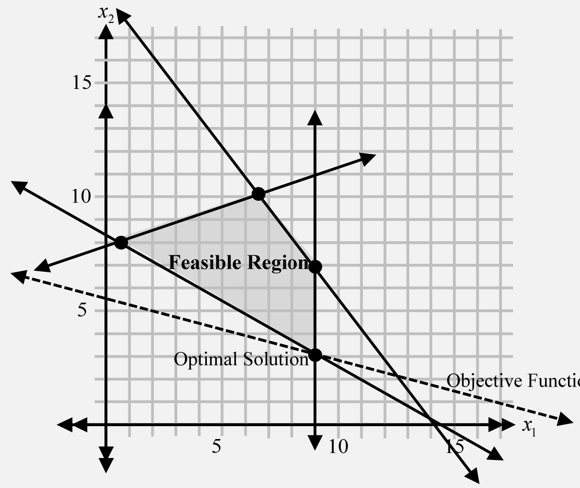


Figure 4.1.4: Example of a feasible region and optimal solution in a minimization problem

4.1.1 Linear Programming Formulation

The first step in the formulation of a linear programming problem is to define the decision variables in the problem. The decision variables are then used to define the objective function. In the Malawian diet problem, Dr. Corr seeks to minimize the daily intake of calories. Therefore, the objective function must represent the total calories in the diet per day. These calories come from the available foods. Thus, each decision variable represents the amount of each type of food in the diet each day.

Q2. How many decision variables should be defined?

Q3. How should the decision variables be defined?

Q4. Use the decision variables and the values in Table 4.1.3 to write the objective function, where z represents the number of calories consumed per day.

The last step in the formulation is writing the system of constraints for the problem. The diet must meet or exceed the minimum daily allowances for each of the key nutrients listed in Table 4.1.1. Therefore, Dr. Corr multiplies the number of grams of the food by its corresponding nutritional content to obtain the amount of that nutrient consumed. For example, if a person eats x_1 grams of maize flour, he/she consumes 0.0612 mg of calcium (see Table 4.1.2). Dr. Corr multiplies these two numbers together to show the number of milligrams of calcium the maize flour serving provides:

(x_1 grams of maize flour)(0.0612 mg of calcium per gram) = $0.0612x_1$ mg of calcium

Dr. Corr needs to determine the total number of milligrams of calcium in the food intake for an entire day. To do so, he must repeat the above process for each decision variable and add the results together. This yields the following expression:.

$$0.0612x_1 + 0.3640x_2 + 0.4290x_3 + 0.1000x_4 + 0.0507x_5 + 1.0500x_6$$

This represents the total number of milligrams of calcium in the food intake for an entire day.
Then, Dr. Corr completes this constraint by making the above expression greater than or equal to 400:

$$0.0612x_1 + 0.3640x_2 + 0.4290x_3 + 0.1000x_4 + 0.0507x_5 + 1.0500x_6 \geq 400$$

In this equation both the left hand side of the equation and right hand side are in milligrams.

Q5. Why is this constraint greater than or equal to 400?

Q6. Continue this process to find a constraint inequality for each of the nutrients in Table 4.1.1.

Next, Dr. Corr builds the notion of a balanced diet into the model. To do this, he takes into account what nutritionists recommend as the minimum and maximum number of calories for a typical Malawian child aged 6–9 years. This recommendation is also broken down into a range of calories for each of the various food groups. Table 4.1.4 contains these recommended minimums and maximums.

Food Group	Food	Minimum	Maximum
Cereals	Maize flour	900	1100
Fruits	Tangerines	15	45
Legumes	Pigeon peas	45	150
Fish, meat, eggs	Matemba	30	90
Roots	Potatoes	60	240
Vegetables	Chinese cabbage	15	45

Table 4.1.4: Minimum and maximum calories per day by food group for children in Malawi

For example, the table shows that the recommended number of calories per day from cereals is between 900 and 1100 calories. In the Malawian children's diets, the food Dr. Corr chooses to represent this group is maize flour, because it is the most available grain in Malawi. The number of grams of maize flour consumed per day is represented by x_1. From Table 4.1.3, its energy content is 3.620 calories per gram. Thus, the total number of calories coming from cereals would be $3.620x_1$. Now, it is recommended that the total be greater than or equal to 900 and less than or equal to 1100 per day. This is really two constraints:

$$3.620x_1 \geq 900 \text{ and } 3.620x_1 \leq 1100$$

In other words, the total number of calories from cereals must be *between* 900 and 1100.

Q7. Continue this process to find a constraint inequality for each of the remaining food groups.

Q8. Using your responses to Q3 through Q7, write the complete problem formulation.

Q9. Based on this problem formulation, write a general prediction for the results of this problem. For example, are there any foods that must be consumed? Are there any foods that may not need to be consumed? Explain your answer.

In the Homework Exercises for this chapter, you will revisit this problem. You will be asked to enter your problem formulation into a spreadsheet solver to find the optimal solution and then to interpret this solution using answer and sensitivity reports.

Section 4.2: Minimizing Cost to Reduce Phosphorus in Watersheds

Water pollution comes from many sources. Water that runs off construction sites following rainstorms—known as *construction runoff*—contributes to water pollution. As water runs off construction sites, it picks up harmful sediment. The sediment might contain lead or mercury, nutrients like nitrogen and phosphorus, as well as oil, grease, and pesticides.

In urban and suburban areas, rain or snowfall that does not evaporate or soak into the ground is called *urban storm runoff*. Urban storm runoff also carries nutrients, sediment, and chemicals as it flows eventually into our waterways.

Fertilizers, pesticides, manure, and tilled soil are beneficial to crops. However, they can become harmful to our water as rains and irrigation wash them away. This is referred to as water pollution from *agricultural sources*.

Finally, industries, such as factories, release water back into the environment that has not been completely relieved of its nutrients and sediments. Much of this water runoff from *industrial sources* contains the chemical phosphorus, which potentially can harm the environment.

Phosphorus is one of the key nutrients necessary for the growth of plants and animals. However, in large amounts, it leads to excessive plant growth and decay. Phosphorus also favors certain weedy species over others. Too much phosphorus is likely to cause severe reductions in water quality. This is known as *excessive phosphorus loading*. Each form of water pollution discussed here (construction runoff, urban storm runoff, agricultural sources, and industrial sources) contributes to the amount of phosphorus in the water.

A challenge for many environmental groups is how to reduce the amount of phosphorus in the water at a minimal cost. Linear programming can be used to determine how to minimize the cost while achieving a specified phosphorus level for the best environmental quality.

In northeastern Wisconsin, there is a watershed system that is an area of concern because of the amount of phosphorus loading. A *watershed system* is the various land areas which drain into a certain lake or river. The communities in northeastern Wisconsin are planning to reduce the pollution in their watersheds. In this region, water drains into either Lake Winnebago or Green Bay (see Figure 4.2.1). In the rest of the state, water drains into the Mississippi River.

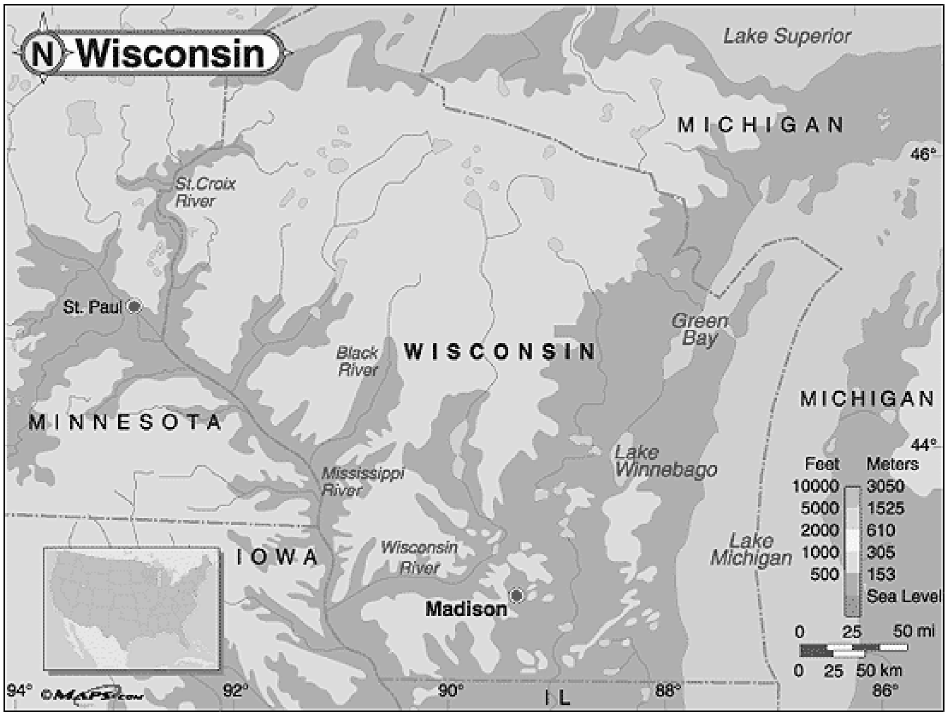


Figure 4.2.1: Map of Wisconsin showing Lake Winnebago and Green Bay

Nadia Manning is leading a group of planners who are focusing on pollution from the four sources discussed above. These are construction site runoff, urban storm runoff, agricultural sources, and industrial sources including municipal treatment plants. Nadia wants to reduce the amount of phosphorus in each watershed. Her goal is to reduce the amount of phosphorus by exactly 40,000 kilograms in Lake Winnebago (approximately 44 tons) and exactly 85,000 kilograms of phosphorus in Green Bay (approximately 93.5 tons), as seen in Table 4.2.1.

	Watershed	Amount of Phosphorus
1	Lake Winnebago	40,000 kg
2	Green Bay	85,000 kg

Table 4.2.1: Amount of phosphorus by watershed

The cost of the phosphorus reduction varies for each source. These costs are found in Table 4.2.2.

	Source	Cost of Phosphorus Reduction per kg
1	Construction runoff	$770
2	Urban storm runoff	$2,025
3	Agricultural sources	$26
4	Industrial sources	$75

Table 4.2.2: Phosphorus reduction costs by source

Table 4.2.1 shows the target reduction goal for each of the two watersheds. Nadia wants to exactly meet these reduction goals. She understands that exceeding the target will always increase the cost. Since the objective of this problem is to minimize the cost, there is no reason to have any more phosphorus than the specified amounts.

Table 4.2.2 shows the cost by source of reducing phosphorus in the water. It is important to notice those quantities are fixed and, thus, not variable. The solution to the water pollution problem requires reducing various sources of pollution in each of the watersheds to meet the reduction goals. Nadia and her team of planners are deciding how to reach those goals while keeping costs to a minimum.

4.2.1 Linear Programming Formulation

Nadia develops the complete linear programming formulation. First, she needs to define the decision variables. One way would be to let g_1, g_2, g_3, and g_4 represent the amount of phosphorus reduction in the Green Bay watershed from each of the four sources. Similarly, w_1, w_2, w_3, and w_4 could represent the amount of phosphorus reduction in Lake Winnebago from each source. Nadia notices she needs a different letter for the variables relating to each watershed.

Nadia and her team decide they need a system that uses just one letter for both watersheds. Therefore, she employs **double-subscripted variables**. As the name suggests, these variables have two subscripts. In this case, the first subscript refers to the watershed (Lake Winnebago or Green Bay). The second subscript refers to the source (construction runoff, urban storm runoff, agricultural sources, or industrial sources).

To help visualize this, Nadia creates a **matrix** (Table 4.2.3). In the matrix, the first subscript refers to the row in which the variable is written. The second subscript refers to its column. In general, $x_{i,j}$ represents the element in row i and column j of the matrix.

In the water pollution example, one decision variable represents the amount of reduction in phosphorus going into the Lake Winnebago watershed from agricultural sources. A single double-subscripted variable ($x_{1,3}$) may be used for this decision variable. The first subscript indicates the first watershed, Lake Winnebago, and the second subscript indicates the third pollution source, agriculture. Table 4.2.3 contains eight decision variables arranged in rows by the two watersheds and in columns by the four sources of pollution.

		Source (*j*)			
		Construction runoff	Urban storm runoff	Agricultural sources	Industrial sources
Watershed (*i*)	Lake Winnebago	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$
	Green Bay	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$

Table 4.2.3: Definition of decision variables

Now that Nadia has defined the decision variables, she writes the objective function. Since the goal is to minimize cost, she uses the values in Table 4.2.2 to develop the following objective function.
Minimize:

$$z = \$770(x_{1,1} + x_{2,1}) + \$2025(x_{1,2} + x_{2,2}) + \$26(x_{1,3} + x_{2,3}) + \$75(x_{1,4} + x_{2,4})$$
$$= \$770x_{1,1} + \$770x_{2,1} + \$2025x_{1,2} + \$2025x_{2,2} + \$26x_{1,3} + \$26x_{2,3} + \$75x_{1,4} + \$75x_{2,4}$$

Notice that each cost appears twice in the objective function. That makes sense, because we are assuming that it costs the same amount per kilogram to remove the pollution from either of the two watersheds. There is, however, a difference in cost to reduce each type of pollution. These cost differences can be an order of magnitude. For example, the cost to remove one kilogram of construction runoff is more than ten times as expensive as the cost to reduce a kilogram of industrial pollution. The reason for this is that the construction runoff arrives from widely dispersed areas while industrial pollution is more concentrated.

Q1. What is the magnitude of difference in cost for urban runoff and agricultural sources? Why do you think it might be less costly to control agricultural pollution as compared to urban pollution?

There are also some constraints Nadia needs to consider. First, the amount of phosphorus must be reduced by the values in Table 4.2.1. Thus, Nadia develops the following constraints:

Lake Winnebago target reduction: $x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 40{,}000$
Green Bay target reduction: $x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 85{,}000$

In addition to meeting the target reductions, Nadia and her team want to ensure that the pollution reductions over all the sources are evenly distributed. Nadia notices that pollution from agricultural sources is the least expensive to reduce. Without setting stipulations, agricultural polluters would be overburdened with phosphorus reductions. On the other hand, urban storm runoff would likely not be reduced at all, because it is by far the most costly. The team of planners agrees to requirements for reductions per source as shown in Tables 4.2.4 and 4.2.5.

Source	Minimum percent reduction of these sources into each watershed
Construction runoff	30%
Urban storm runoff	30%

Table 4.2.4: Minimum proportions of reductions per source

Source	Maximum percent reduction of these sources into each watershed
Agricultural sources	15%
Industrial sources	15%

Table 4.2.5: Maximum proportions of reductions per source

Based on this information, Nadia and her team develop the following constraints.

Construction runoff for Lake Winnebago: $x_{1,1} \geq 12{,}000=(0.3)(40{,}000)$
Construction runoff for Green Bay: $x_{2,1} \geq 25{,}500= (0.3)(85{,}000)$
Urban storm runoff for Lake Winnebago: $x_{1,2} \geq 12{,}000= (0.3)(40{,}000)$
Urban storm runoff for Green Bay: $x_{2,2} \geq 25{,}500= (0.3)(85{,}000)$
Agricultural sources for Lake Winnebago: $x_{1,3} \leq 6{,}000=(0.15)(40{,}000)$
Agricultural sources for Green Bay: $x_{2,3} \leq 12{,}750=(0.15)(85{,}000)$
Industrial sources for Lake Winnebago: $x_{1,4} \leq 6{,}000=(0.15)(40{,}000)$
Industrial sources for Green Bay: $x_{2,4} \leq 12{,}750= (0.15)(85{,}000)$

Q2. Why do the first four constraints use "≥"?

Q3. Why do the last four constraints use "≤"?

Q4. Why is the right-hand side of the first constraint 12,000?

Q5. Why is the right-hand side of the last constraint 25,500?

Q6. Without looking at any table or chart explain how you could tell that the constraint containing the decision variable $x_{2,3}$ refers to reducing pollution from agricultural sources in Green Bay?

Q7. Which constraint sets a limit on pollution reduction from urban storm runoff in Lake Winnebago?

Now, Nadia and her team have the complete linear programming formulation:

Decision Variables

Let: $x_{1,1}$ = amount of phosphorus reduction from construction runoff in Lake Winnebago (in kg)
$x_{1,2}$ = amount of phosphorus reduction from urban storm runoff in Lake Winnebago (in kg)
$x_{1,3}$ = amount of phosphorus reduction from agricultural sources in Lake Winnebago (in kg)
$x_{1,4}$ = amount of phosphorus reduction from industrial sources in Lake Winnebago (in kg)
$x_{2,1}$ = amount of phosphorus reduction from construction runoff in Green Bay (in kg)
$x_{2,2}$ = amount of phosphorus reduction from urban storm runoff in Green Bay (in kg)
$x_{2,3}$ = amount of phosphorus reduction from agricultural sources in Green Bay (in kg)
$x_{2,4}$ = amount of phosphorus reduction from industrial sources in Green Bay (in kg)
z = the total cost of reducing the amount of phosphorus in the watersheds

Objective Function

Maximize: $z = \$770x_{1,1} + \$770x_{2,1} + \$2025x_{1,2} + \$2025x_{2,2} + \$26x_{1,3} + \$26x_{2,3} + \$75x_{1,4} + \$75x_{2,4}$

Constraints

Subject to:

Lake Winnebago target reduction:	$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 40{,}000$
Construction runoff for Lake Winnebago:	$x_{1,1} \geq 12{,}000$
Urban storm runoff for Lake Winnebago:	$x_{1,2} \geq 12{,}000$
Agricultural sources for Lake Winnebago:	$x_{1,3} \leq 6{,}000$
Industrial sources for Lake Winnebago:	$x_{1,4} \leq 6{,}000$
Green Bay target reduction:	$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 85{,}000$
Construction runoff for Green Bay:	$x_{2,1} \geq 25{,}500$
Urban storm runoff for Green Bay:	$x_{2,2} \geq 25{,}500$
Agricultural sources for Green Bay:	$x_{2,3} \leq 12{,}750$
Industrial sources for Green Bay:	$x_{2,4} \leq 12{,}750$
Non-Negativity:	$x_{1,1} \geq 0$, $x_{1,2} \geq 0$, $x_{1,3} \geq 0$, $x_{1,4} \geq 0$, $x_{2,1} \geq 0$, $x_{2,2} \geq 0$, , $x_{2,3} \geq 0$, and $x_{2,4} \geq 0$

4.2.2 Using the Excel Solver

To solve this problem, Nadia and her team rely on Excel Solver. The spreadsheet is shown in Figure 4.2.2.

	A	B	C	D	E	F	G	H
1	Chapter 4: LP Minimization							
2	4.2 WI Watershed							
3	Reduction Cost Minimization							
4								
5	Decision Variable Values [# of kg of Phosphorus]	**Construction**	**Urban Storm**	**Agricultural**	**Industrial**			
6		**Runoff**	**Runoff**	**Sources**	**Sources**			
7	**Lake Winnebago (LW)**							
8	**Green Bay (GB)**							
9								**Total Cost**
10	Objective Function [Cost ($/kg)]	770	2025	26	75			$0
11								
12	**Constraints**							
13	Target Reduction for LW (kg)	1	1	1	1	0	=	40,000
14	Total Reduction for Construction Runoff_LW	1				0	≥	12,000
15	Total Reduction for Urban Storm Runoff_LW		1			0	≥	12,000
16	Total Reduction for Agricultural Sources_LW			1		0	≤	6,000
17	Total Reduction for Industrial Sources_LW				1	0	≤	6,000
18	Target Reduction for GB (kg)	1	1	1	1	0	=	85,000
19	Total Reduction for Construction Runoff_GB	1				0	≥	25,500
20	Total Reduction for Urban Storm Runoff_GB		1			0	≥	25,500
21	Total Reduction for Agricultural Sources_GB			1		0	≤	12,750
22	Total Reduction for Industrial Sources_GB				1	0	≤	12,750

Figure 4.2.2: Complete linear programming formulation in Excel

Nadia sets up this linear programming formulation in Excel with the decision variables in the form of a matrix, as in Table 4.2.3. The decision variables are, therefore, in two rows, rows 7 and 8, rather than in one long row. The double subscript makes it possible to present the decision variables in a compact form. Row 7 corresponds to Lake Winnebago and row 8 corresponds to Green Bay. The constraints are grouped in a similar fashion. By grouping the constraints for watershed, it is possible to efficiently use the copy and paste command.

Using the cell references from Figure 4.2.2, Nadia types the objective function into cell H10.
= B10 * B8 + B10 * B7 + C10 * C8 + C10 * C7 + D10 * D8 + D10 * D7 + E10 * E8 + E10 * E7

Alternatively, she could have used a more compact notation based on the distributive law.
= B10 * (B8 + B7) + C10 * (C8 + C7) + D10 * (D8 + D7) + E10 * (E8 + E7)

All of the constraints for Lake Winnebago appear in rows 13 through 17. The constraints for Green Bay appear in rows 18 through 22. Column F contains the SUMPRODUCT expression. For example, cell F13 is SUMPRODUCT($B$7:$E$7,B13:E13). This expression can then be copied into the next four rows.

Q8. Write an expression that can be used to determine the left hand side of the constraint for the target reduction for Green Bay (i.e., the expression in cell F18).

Solving a minimization problem in Excel is very similar to solving a maximization problem. Nadia simply needs to tell Solver to minimize the objective function. She follows the same procedures developed in earlier problems to set up the parameters of the model. The critical difference for this problem is that it is a minimization problem. Solver needs to be instructed to solve it accordingly (see Figure 4.2.3). The options are also going to remain the same as in earlier examples (see Figure 4.2.4).

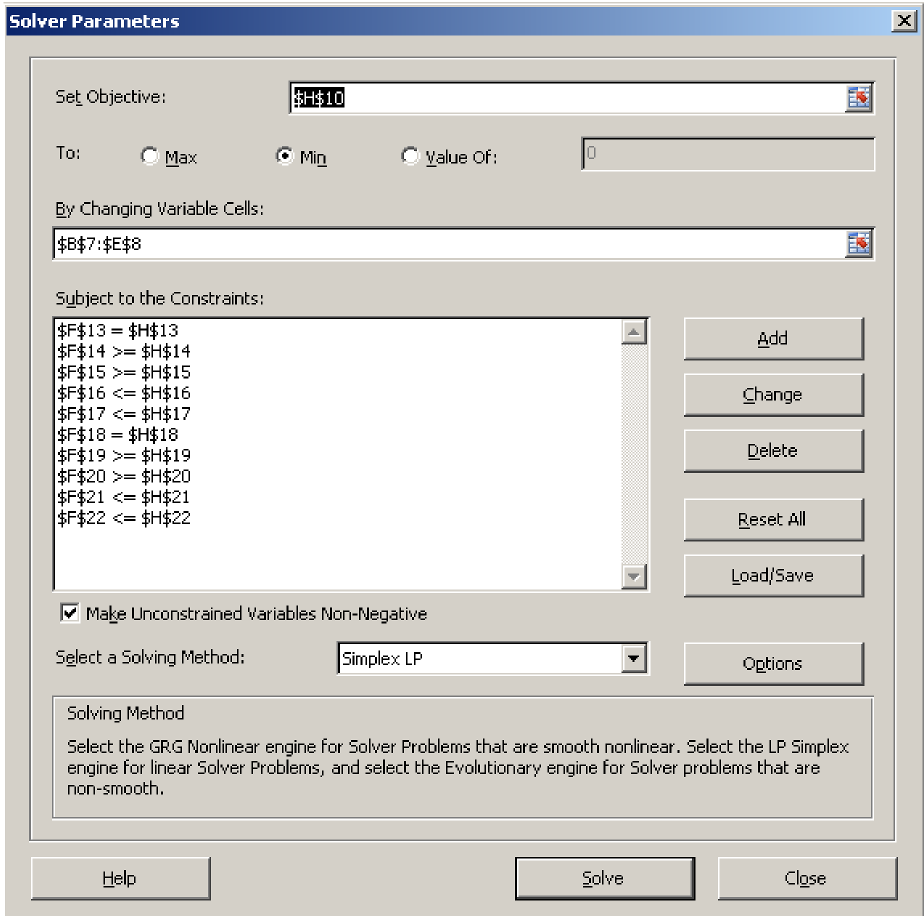


Figure 4.2.3: Setting up the parameters for this minimization linear programming problem

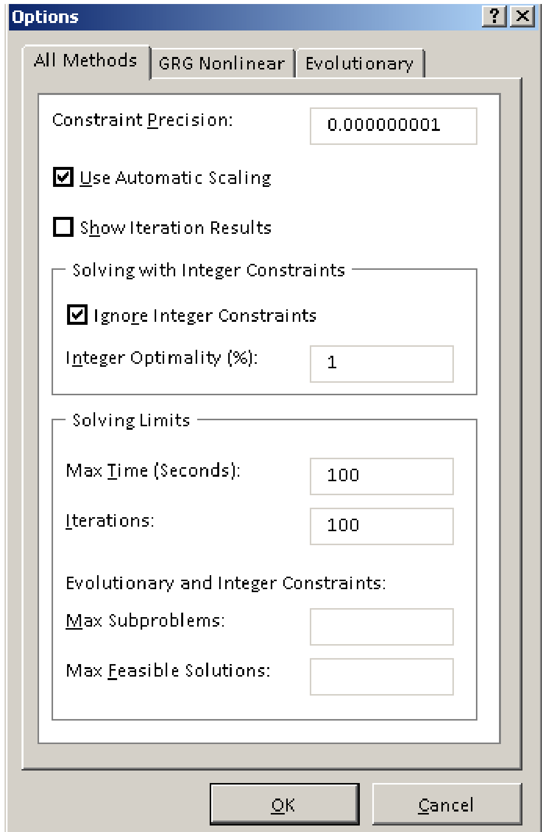


Figure 4.2.4: Solver Options

Nadia uses Excel Solver to come up with the optimal solution in Figure 4.2.5.

	A	B	C	D	E	F	G	H
1	Chapter 4: LP Minimization							
2	4.2 WI Watershed							
3	Reduction Cost Minimization							
4								
5	Decision Variable Values [# of kg of Phosphorus]	**Construction**	**Urban Storm**	**Agricultural**	**Industrial**			
6		**Runoff**	**Runoff**	**Sources**	**Sources**			
7	**Lake Winnebago (LW)**	16000	12000	6000	6000			
8	**Green Bay (GB)**	34000	25500	12750	12750			
9								**Total Cost**
10	Objective Function [Cost ($/kg)]	770	2025	26	75			$116,331,250
11								
12	**Constraints**							
13	Target Reduction for LW (kg)	1	1	1	1	40000	=	40,000
14	Total Reduction for Construction Runoff_LW	1				16000	≥	12,000
15	Total Reduction for Urban Storm Runoff_LW		1			12000	≥	12,000
16	Total Reduction for Agricultural Sources_LW			1		6000	≤	6,000
17	Total Reduction for Industrial Sources_LW				1	6000	≤	6,000
18	Target Reduction for GB (kg)	1	1	1	1	85000	=	85,000
19	Total Reduction for Construction Runoff_GB	1				34000	≥	25,500
20	Total Reduction for Urban Storm Runoff_GB		1			25500	≥	25,500
21	Total Reduction for Agricultural Sources_GB			1		12750	≤	12,750
22	Total Reduction for Industrial Sources_GB				1	12750	≤	12,750

Figure 4.2.5: Spreadsheet with optimal solution

The optimal solution shown in Figure 4.2.5 indicates the kilograms of phosphorus reduction from each source in each watershed. These values meet all of the constraints for the reduction of phosphorus. The solution achieves the goal of meeting these constraints for reduction while keeping cost to a minimum.

Q9. What is the optimal solution?

Q10. What is the value of the objective function for that optimal solution?

Q11. Which of the constraints are binding?

4.2.3 Interpreting Results

Nadia and her team examine the Answer Report shown in Figure 4.2.6 and the Sensitivity Report shown in Figure 4.2.7.

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$H$10	Objective Function [Cost ($/kg)] Total Cost	$0	$116,331,250

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$7	Lake Winnebago (LW) Construction Runoff	0	16000	Contin
$C$7	Lake Winnebago (LW) Urban Storm Runoff	0	12000	Contin
$D$7	Lake Winnebago (LW) Agricultural Sources	0	6000	Contin
$E$7	Lake Winnebago (LW) Industrial Sources	0	6000	Contin
$B$8	Green Bay (GB) Construction Runoff	0	34000	Contin
$C$8	Green Bay (GB) Urban Storm Runoff	0	25500	Contin
$D$8	Green Bay (GB) Agricultural Sources	0	12750	Contin
$E$8	Green Bay (GB) Industrial Sources	0	12750	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$13	Target Reduction for LW (kg)	40000	$F$13=$H$13	Binding	0
$F$14	Total Reduction for Construction Runoff_LW	16000	$F$14>=$H$14	Not Binding	4000
$F$15	Total Reduction for Urban Storm Runoff_LW	12000	$F$15>=$H$15	Binding	0
$F$16	Total Reduction for Agricultural Sources_LW	6000	$F$16<=$H$16	Binding	0
$F$17	Total Reduction for Industrial Sources_LW	6000	$F$17<=$H$17	Binding	0
$F$18	Target Reduction for GB (kg)	85000	$F$18=$H$18	Binding	0
$F$19	Total Reduction for Construction Runoff_GB	34000	$F$19>=$H$19	Not Binding	8500
$F$20	Total Reduction for Urban Storm Runoff_GB	25500	$F$20>=$H$20	Binding	0
$F$21	Total Reduction for Agricultural Sources_GB	12750	$F$21<=$H$21	Binding	0
$F$22	Total Reduction for Industrial Sources_GB	12750	$F$22<=$H$22	Binding	0

Figure 4.2.6: Answer Report for optimal solution

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$7	Lake Winnebago (LW) Construction Runoff	16000	0	770	1255	695
$C$7	Lake Winnebago (LW) Urban Storm Runoff	12000	0	2025	1E+30	1255
$D$7	Lake Winnebago (LW) Agricultural Sources	6000	0	26	744	1E+30
$E$7	Lake Winnebago (LW) Industrial Sources	6000	0	75	695	1E+30
$B$8	Green Bay (GB) Construction Runoff	34000	0	770	1255	695
$C$8	Green Bay (GB) Urban Storm Runoff	25500	0	2025	1E+30	1255
$D$8	Green Bay (GB) Agricultural Sources	12750	0	26	744	1E+30
$E$8	Green Bay (GB) Industrial Sources	12750	0	75	695	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$13	Target Reduction for LW (kg)	40000	770	40000	1E+30	4000
$F$14	Total Reduction for Construction Runoff_LW	16000	0	12000	4000	1E+30
$F$15	Total Reduction for Urban Storm Runoff_LW	12000	1255	12000	4000	12000
$F$16	Total Reduction for Agricultural Sources_LW	6000	-744	6000	4000	6000
$F$17	Total Reduction for Industrial Sources_LW	6000	-695	6000	4000	6000
$F$18	Target Reduction for GB (kg)	85000	770	85000	1E+30	8500
$F$19	Total Reduction for Construction Runoff_GB	34000	0	25500	8500	1E+30
$F$20	Total Reduction for Urban Storm Runoff_GB	25500	1255	25500	8500	25500
$F$21	Total Reduction for Agricultural Sources_GB	12750	-744	12750	8500	12750
$F$22	Total Reduction for Industrial Sources_GB	12750	-695	12750	8500	12750

Figure 4.2.7: Sensitivity Report for optimal solution

Q12. From the Sensitivity Report, which of the constraints have a shadow price?

Q13. Explain how the two previous answers are related.

Recall that *shadow price* refers to the amount by which the objective function value changes given a unit increase or decrease in one right-hand side (RHS) value of a constraint.

Q14. Interpret the meaning of the shadow price for the "Target Reduction for GB" constraint.

Q15. Interpret the meaning of the shadow price for the "Total Reduction for Agricultural Souces_GB" constraint. Why is it negative?

Q16. How does the interpretation of the shadow price in a minimization problem differ from the the interpretation of the shadow price in a maximization problem?

While the answer and sensitivity reports give Nadia and her team a lot of important information, they do not give them the entire picture. They must use their knowledge of the problem. It is important to remember the units for the various constraints and decision variables. For instance, the decision variables are in kilograms of phosphorus.

Q17. Which of the sources provides the highest amount of phosphorus reduction? Why does this make sense?

Q18. Which of the sources provides the least amount of phosphorus reduction? Why does this make sense?

Nadia's team notices that the Sensitivity Report for the Watersheds Problem lists an Allowable Decrease in the objective coefficients of four of the decision variables as 1E+30. In other words, these coefficients may be decreased as much as they want without affecting the optimal solution.

They also notice that these four decision variables (that have an Allowable Decrease of 1E+30) are the decision variables for agricultural and industrial sources of pollution. These two sources are the least costly of the four sources from which to reduce pollution. As a result, Nadia's team sees why the four constraints related to agricultural and industrial sources are binding. They want to reduce as much pollution as possible from the least expensive sources in order to minimize the total cost.

Then they notice that the two constraints for urban storm runoff are also binding, but reducing urban storm runoff is the *most* expensive source to reduce. Nadia calls the team's attention to the direction of the constraint inequalities. For urban storm runoff, both constraint inequalities are "greater than or equal to." One way to think of "greater than or equal to" is "at least." Since the goal is to keep costs at a minimum, the most expensive source of pollution should be reduced as little as required. For example, reducing pollution from urban storm runoff by "at least 25,500 kg" means reduce it by exactly 25,500 kg. To reduce it more while staying within the total target reduction would be more costly. Now they understand why those two constraints are also binding,

Returning to the Sensitivity Report, the team notices that Solver reports a Shadow Price of 1255 for both of the urban storm runoff constraints. That means that if the right-hand side of either of those constraints increases by 1 kg, the total cost of the project would increase by $1,255. They understand why the cost would increase, but they do not see where the 1255 came from. To see why that makes sense, Nadia changes the spreadsheet formulation, as shown in Figure 4.2.8. The spreadsheet shows an increase to 12,001 in the right-hand side of the Lake Winnebago urban storm runoff constraint, as well as the decision variable values for the resulting optimal solution.

	A	B	C	D	E	F	G	H
1	Chapter 4: LP Minimization							
2	4.2 WI Watershed							
3	Reduction Cost Minimization							
4								
5	Decision Variable Values [# of kg of Phosphorus]	**Construction**	**Urban Storm**	**Agricultural**	**Industrial**			
6		**Runoff**	**Runoff**	**Sources**	**Sources**			
7	**Lake Winnebago (LW)**	15999	12001	6000	6000			
8	**Green Bay (GB)**	34000	25500	12750	12750			
9								**Total Cost**
10	Objective Function [Cost ($/kg)]	770	2025	26	75			$116,332,505
11								
12	**Constraints**							
13	Target Reduction for LW (kg)	1	1	1	1	40000	=	40,000
14	Total Reduction for Construction Runoff_LW	1				15999	≥	12,000
15	Total Reduction for Urban Storm Runoff_LW		1			12001	≥	12,001
16	Total Reduction for Agricultural Sources_LW			1		6000	≤	6,000
17	Total Reduction for Industrial Sources_LW				1	6000	≤	6,000
18	Target Reduction for GB (kg)	1	1	1	1	85000	=	85,000
19	Total Reduction for Construction Runoff_GB	1				34000	≥	25,500
20	Total Reduction for Urban Storm Runoff_GB		1			25500	≥	25,500
21	Total Reduction for Agricultural Sources_GB			1		12750	≤	12,750
22	Total Reduction for Industrial Sources_GB				1	12750	≤	12,750

Figure 4.2.8: Increasing the amount of reduction in Lake Winnebago urban storm runoff by 1 kg

By increasing the amount of reduction in urban storm runoff in the Lake Winnebago watershed by 1 kg, the final value of only two of the decision variables changes. Urban storm runoff in Lake Winnebago increases from 12,000 kg to 12,001 kg, and construction runoff in Lake Winnebago decreases from 16,000 kg to 15,999 kg.

The increase in urban storm runoff in Lake Winnebago by 1 kg causes the total cost of the project to increase by $2,025 (see Table 4.2.2). In addition, reducing the construction runoff in Lake Winnebago by 1 kg causes the total cost of the project to decrease by $770. Thus, Nadia saw that the net effect of the change she made is $2,025 – $770 = $1,255, which is exactly the amount of increase in the total cost of the project.

Next, Nadia and her team notice that four of the Shadow Prices in the Sensitivity Report are negative numbers (e.g., the Shadow price for the constraint for agricultural sources in Lake Winnebago is -$744). This means that increasing the right-hand side of the constraint will decrease the total cost of the project. The negative shadow prices are all linked to less than or equal to constraints. These constraints limit the use of lower cost reduction strategies. Increasing the right hand side of any of these constraints expands the size of the feasible region. As a result, it is possible to improve the optimal solution. In a cost minimization problem improvements result in a reduction in total cost. This reduction appears as a negative shadow price. For example, Figure 4.2.9 shows the spreadsheet Nadia has after increasing the right-hand side of the Lake Winnebago agricultural sources constraint by 1 kg to 6,001. (Note: the right-hand side of the constraint for urban storm runoff for Lake Winnebago has been changed back to 12,000 kg.)

	A	B	C	D	E	F	G	H
1	Chapter 4: LP Minimization							
2	4.2 WI Watershed							
3	Reduction Cost Minimization							
4								
5	Decision Variable Values [# of kg of Phosphorus]	**Construction**	**Urban Storm**	**Agricultural**	**Industrial**			
6		**Runoff**	**Runoff**	**Sources**	**Sources**			
7	**Lake Winnebago (LW)**	15999	12000	6001	6000			
8	**Green Bay (GB)**	34000	25500	12750	12750			
9								**Total Cost**
10	Objective Function [Cost ($/kg)]	770	2025	26	75			$116,330,506
11								
12	**Constraints**							
13	Target Reduction for LW (kg)	1	1	1	1	40000	=	40,000
14	Total Reduction for Construction Runoff_LW	1				15999	≥	12,000
15	Total Reduction for Urban Storm Runoff_LW		1			12000	≥	12,000
16	Total Reduction for Agricultural Sources_LW			1		6001	≤	6,001
17	Total Reduction for Industrial Sources_LW				1	6000	≤	6,000
18	Target Reduction for GB (kg)	1	1	1	1	85000	=	85,000
19	Total Reduction for Construction Runoff_GB	1				34000	≥	25,500
20	Total Reduction for Urban Storm Runoff_GB		1			25500	≥	25,500
21	Total Reduction for Agricultural Sources_GB			1		12750	≤	12,750
22	Total Reduction for Industrial Sources_GB				1	12750	≤	12,750

Figure 4.2.9: Increasing the amount of reduction from Lake Winnebago agricultural sources by 1 kg

The team notices that this change in the constraint causes two changes in the final values of the decision variables. The amount of pollution reduction from agricultural sources in the Lake Winnebago watershed increases by 1 kg, from 6,000 to 6,001. At the same time, the amount of pollution reduction from construction runoff in the Lake Winnebago watershed decreases by 1 kg, from 16,000 to 15,999. The net effect of these two changes is $26 – $770 = -$744, which matches the reported Shadow Price.

Finally, after a lot of hard work, Nadia and her team of planners know how much reduction in phosphorus should be coming from each source and how much it will cost to do the entire reduction process. Based on this information, they are now ready to move forward on this project.

Section 4.3: Disk Gasoline Distributors, Inc.

Disk Gasoline Distributors, Inc. obtains gasoline wholesale from three refineries. It then blends this gasoline and introduces additives. These additives are designed to improve vehicle performance. Disk delivers the finished product to various gasoline retailers.

During the first quarter of the year, the management at Disk wants to produce a blend of gasoline meeting a particular set of specifications. The product will be delivered to retailers in the Southeast. Table 4.3.1 contains those specifications. To meet the company's goals for profitability, it must produce 500,000 gallons per week of the blend.

Octane rating:	Greater than or equal to 87
Vapor pressure:	Less than 7.2 pounds per square inch (psi)
Sulfur content:	Less than 75 parts per million (ppm)
Olefins (a family of toxic pollutants):	Less than 10% by volume (%v)

Table 4.3.1: Gasoline blend specifications at Disk Gasoline Distributors

The octane rating is a performance measure of a gasoline. The higher the octane rating, the better the gasoline performs. Some vehicles require gasoline with an octane rating higher than 89. Vapor pressure is a measure of the extent to which a gasoline is subject to evaporation. Sulfur content and olefin content determine how cleanly a gasoline blend burns in a vehicle. A lower content of either produces cleaner gasoline.

Disk buys gasoline in 100-gallon units directly from three refineries in the southeast:

- Vicksburg, MS,
- Norco, LA, and
- Mobile, AL.

The cost per 100-gallon unit from the refineries is $274.90 from Vicksburg, $265.90 from Norco, and $249.90 from Mobile. These costs include delivery. The characteristics of the gasoline that Disk can obtain from these three refineries are contained in Table 4.3.2.

	Refinery		
Characteristic	**Vicksburg**	**Norco**	**Mobile**
Octane rating	89	88	85
Vapor pressure	7.23	7.09	7.32
Sulfur content	72	86	58
Olefins	7.52	8.97	13.38

Table 4.3.2: Characteristics of gasoline from three refineries

The managers at Disk would like to minimize the cost of the gasoline used in this blend. However, they are further constrained by the capacities of the three refineries. They can obtain no more than 210,000 gallons per week from Vicksburg, no more than 190,000 gallons per week from Norco, and no more than 200,000 gallons per week from Mobile. Edward Thompson, the production manager at Disk, wants to know how much gasoline to purchase from each refinery in order to keep the cost of the blend at a minimum.

4.3.1 Linear Programming Formulation

Edward Thompson begins by defining the decision variables. He must decide how much gasoline to obtain from each of the three refineries per week. So he lets:

x_1 = the number of hundred-gallon units of gasoline purchased from Vicksburg per week,
x_2 = the number of hundred-gallon units of gasoline purchased from Norco per week, and
x_3 = the number of hundred-gallon units of gasoline purchased from Mobile per week.

Now he must define a function that represents the objective of the problem in terms of the decision variables. Because the goal is to minimize the cost of the gasoline in the blend, Edward Thompson lets:

$$z = 274.9x_1 + 265.9x_2 + 249.9x_3$$

Edward Thompson wants to minimize z, subject to all of the constraints. First, there is a production constraint. The company wants to produce 500,000 gallons of the blend per week. However, Edward Thompson notices that the units of purchase are hundred-gallons. There are 5,000 hundred-gallon units in 500,000 gallons. Therefore, he writes the following constraint:

$x_1 + x_2 + x_3 = 5{,}000.$

Next, he must account for the capacities at each of the refineries:

$x_1 \leq 2{,}100,$
$x_2 \leq 1{,}900,$ and
$x_3 \leq 2{,}000.$

Notice that these constraints have also been expressed in hundred-gallon units.

Next, Edward Thompson must include all of the constraints that derive from the specifications of the blend.

He knows the octane rating of the blend must be greater than or equal to 87. He also knows the octane ratings of the gasoline coming from all three refineries. However, he cannot just average those numbers. The amount of gasoline from each of the refineries might not be the same. To account for this possibility, he must use a **weighted average** to calculate the octane rating of the blend.

To compute this weighted average, Edward Thompson first multiplies the octane rating of the gasoline from each of the refineries by the amount purchased from that refinery. Then, he adds the three products and divides the sum by the total amount of the blend:

$$\frac{89x_1 + 88x_2 + 85x_3}{x_1 + x_2 + x_3} = \text{the octane rating of the blend.}$$

Now the octane rating of the blend must be greater than or equal to 87, so he writes:

$$\frac{89x_1 + 88x_2 + 85x_3}{x_1 + x_2 + x_3} \geq 87$$

Then, Edward Thompson manipulates the original inequality to remove the rational expression on the left-hand side. Doing so replaces the fraction with a linear expression:

$89x_1 + 88x_2 + 85x_3 \geq 87(x_1 + x_2 + x_3)$
$89x_1 + 88x_2 + 85x_3 \geq 87x_1 + 87x_2 + 87x_3$
$2x_1 + x_2 - 2x_3 \geq 0$

Similarly, the constraints on vapor pressure, sulfur content, and olefin content can be found using this weighted average approach.

Vapor pressure: $\frac{7.23x_1 + 7.09x_2 + 7.32x_3}{x_1 + x_2 + x_3} \le 7.2$

$7.23x_1 + 7.09x_2 + 7.32x_3 \le 7.2(x_1 + x_2 + x_3)$
$7.23x_1 + 7.09x_2 + 7.32x_3 \le 7.2x_1 + 7.2x_2 + 7.2x_3$
$0.03x_1 - 0.11x_2 + 0.12x_3 \le 0$

Sulfur content: $\frac{72x_1 + 86x_2 + 58x_3}{x_1 + x_2 + x_3} \le 75$

$72x_1 + 86x_2 + 58x_3 \le 75(x_1 + x_2 + x_3)$
$72x_1 + 86x_2 + 58x_3 \le 75x_1 + 75x_2 + 75x_3$
$-3x_1 + 11x_2 - 17x_3 \le 0$

Olefin content: $\frac{7.52x_1 + 8.97x_2 + 13.38x_3}{x_1 + x_2 + x_3} \le 10$

$7.52x_1 + 8.97x_2 + 13.38x_3 \le 10(x_1 + x_2 + x_3)$
$7.52x_1 + 8.97x_2 + 13.38x_3 \le 10x_1 + 10x_2 + 10x_3$
$-2.48x_1 - 1.03x_2 + 3.38x_3 \le 0$

Finally, Edward Thompson includes the non-negativity constraints: $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

Complete Linear Programming Formulation
Now, Edward Thompson writes the complete formulation.

Decision Variables
Let: x_1 = the number of hundred-gallon units of gasoline purchased from Vicksburg per week,
x_2 = the number of hundred-gallon units of gasoline purchased from Norco per week, and
x_3 = the number of hundred-gallon units of gasoline purchased from Mobile per week.
z = the cost of the gasoline blend

Objective Function
Minimize: $z = 274.9x_1 + 265.9x_2 + 249.9x_3$

Subject to the following constraints:

Production:	$x_1 + x_2 + x_3 = 5{,}000$
Vicksburg Refinery Capacity:	$x_1 \le 2{,}100$
Norco Refinery Capacity:	$x_2 \le 1{,}900$
Mobile Refinery Capacity:	$x_3 \le 2{,}000$.
Octane Rating:	$2x_1 + x_2 - 2x_3 \ge 0$
Vapor Pressure:	$0.03x_1 - 0.11x_2 + 0.12x_3 \le 0$
Sulfur Content:	$-3x_1 + 11x_2 - 17x_3 \le 0$
Olefin Content:	$-2.48x_1 - 1.03x_2 + 3.38x_3 \le 0$
Non-Negativity:	$x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$

4.3.2 Excel Solver

Figure 4.3.1 contains a spreadsheet formulation for the Disk Gasoline Distributors problem. Figures 4.3.2 and 4.3.3 contain the Answer Report and the Sensitivity Report, respectively, which were generated by Solver.

	A	B	C	D	E	F	G
1	Chapter 4: LP Minimization						
2	4.3 Disk Gasoline Distributors						
3	Gasoline Blending Cost Minimization						
4							
5	Decision Variable	**Vicksburg** (x_1)	**Norco** (x_2)	**Mobile** (x_3)			
6	Decision Value [# of 100-gal per week]	1811.111111	1900	1288.888889			
7							**Total Cost**
8	Objective Function [Cost (100-gal)]	$274.90	$265.90	$249.90			$1,325,177.78
9							
10	**Constraints**						
11	Production (100-gal)	1	1	1	5000	=	5,000
12	Vicksburg Capacity (100-gal)	1	0	0	1811.1111	≤	2,100
13	Norco Capacity (100-gal)	0	1	0	1900	≤	1,900
14	Mobile Capacity (100-gal)	0	0	1	1288.8889	≤	2,000
15	Octane Rating	2	1	-2	2944.4444	≥	0
16	Vapor Pressure (psi)	0.03	-0.11	0.12	0	≤	0
17	Sulfur Content (ppm)	-3	11	-17	-6444.444	≤	0
18	Olefin (%Vol)	-2.48	-1.03	3.38	-2092.111	≤	0

Figure 4.3.1: Spreadsheet formulation of the Disk Gasoline Distributors problem

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Cost (100-gal)] Total Cost	$0.00	$1,325,177.78

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Value [# of 100-gal per week] Vicksburg (x1)	0	1811.111111	Contin
$C$6	Decision Value [# of 100-gal per week] Norco (x2)	0	1900	Contin
$D$6	Decision Value [# of 100-gal per week] Mobile (x3)	0	1288.888889	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Production (100-gal)	5000	$E$11=$G$11	Binding	0
$E$12	Vicksburg Capacity (100-gal)	1811.111111	$E$12<=$G$12	Not Binding	288.8888889
$E$13	Norco Capacity (100-gal)	1900	$E$13<=$G$13	Binding	0
$E$14	Mobile Capacity (100-gal)	1288.888889	$E$14<=$G$14	Not Binding	711.1111111
$E$15	Octane Rating	2944.444444	$E$15>=$G$15	Not Binding	2944.444444
$E$16	Vapor Pressure (psi)	0	$E$16<=$G$16	Binding	0
$E$17	Sulfur Content (ppm)	-6444.444444	$E$17<=$G$17	Not Binding	6444.444444
$E$18	Olefin (%Vol)	-2092.111111	$E$18<=$G$18	Not Binding	2092.111111

Figure 4.3.2: Solver Answer Report for the Disk Gasoline Distributors problem

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Decision Value [# of 100-gal per week] Vicksburg (x1)	1811.111111	0	274.9	1E+30	18.73913043
$C$6	Decision Value [# of 100-gal per week] Norco (x2)	1900	0	265.9	47.88888889	1E+30
$D$6	Decision Value [# of 100-gal per week] Mobile (x3)	1288.888889	0	249.9	25	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$11	Production (100-gal)	5000	283.2333333	5000	216.6666667	471.9047619
$E$12	Vicksburg Capacity (100-gal)	1811.111111	0	2100	1E+30	288.8888889
$E$13	Norco Capacity (100-gal)	1900	-47.88888889	1900	198.0124093	113.0434783
$E$14	Mobile Capacity (100-gal)	1288.888889	0	2000	1E+30	711.1111111
$E$15	Octane Rating	2944.444444	0	0	2944.444444	1E+30
$E$16	Vapor Pressure (psi)	0	-277.7777778	0	32.13139932	26
$E$17	Sulfur Content (ppm)	-6444.444444	0	0	1E+30	6444.444444
$E$18	Olefin (%Vol)	-2092.111111	0	0	1E+30	2092.111111

Figure 4.3.3: Solver Sensitivity Report for the Disk Gasoline Distributors problem

4.3.3 Interpreting Results

Edward Thompson now knows the optimal gasoline blend and how much it will cost. However, he still needs to spend some time interpreting these results.

Carefully examine the Answer and Sensitivity Reports in Figures 4.3.2 and 4.3.3. Then answer each of the following questions about the solution to this problem.

Q1. What is the optimal solution, and what is the cost of producing that blend?

Q2. Calculate the cost per hundred gallons of the optimal blend. How does the cost per hundred gallons of the blend compare to the cost per hundred gallons of each of the components?

Q3. Why does the Sensitivity Report list the Reduced Cost for each of the three decision variables as 0?

Notice that the Allowable Increase in the objective coefficient (the cost per hundred gallons) for x_3 (the number of hundred-gallon units of gasoline purchased from the Mobile refinery per week) is 25. Suppose that this objective coefficient increases by 24.9 to 274.8.

Q4. What do you think would happen to the optimal solution?

Q5. What do you think would happen to the optimal solution if the price per hundred-gallon at the Mobile refinery increased to $275.90?

Q6. Similarly, assume the objective coefficient of x_2 decreased by 18 to 247.9. What do you think would happen to the optimal solution?

The capacity of each of the three refineries forms a constraint in the problem formulation. According to the Answer Report, the constraint is binding for the Norco refinery, but not for the Vicksburg and Mobile refineries.

Q7. In the context of the problem, what does it mean that the Mobile refinery constraint is nonbinding?

Q8. The slack value for the Mobile refinery is given as 711.1111.
a. What does that slack value tell you about the optimal solution?
b. Based on other information in the problem, why do you think this has happened?

Q9. Sometimes Solver reports an allowable increase or decrease of 1E+30. How should that number be interpreted?

Q10. Why does an allowable increase in the right-hand side of the Mobile capacity constraint of 1E+30 make sense?

Next, the shadow price for the production constraint is given as 283.2333333. To see the effect of this shadow price, Edward Thompson increases the production constraint by 100 gallons. Figure 4.3.4 shows the new optimal solution when the production is 5,001 100-gallons (i.e., 500,100 gallons) instead of 5,000 100-gallons.

	A	B	C	D	E	F	G
1	Chapter 4: LP Minimization						
2	4.3 Disk Gasoline Distributors						
3	Gasoline Blending Cost Minimization						
4							
5	Decision Variable	**Vicksburg** (x_1)	**Norco** (x_2)	**Mobile** (x_3)			
6	Decision Value [# of 100-gal per week]	1812.444444	1900	1288.555556			
7							**Total Cost**
8	Objective Function [Cost (100-gal)]	$274.90	$265.90	$249.90			$1,325,461.01
9							
10	**Constraints**						
11	Production (100-gal)	1	1	1	5001	=	5,001
12	Vicksburg Capacity (100-gal)	1	0	0	1812.4444	≤	2,100
13	Norco Capacity (100-gal)	0	1	0	1900	≤	1,900
14	Mobile Capacity (100-gal)	0	0	1	1288.5556	≤	2,000
15	Octane Rating	2	1	-2	2947.7778	≥	0
16	Vapor Pressure (psi)	0.03	-0.11	0.12	0	≤	0
17	Sulfur Content (ppm)	-3	11	-17	-6442.778	≤	0
18	Olefin (%Vol)	-2.48	-1.03	3.38	-2096.544	≤	0

Figure 4.3.4: Spreadsheet formulation with a new production constraint

Q11. What do you observe about the total cost now?

Q12. What other changes do you observe?

Finally, the Shadow Prices for the capacity of each refinery are reported as 0 for Vicksberg, -47.89 for Norco, and 0 for Mobile.

Q13. Why are the Vicksberg and Mobile Shadow Prices 0?

Q14. How do you interpret the negative Shadow Price?

Q15. Why should Disk try to obtain more gasoline per week from Norco to use in the blend?

Based on these results and his interpretation of the results, Edward Thompson feels confident in moving forward with this project. He knows the optimal amounts of gasoline to obtain from each refinery, but he is also aware of the changes that could occur in the constraints and the objective function that could result in different amounts of gasoline and/or a different total cost.

Section 4.4: Chapter 4 (LP Minimization) Homework Questions

1. Sue's uncle Bob grows strawberries on his 500-acre farm. An agricultural engineer tested the soil on his farm and found the level of phosphorus (P) and potassium (K) to be 35 parts per million (ppm) and 80 ppm, respectively. The two tables below show the recommended usage of fertilizer for growing strawberries in his climate.

Phosphorus concentration (ppm)	Recommended additional phosphorus (lbs/acre)
0-15	46-55
16-45	28-46
over 45	0-28

Table 1: Phosphorus fertilization rates for strawberries

Potassium concentration (ppm)	Recommended additional potassium (lbs/acre)
0-75	60-72
76-175	48-60
over 175	0-48

Table 2: Potassium fertilization rates for strawberries

The engineer also suggests applying 30-40 pounds of nitrogen (N) per acre and 15-20 pounds of sulfur (S) per acre.

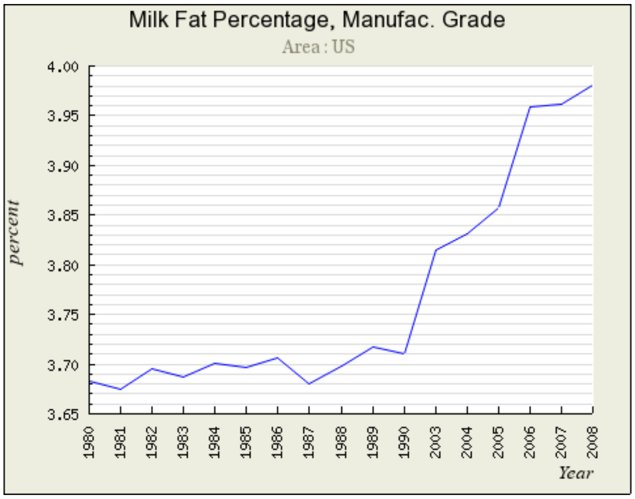
Uncle Bob has six commercially available fertilizers from which to choose. An analysis of the contents and costs of these fertilizers is given in Table 3.

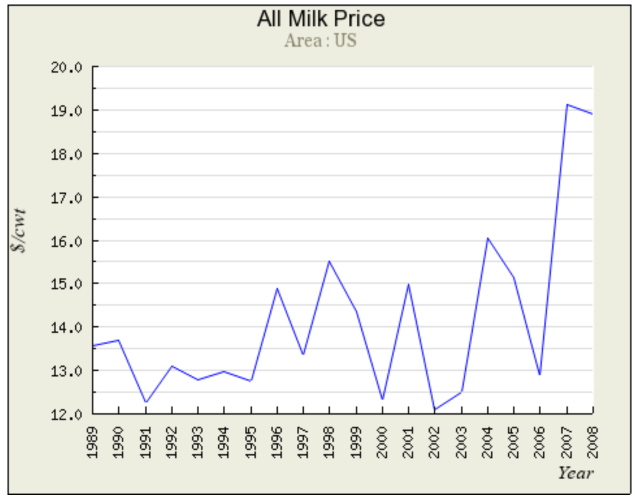
Product	Nutrient (%/lb)				Cost ($/lb)
	P	K	N	S	
Ammonium sulfate	0	0	21	24	0.1275
Potassium sulfate	0	52	0	12	0.1045
Urea	0	0	46	0	0.2025
Urea ammonium phosphate	17	0	34	0	0.1530
Potash chloride	0	60	0	0	0.1375
Ammonium phosphate sulfate	20	0	16	14	0.1615

Table 3: Analysis of fertilizer products

Uncle Bob needs to choose a combination of the above fertilizers that will meet the engineer's recommendations while minimizing the cost per acre. Formulate the problem to help Uncle Bob make this decision.

2. Each week, the DeeLite Milk Company gets milk from three dairies and then blends the milk to get the desired amount of butterfat for the company's premier product. Dairy A can supply at most 600 cwt of milk averaging 3.9% butterfat and costing \$17.0 per cwt. Dairy B can supply milk averaging 3.5% butterfat costing \$13.0 per cwt. Dairy C can supply at most 700 cwt of milk averaging 3.7% butterfat costing \$15.45 per cwt. How much milk from each supplier should DeeLite use to get 1500 cwt of milk with minimum cost and at least 3.7% butterfat?[1]





3. Use a spreadsheet solver to solve the nutrition problem in Malawi that you formulated in section 4.1.

4. Howie, Billy, and Don are three brothers who recently inherited some money. They decide to pool some of their newly acquired resources by forming a cooperative. They agree to each place \$35,000 into the cooperative pool to be invested in three different mutual funds. They want to minimize their risk while at the same time achieving a 10% gain over the course of one year. Table 4 contains a risk factor based on a 5% Value at Risk (VaR) for 1 year, as well as the estimated percentage of income over the first year.

Fund	5% VaR factor	Estimated % of Income
Loyalty	1	9.2%
Value	1.5	10.1%
Spearhead	2	10.9%

Table 4: Risk and income factors for three mutual funds

a) Define a set of decision variables for this problem.

b) Define an objective function in terms of the decision variables.

c) Formulate all of the problem constraints.

d) Use a spreadsheet solver to obtain the optimal solution.

e) How much money should they invest in each fund?

[1] In U.S. customary units, cwt is the abbreviation for "hundred-weight", meaning 100 lbs.

f) What is the estimated amount their investments will earn in the first year?

5. Obtain an answer and sensitivity report for your optimal solution to the previous problem.

   a) By how much could the risk factor for the Loyalty Fund increase or decrease before the optimal solution would change? For the Value Fund? The Spearhead Fund?

   b) Suppose the three brothers decide that they are willing to accept a $10,000 in the first year. In what ways, if any, would the optimal solution change?

   c) Suppose they decide to require at least $35,000 be invested in the least risky fund. In what ways, if any, would that change the optimal solution? What if they required at least $40,000 invested in the least risky fund?

   d) What would be the effect, if any, on the optimal solution if they decided to require that no more than $25,000 be invested in the riskiest fund? What if they required no more than $20,000 be invested in that fund?

Chapter 4 Summary

What have we learned?

As we learned in chapter 2, Linear Programming is used to make decisions that will lead to the optimal solution for a situation. However, this does not always mean making something as big as possible. Often, we want to minimize something undesirable such as cost, risk, or pollution.

The process to follow is similar to the one we used in the previous chapter:

Formulate the linear model representing the situation

- Identify the decision variables
- Write the objective function
- Define the constraints including maximum, minimum, and combination of variable constraints.

Use a spreadsheet program such as Excel

- Specify the decision variables, objective function, and constraints
- Use Solver to find the optimal solution to the problem – change the default selection for "Equal To:" from Max to Min

Analyze the results

- What is the optimal plan?
- What is the minimum value of the objective function?
- What values of the decision values will result in this optimal solution?
- What effect do changes to the situation have on the optimal solutions? Changes that make the situation worse will increase the final value rather than decrease it as in the previous chapter.

Terms

Matrix	A rectangular arrangement of values and/or variables in a table
Double-Subscripted Variable	A variable with two subscripts to represent the location in a matrix, where the first subscript refers to the row in the matrix and the second subscript refers to the column in the matrix. Often the rows and columns can be matched with key features of a problem, such as in the water pollution problem where the rows were the watersheds and the columns were the pollution sources.
Weighted Average	An average in which each quantity to be averaged is assigned a weight; the weighted average is found by first adding together the products of each variable and its weight value and then by dividing this sum by the total of the weights

Chapter 4 (LP Minimization) Objectives

You should be able to:

- Use Solver to optimize a situation in which the goal is to minimize a value
- Use a matrix and double-subscripts to represent the combination of two factors into one variable.
- Manipulate an inequality to remove variables from the denominator of a fraction and move all the decision variables to the left hand side.

Chapter 4 Study Guide

1. What is the objective function? Why are we trying to minimize the objective function in this chapter?

2. Why are some constraints shown with <= and others with >=? Give examples.

3. What is a matrix? Give an example.

4. What is a double-subscripted variable? Give an example.

5. What is a weighted average? Give an example.

6. What is shadow price?

7. What does it mean if a shadow price is negative?

Section 5.1: An Advertising Problem

Cynthia Brown, who is running for the office of governor in Michigan, is considering purchasing TV ads costing $60,000 per day or ads in several newspapers costing a total of $80,000 per day. How many of each type of ad could she purchase using $600,000?

Q1. If we let x_1 represent the number of newspaper ads purchased and x_2 represent the number of TV ads purchased, explain why the equation $80x_1 + 60x_2 = 600$ models this situation.

Q2. Why is 80 used instead of 80,000, 60 instead of 60,000, and 600 instead of 600,000?

Using algebra, we can find a solution to the equation $80x_1 + 60x_2 = 600$ by choosing any number of newspaper ads and solving for the number of TV ads. For example, if we let $x_1 = 5$, then we have $80(5) + 60x_2 = 600$. Then

$$\begin{aligned} 400 + 60x_2 &= 600 \\ 60x_2 &= 200 \\ x &= \frac{200}{60} = 3\tfrac{1}{3} \end{aligned}$$

Q3. Why is this solution not feasible?

Q4. Does this one infeasible solution mean that there are no solutions at all?

Q5. Find a feasible solution to this equation.

The key to understanding this problem is identifying the need for our solutions to be integers. The candidate must purchase a whole number of ads; she cannot purchase half an ad. Although newspaper ads may be purchased in fractions of a page, not all fractions are possible. For example, a half-page add could be purchased, but not a 0.345-page ad. Thus, this fractional possibility does not make one of the decision variables continuous. If we wanted to include the possibility of half page and quarter page ads, we would need to introduce another integer decision variable for each sized ad. We chose not to increase the problem size by adding these decision variables but do include these possibilities in the homework.

A linear equation that has two variables with integer coefficients in which we seek integer solutions is called a linear Diophantine equation. Requiring an integer solution actually makes the problem more complicated. Our goal is to find an easy criterion to test whether an integer solution exists.

Consider again the equation $80x_1 +60x_2 = 600$. If we let $x_1 = 3$, then

$$\begin{aligned} 80(3) + 60x_2 &= 600 \\ 240 + 60x_2 &= 600 \\ 60x_2 &= 360 \\ x_2 &= 6 \end{aligned}$$

So the equation has the solution $x_1 = 3$ and $x_2 = 6$, and purchasing 3 newspaper ads and 6 TV ads is feasible. Now, suppose the candidate decides that she can afford to spend $650,000 on advertising.

Q6. How does that change the equation?

Q7. Is there an integer solution to this equation? Why or why not?

Q8. Under what circumstances will there be an integer solution?

Q9. Is it possible to determine why $80x_1 + 60x_2 = 600$ has an integer solution, while $80x_1 + 60x_2 = 650$ does not have an integer solution?

Since the left-hand sides of the two equations are identical, the key lies in the value on the right-hand side.

Q10. What do the coefficients of x_1 and x_2 have in common?

Q11. How does that common feature relate to 600 and 650?

The observation we just made is the pivotal requirement for the existence of an integer solution to a linear Diophantine equation.

There is an integer solution to a linear Diophantine equation *if and only if* the greatest common divisor (GCD) of the coefficients of x_1 and x_2 divides the constant term.

Therefore, the equation $80x_1 + 60x_2 = 600$ has an integer solution, because the GCD of 80 and 60 is 20, and 600 is divisible by 20. Algebraically, this means we can factor a 20 out of the left-hand side of the equation and then divide both sides by 20 to yield a simpler equation.

$$\begin{aligned} 80x_1 + 60x_2 &= 600 \\ 20\left(4x_1 + 3x_2\right) &= 20\left(30\right) \\ 4x_1 + 3x_2 &= 30 \end{aligned}$$

Of course, we can do the same thing with the other equation, $80x_1 + 60x_2 = 650$, but watch what happens when we do.

$$\begin{aligned} 80x_1 + 60x_2 &= 650 \\ 20\left(4x_1 + 3x_2\right) &= 650 \\ 4x_1 + 3x_2 &= \frac{650}{20} = \frac{65}{2} \end{aligned}$$

Q12. Now, if x_1 and x_2 are both integers, what does that say about the number on the left-hand side of the equation?

Q13. How does this observation about the left-hand side of the equation compare with the right-hand side of the equation?

Notice that although we have simplified our original equation, we have yet to find an actual solution. Once we do find a solution, we can ask many more questions: Will there be more than one integer solution? What is the largest number of ads that could be purchased? What is the least number of ads that could be purchased? Is it possible to purchase the same number of TV ads as newspaper ads?

Restricting the decision variables to take on only integer values transforms the linear programming problem into an **integer programming** (IP) problem. An integer programming problem is more difficult to solve, because as we have already seen, a linear equation need not have any integer solutions. Geometrically, that means there might not be any points on the graph of a line with integer coordinates. If such a line happens to be part of the boundary of an integer programming formulation, that might lead to an optimal solution that lies not at a corner point, nor even on the boundary of the feasible region, but in the interior of the feasible region.

The points that we would consider for a solution to an integer programming problem must have integer coordinates. Points with integer coordinates are called **lattice points** and are defined as the points of intersection of the grid lines of a coordinate plane. Figure 5.1.1 shows the lattice points in the interior of the feasible region for our integer programming problem.

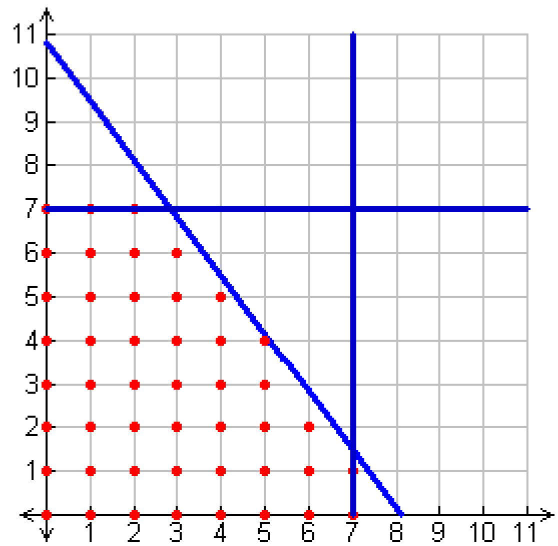


Figure 5.1.1: Graph of feasible region showing lattice points

Figure 5.1.2 highlights the lattice points in the feasible region that have the maximum *x*- and *y*-coordinates for their row and column. These points are referred to as the **kernel** of the problem. Their importance will be discussed further in Section 5.2.

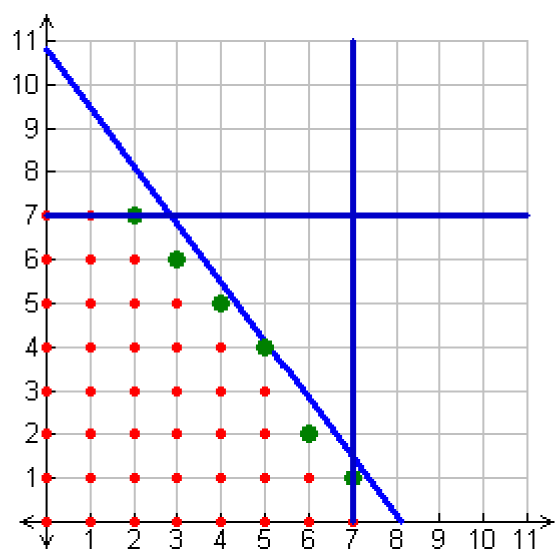


Figure 5.1.2: Graph of feasible region, lattice points, and kernel

Section 5.2: The Effectiveness of Political Advertising

Ms. Cynthia Brown is running for the office of governor in the next Michigan election. She wants to have an effective advertising strategy and has assigned Mr. Response to the advertising task. Ms. Brown has a weekly budget of $650,000. Mr. Response is planning to develop a linear model that determines the most **effective** and **cost-efficient** advertising media.

The advertising media that can be used in the campaign are newspaper, TV, radio, the Internet, brochures and mailings, and billboards. Mr. Response has gathered some information related to these media for Michigan.

In a survey, probable voters were asked to evaluate each medium. Of those surveyed, 35% said political ads in newspapers are influential. The other media were rated as influential in the following proportions: TV ads: 25%, brochures and mailings: 15%, radio: 10%, the Internet: 5%, and billboards: 3%.

Mr. Response decides to purchase ads only from all local TV stations and newspapers, because these two media have the highest proportions in the survey.

Listed below are some additional data that Mr. Response has deemed important:

- 60% of Michigan's 10 million people are eligible to vote.
- TV ads reach 85% of the electorate, with a cost of $60,000 per ad per day.
- Newspaper ads reach 70% of the electorate, with a cost of $80,000 per ad per day.

Mr. Response does not want to order more than one ad per day on any medium (that is, no more than seven ads per week for any particular medium).

Q1. What is Mr. Response's objective?

Q2. What decision variables should Mr. Response use?

Q3. Who are the people being targeted for the advertisements?

Q4. How would the number of people targeted in Michigan for the political advertisements be determined?

Q5. How would the number of these targeted people who are reached through newspaper advertisements be determined?

Q6. Of the targeted people who are reached by a newspaper ad, how many people are influenced by the newspaper ad?

Q7. Determine the coefficient for this decision variable. This coefficient represents the effectiveness rate for a newspaper ad.

Q8. Use the same process, based on the "most influential" percentages, to compute the effectiveness rate (coefficient) for the TV advertisement decision variable.

Q9. Use the decision variables and the coefficients from above to define the objective function.

Q10. Identify any constraints Mr. Response faces.

Q11. How should the constraint related to the weekly budget be represented in the problem?

Q12. How should the other constraints be represented in the problem?

Q13. Develop a spreadsheet representation of this LP formulation.

The problem formulation is shown below.

Decision Variables

Let: x_1 = the number of ads in newspapers per week
x_2 = the number of ads on TV per week
z = the number of people affected by the ads

Objective Function

Maximize: $z = (10 \cdot 0.60 \cdot 0.70 \cdot 0.35)x_1 + (10 \cdot 0.60 \cdot 0.85 \cdot 0.25)x_2$

Subject to:

Constraints

Weekly Budget: $80x_1 + 60x_2 \leq 650$
Newspaper Advertising Limit: $x_1 \leq 7$
TV Advertising Limit: $x_2 \leq 7$
Non-Negativity: $x_1 \geq 0$ and $x_2 \geq 0$

The problem formulation in Excel is shown in Figure 5.2.1.

	A	B	C	D	E	F
1	Chapter 5: Integer Programming					
2	5.2 Political Advertising					
3	Advertising Effectiveness Maximization					
4						
5	Decision Variable	**Newspaper Ads (x_1)**	**TV Ads (x_2)**			
6	Decision Values [# of ads]					
7						**Total # Affected**
8	Objective Function [Ad Effectiveness]	1.47	1.275			0
9						
10	**Constraints**					
11	Weekly Budget	80	60	0	≤	650
12	Newspaper Advertising Limit	1		0	≤	7
13	TV Advertising Limit		1	0	≤	7

Figure 5.2.1: The political advertising problem formulation

The graph in Figure 5.2.2 shows the optimal solution to this LP formulation. The optimal solution is $x_1 = 2.875$ and $x_2 = 7$.

Q14. Are these values for the decision variables feasible? Explain.

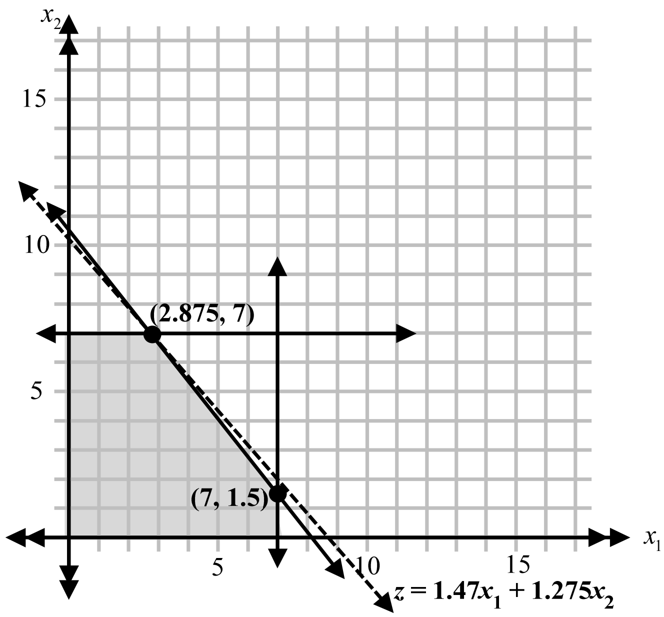


Figure 5.2.2: The political advertising solution

5.2.1 The Kernel of the Problem

Recall the LP formulation for our political advertising problem.

Decision Variables

Let: x_1 = the number of ads in newspapers per week
x_2 = the number of ads on TV per week
z = the number of people affected by the ads

Objective Function

Maximize: $z = (10 \cdot 0.60 \cdot 0.70 \cdot 0.35)x_1 + (10 \cdot 0.60 \cdot 0.85 \cdot 0.25)x_2$

Subject to:

Constraints

Weekly Budget:	$80x_1 + 60x_2 \leq 650$
Newspaper Advertising Limit:	$x_1 \leq 7$
TV Advertising Limit:	$x_2 \leq 7$
Non-Negativity:	$x_1 \geq 0$ and $x_2 \geq 0$

If we consider an additional constraint, that our decision variables take on only integer values, our problem becomes an integer programming (IP) problem. A common misconception is that you can find the solution to an IP problem simply by taking the LP solution and rounding the optimal solution when necessary. Consider the optimal solution for our political advertising problem, which is 2.875 newspaper ads per week and 7 TV ads per week. If we try to round this solution, we may guess that 3 newspaper ads and 7 TV ads would be the optimal integer solution. Unfortunately, if we substitute $x_1 = 3$ and $x_2 = 7$ into the budget constraint, we will obtain 80(3) + 60(7) = 240 + 420 = 660, which violates the budget constraint. Thus, this solution is not even feasible. In general, rounding the optimal solution to an LP problem will not produce an optimal solution to an IP problem. There are two difficulties with the rounding strategy, namely, the rounded solution may not be feasible, and if it is feasible, it may not be optimal. Thus, the rounding strategy applied to an IP problem may not generate the optimal solution.

The feasible region for an IP problem contains only the integer ordered pairs within the constraint boundaries. These ordered pairs of integers are called lattice points. In our political advertising problem, there are nearly 50 lattice points in the feasible region.

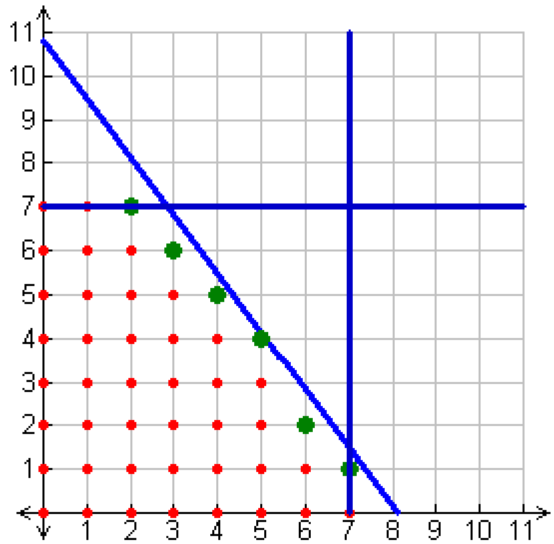


Figure 5.2.3: The feasible region showing the lattice points and kernel

A brute force attempt will allow us to check all of these points in order to find our optimal solution. For example, the top horizontal boundary line contains three lattice points: (0, 7), (1, 7) and (2, 7). If we check these in the objective function, we find that the lattice point closest to the LP optimal line of constant effectiveness will maximize our objective function. But that point is also the one that is farthest to the right of those three points. Since they are on the same horizontal line, all three points have the same value of x_2, namely 7. So it makes sense that the one that yields the largest value of the objective function is the one farthest to the right, because that point has the largest value of x_1.

In general, as we consider lattice points closer to that line of constant effectiveness, we will always get a better value for our objective function. Therefore, we can minimize the number of lattice points to check by only considering the lattice points closest to the boundary. This collection of lattice points is called the **kernel** of the feasible region. Figure 5.2.3 shows the kernel of the feasible region.

In our problem, the kernel consists of (2, 7), (3, 6), (4, 5), (5, 4), (6, 2), and (7, 1). Using brute force on these six points is significantly less work than applying brute force to the original 50 lattice points.

x_1	x_2	$z = 1.47x_1 + 1.275x_2$
2	7	11.865
3	6	12.06
4	5	12.255
5	4	12.45
6	2	11.37
7	1	11.565

Table 5.2.1: Evaluating the objective function for the kernel points

Table 5.2.1 displays the result of evaluating our objective function at these points. Notice that the optimal integer solution is (5, 4)—that is, 5 newspaper ads per week and 4 TV ads per week, with an advertising effectiveness of 12.45.

Q15. What are the units of measure for advertising effectiveness?

Q16. What does an advertising effectiveness of 12.45 mean?

Q17. Is that answer reasonable?

We should also note how the IP solution differs from the LP solution. We tried rounding up and found a solution that is not even feasible. If we had rounded down to 2 newspaper ads and 7 TV ads, we would have a feasible solution, just not the optimal solution. The IP optimal solution, (5, 4), not only fails to be the point we would obtain by rounding the LP optimal solution, which is not even feasible, it is not even the feasible point closest to the LP corner point.

Using the Excel Solver, the integer constraints are easily handled. In the Add Constraint dialog box, simply select the cells containing the decision variables for both the Cell Reference and Constraint boxes. Instead of an inequality between them, we will select "int," which is how the Solver will know that the constraint is that these variables must be integers (see Figure 5.2.4).

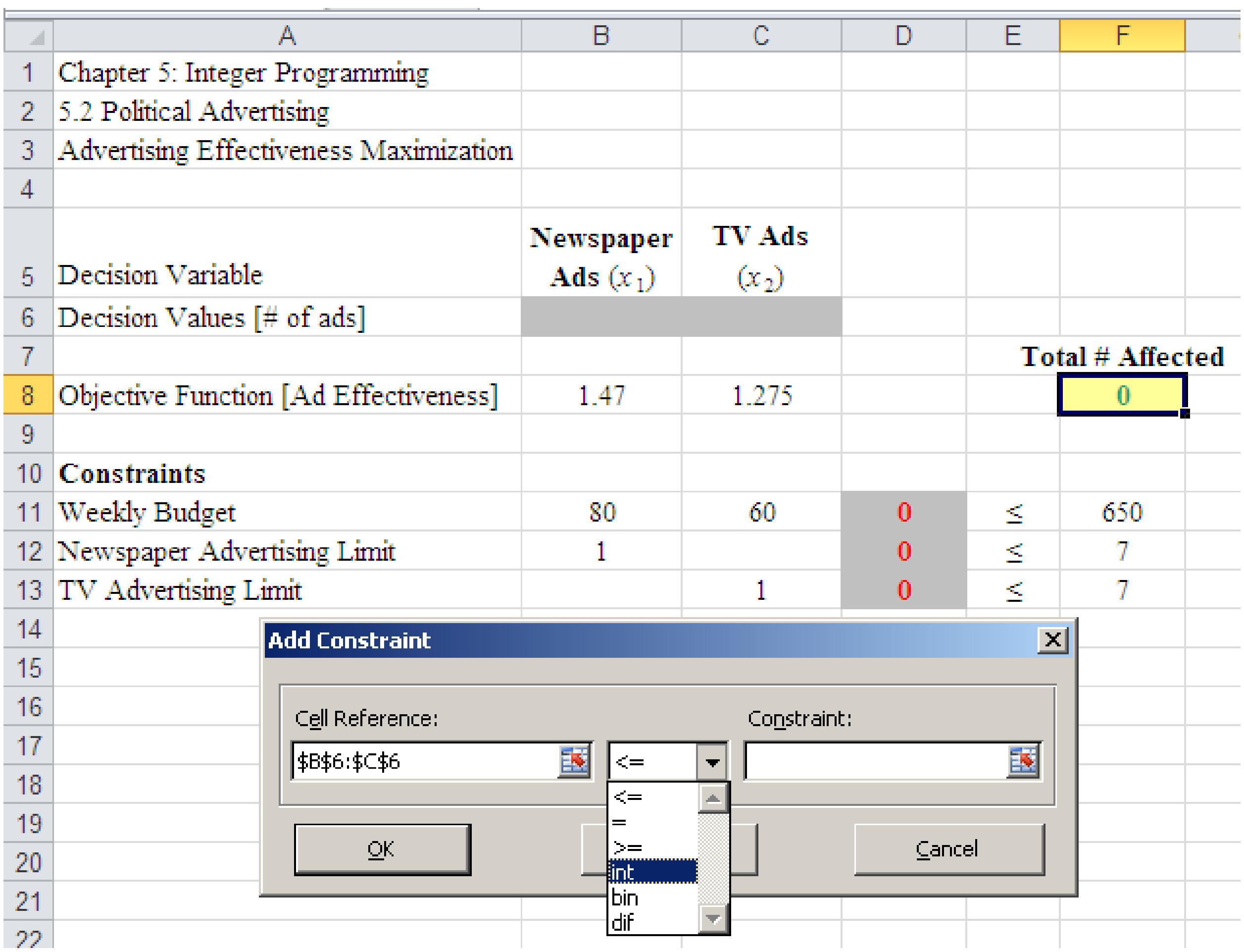


Figure 5.2.4: Specifying integer constraints

The optimal solution to purchase 5 newspaper ads and 4 TV ads uses up some of the advertising budget, but does it use all of the available funds? Compute the cost of the optimal solution. What should Ms. Brown do?

5.2.2 Gearing Up for the Election

After a very successful round of fund-raising, Ms. Brown asks Mr. Response to purchase ads using other possible media: all local radio stations, several Web sites, direct mailings to all residents, and billboards on several main roads. Ms. Brown increases the weekly budget to $900,000. Use the data given in Table 5.2.2 below.

Medium	Effectiveness	Achieved reach	Cost (per ad-day)
Radio (x_3)	10%	90%	$20K
Internet (x_4)	5%	60%	$10K
Direct mailings (x_5)	15%	95%	$180K
Billboards (x_6)	3%	40%	$15K

Table 5.2.2: Effectiveness, reach, and cost of various advertising media

Q18. What decision variables are needed? (Don't forget the first two media!)

Q19. What data will be required to determine the coefficient for each decision variable?

Q20. Compute the coefficient for each decision variable. Apply the coefficient and the corresponding decision variable to construct the objective function.

Q21. Determine how to represent the constraint based on the weekly budget.

Q22. Represent all the other constraints that must be considered.

Q23. What are the impossible number values for each decision variable?

The integer programming problem formulation should look something like the following.

Decision Variables

Let:
x_1 = the number of ads in newspapers per week
x_2 = the number of ads on TV per week
x_3 = the number of ads on the radio per week
x_4 = the number of ads on the Internet per week
x_5 = the number of ads in direct mailings per week
x_6 = the number of ads on billboards per week
z = the number of people affected by the ads

Objective Function

Maximize: $z = 1.47x_1 + 1.275x_2 + 0.54x_3 + 0.18x_4 + 0.855x_5 + 0.072x_6$

Subject to:

Constraints

Weekly Budget: $80x_1 + 60x_2 + 20x_3 + 10x_4 + 180x_5 + 15x_6 \leq 900$
Newspaper Advertising Limit: $x_1 \leq 7$
TV Advertising Limit: $x_2 \leq 7$
Radio Advertising Limit: $x_3 \leq 7$

Internet Advertising Limit:	$x_4 \leq 7$
Direct Mailings Advertising Limit:	$x_5 \leq 7$
Billboards Advertising Limit:	$x_6 \leq 7$
Non-Negativity:	$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$, and $x_6 \geq 0$
Integer Constraints:	x_1, x_2, x_3, x_4, x_5, and x_6 are integers

The problem formulation and solution are shown in Figure 5.2.5.

	A	B	C	D	E	F	G	H	I	J
1	Chapter 5: Integer Programming									
2	5.2 Political Advertising									
3	Advertising Effectiveness Maximization									
4										
5	Decision Variables	Newspaper Ads (x_1)	TV Ads (x_2)	Radio Ads (x_3)	Internet Ads (x_4)	Direct Mailing Ads (x_5)	Billboard Ads (x_6)			
6	Decision Values [# of ads]	4	7	7	2	0	0			
7										Total # Affected
8	Objective Function [Ad Effectiveness]	1.47	1.275	0.54	0.18	0.855	0.072			18.945
9										
10	Weekly Budget	80	60	20	10	180	15	900	≤	900
11	Newspaper Advertising Limit	1						4	≤	7
12	TV Advertising Limit		1					7	≤	7
13	Radio Advertising Limit			1				7	≤	7
14	Internet Advertising Limit				1			2	≤	7
15	Direct Mailings Advertising Limit					1		0	≤	7
16	Billboard Advertising Limit						1	0	≤	7

Figure 5.2.5: IP Formulation and solution to the Political Advertising problem

Q24. How many of each type of ad should Mr. Response purchase?

Q25. What does the value 18.945 in the far right-hand column of the spreadsheet mean?

Q26. Which constraints are binding?

5.2.3 Adding Another Requirement

Ms. Brown has decided that she wants to reach the electorate with direct mailings *at least once a week*.

Q27. What aspect(s) of the problem will be affected by this additional change?

Q28. How does this modification affect the objective function?

Q29. In what way does this feature affect the constraints?

The formulation below accounts for this newest modification, and the spreadsheet that follows in Figure 5.2.6 contains the optimal solution following that modification.

<u>Decision Variables</u>

Let:
x_1 = the number of ads in newspapers per week
x_2 = the number of ads on TV per week
x_3 = the number of ads on the radio per week
x_4 = the number of ads on the Internet per week
x_5 = the number of ads in direct mailings per week
x_6 = the number of ads on billboards per week

z = the number of people affected by the ads

Objective Function

Maximize: $z = 1.47x_1 + 1.275x_2 + 0.54x_3 + 0.18x_4 + 0.855x_5 + 0.072x_6$

Subject to:

Constraints

Weekly Budget:	$80x_1 + 60x_2 + 20x_3 + 10x_4 + 180x_5 + 15x_6 \le 900$
Newspaper Advertising Limit:	$x_1 \le 7$
TV Advertising Limit:	$x_2 \le 7$
Radio Advertising Limit:	$x_3 \le 7$
Internet Advertising Limit:	$x_4 \le 7$
Direct Mailings Advertising Limit:	$x_5 \le 7$
Billboards Advertising Limit:	$x_6 \le 7$
Direct Mailing Advertising Minimum:	$x_5 \ge 1$
Non-Negativity:	$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0$, and $x_6 \ge 0$
Integer Constraints:	x_1, x_2, x_3, x_4, x_5, and x_6 are integers

	A	B	C	D	E	F	G	H	I	J
1	Chapter 5: Integer Programming									
2	5.2 Political Advertising									
3	Advertising Effectiveness Maximization									
4										
5	Decision Variables	**Newspaper Ads** (x_1)	**TV Ads** (x_2)	**Radio Ads** (x_3)	**Internet Ads** (x_4)	**Direct Mailing Ads** (x_5)	**Billboard Ads** (x_6)			
6	Decision Values [# of ads]	2	7	7	0	1	0			
7										**Total # Affected**
8	Objective Function [Ad Effectiveness]	1.47	1.275	0.54	0.18	0.855	0.072			16.5
9										
10	Weekly Budget	80	60	20	10	180	15	900	≤	900
11	Newspaper Advertising Limit	1						2	≤	7
12	TV Advertising Limit		1					7	≤	7
13	Radio Advertising Limit			1				7	≤	7
14	Internet Advertising Limit				1			0	≤	7
15	Direct Mailings Advertising Limit					1		1	≤	7
16	Billboard Advertising Limit						1	0	≤	7
17	Direct Mailings Advertising Minimum					1		1	≥	1

Figure 5.2.6: IP Solution to the Political Advertising problem after Ms. Brown's modification

Q30. How does the requirement to have at least one direct mailing per week affect the optimal solution?

Q31. The value of 18.945 from the previous solution is now 16.5. What does that mean?

Q32. Why might Ms. Brown want to stay with her decision regarding direct mailings?

Section 5.3: Opening and Operating the Pizza Palace

A new shopping district has recently been developed downtown. Juanita Johnson just purchased a restaurant that is located in the district. She plans to open a pizza eatery and name it Pizza Palace. As owner, Ms. Johnson wants this venture to be as successful and as profitable as possible. She needs to determine how many employees to schedule for a given shift and must keep the wage cost as low as possible.

There are three categories of employee for any given shift: crew member, shift supervisor, and assistant manager. Crew members take orders, prepare and cook the pizza, clean the facility, restock materials, and handle customer transactions. Shift supervisors perform all of the tasks of crew members, as well as supervise, train, and assist the crew members, and aid the assistant manager. Assistant managers perform all of the tasks of shift supervisors. In addition, they set the schedules, complete inventory, supervise all workers, manage productivity, make deposits, and settle customer issues. They also direct the operation of the restaurant during their shifts, ensure compliance with company standards, ensure compliance with health and safety codes, maintain fast and accurate service, interview job applicants, and motivate and discipline employees as necessary.

Assistant managers are the most productive employees. Shift supervisors are 80% as productive as assistant managers. Compared to shift supervisors, crew members are 75% as productive. Ms. Johnson decides on the following pay rates:

- Crew members: $8.00/hr
- Shift supervisors: $10.00/hr
- Assistant managers: $15.00/hr

After thorough research of similar restaurants, Ms. Johnson arrives at some significant conclusions. There are certain necessities for any given shift. In order to have a profitable, safe, and efficiently run eatery, there must be at least one assistant manager working each shift. There has to be at least one assistant manager or shift supervisor for every five crew members. The makeup of any scheduled shift requires a level of productivity equivalent to at least ten assistant managers. Ms. Johnson also realizes that the union representing the crew members requires at least seven crew members to work each shift.

Q1. Adhering to the required conditions, how many employees of each category must Ms. Johnson hire for a given shift at Pizza Palace while keeping the wage cost as low as possible?

5.3.1 Problem Formulation

Ms. Johnson begins by defining the decision variables and the objective function.

Decision Variables

Let: x_1 = number of crew members working per shift
x_2 = number of shift supervisors working per shift
x_3 = number of assistant managers working per shift
z = the cost of wages

Objective Function

Minimize: $z = 8x_1 + 10x_2 + 15x_3$

Two of the four constraints are straightforward: there must always be at least one assistant manager and at least seven crew members working.

Subject to:

Constraints

Assistant Manager Minimum: $x_3 \geq 1$

Crew Member Minimum: $x_1 \geq 7$

Formulating the supervision constraint takes some careful thought. There must be at least one shift supervisor or assistant manager for every five crew workers. This situation represents a ratio.

$$\frac{\text{supervisors}}{\text{crew members}} \geq \frac{1}{5}$$

$$\frac{x_2 + x_3}{x_1} \geq \frac{1}{5}$$

$$5\left(\frac{x_2 + x_3}{x_1}\right) \geq 5\left(\frac{1}{5}\right)$$

$$\frac{5x_2 + 5x_3}{x_1} \geq 1$$

$$x_1\left(\frac{5x_2 + 5x_3}{x_1}\right) \geq x_1(1)$$

$$5x_2 + 5x_3 \geq x_1$$

Putting all of the decision variables on the left hand side, we get the following constraint:

Supervision Minimum: $-x_1 + 5x_2 + 5x_3 \geq 0$

The productivity constraint is based on the productivity of each type of employee, stated above, and must be at least ten.

Productivity: $0.6x_1 + 0.8x_2 + 5x_3 \geq 10$

Figure 5.3.1a contains a spreadsheet formulation and solution of the Pizza Palace problem. Figure 5.3.1b shows an Answer Report.

	A	B	C	D	E	F	G
1	Chapter 5: Integer Programming						
2	5.3 Pizza Palace						
3	Cost Minimization						
4							
5	Decision Variable	**Crew Members** (x_1)	**Supervisors** (x_2)	**Assistant Mangers** (x_3)			
6	Decision Values [# of workers]	7	6	1			
7							**Total Cost**
8	Objective Function [Cost of Wages]	8	10	15			131
9							
10	**Constraints**						
11	Assistant Managers			1	1	≥	1
12	Crew Members	1			7	≥	7
13	Supervision	-1	5	5	28	≥	0
14	Productivity	0.6	0.8	1	10	≥	10

Figure 5.3.1a: Formulation for the Pizza Palace problem

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$G$8	Objective Function [Cost of Wages] Total Cost	0	131

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Decision Values [# of workers] Crew Members (x1)	0	7	Integer
$C$6	Decision Values [# of workers] Supervisors (x2)	0	6	Integer
$D$6	Decision Values [# of workers] Assistant Mangers (x3)	0	1	Integer

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$11	Assistant Managers	1	$E$11>=$G$11	Binding	0
$E$12	Crew Members	7	$E$12>=$G$12	Binding	0
$E$13	Supervision	28	$E$13>=$G$13	Not Binding	28
$E$14	Productivity	10	$E$14>=$G$14	Binding	0
$B$6:$D$6=Integer					

Figure 5.3.1b: Answer Report for the Pizza Palace problem

Q2. Identify the binding and nonbinding constraints, and interpret them in the context of the problem. How are your answers related to the slack for each variable?

Q3. Determine the sensitivity of the optimal solution to pay raises by considering the following three options.

Option A: Offer a raise to the shift supervisors to \$10.25, \$10.50, \$10.75, and so forth in \$0.25 increments. This will naturally cause the value of the objective function to increase, but at what point will the optimal number of each type of employee change?

Option B: Offer a raise to all employees in \$1.00 increments. At what uniform wage increase for all employees will the optimal number of employees change? Why do you think that happens?

Option C: Offer a raise to all employees, but make the increase proportional to the current rate; that is, increase each wage by 10%, 20%, 30% and so forth in 10% increments. At what proportional wage increase for all employees, if any, will the optimal number of employees change? Why do you think that happens?

As Ms. Johnson's business grows, she finds that her productivity needs change. Pizza Palace now requires that the makeup of any scheduled shift include a level of productivity equivalent to twelve assistant managers instead of ten.

Q4. Use the Solver to determine how this affects the optimal solution.

Currently, the ratio of assistant mangers or shift supervisors to crew members must be no lower than 1:5. Over the first few months, her crew members have become more experienced and need less supervision.

Q5. Use the Solver to learn what happens to the optimal solution if that ratio changes to 1:7; that is, there must be at least one assistant manager or shift supervisor for every *seven* crew members.

After three years, the union representing the crew members negotiates a new contract that stipulates that at least eight crew members must work each shift.

Q6. In what ways will this change affect the optimal solution?

Q7. Which category or categories of employee increase in number? Which category or categories decrease?

Q8. Does the situation change if you require nine crew members?

Section 5.4: Transporting Oranges to Midwest Markets

Efficient delivery of goods and services is a critical element of the U.S. economy. The large trucks that transport a wide range of goods on our highways are just one highly visible piece of a broader transportation network. This includes highways, railroads, airports, and pipelines. Consider the steps in bringing gasoline to your neighborhood. The oil is extracted from the ground in some distant land. It is transported by pipeline to a storage facility and then loaded onto vast super tankers. These ships bring the oil to a US port where it is then transported to a refinery to be turned into gasoline. The gasoline is then stored until it is loaded onto specialized trucks. Each truck transports the gasoline to several of the more than one hundred thousand gas stations spread throughout the country.

Physical transport is part of a broader industry that is labeled logistics. Logistics is defined as

> "the process of planning, carrying out, and controlling the efficient, effective flow and storage of goods, services, and related information from point of origin to point of consumption to meet the needs of the final consumer."

It starts with collecting all of the raw materials and transporting them to one or more facilities that converts them into products. This is called *inbound* logistics. These products or components are later assembled or mixed into the final product. It is then transported through a variety of means to a point where the final consumer can purchase it. These are called *outbound* logistics. The customer may bring the product home or have it delivered to his/her home. Logistics does not end there if the final product eventually gets recycled. The used up product must be picked up and brought into a recycling center.

The term logistics has its modern origins in the military. In the military, logistics refers to all of the support services required to field an army. The personnel at the front had to first be transported there. Logistics then provides for their daily needs of food, fuel, clothing, equipment, housing, ammunition, and medical treatment.

It is estimated logistics accounts for between 8% and 10% of the U.S. Gross Domestic Product. It contributes more than $1 trillion to the US economy and provides more than 10 million jobs. It includes low skilled workers who restock inventory and highly skilled industrial engineers and operations researchers who develop sophisticated procedures to improve and track efficiency of the whole system.

Let's consider the logistics for simply sending a package from a business to be delivered overnight to a customer 1,500 miles away.

- The sender may already have a contract with the delivery service. This includes a computer account, pricing, and packaging material.
- The sender inputs the relevant data into an information system that is used to track the package and also bill the sender. The sender attaches an information-rich label to the package.
- The package is deposited into a box for collection.
- A local truck collects all of the packages from collection boxes and brings them to a local processing center where the package is logged in.
- The package is shipped to a nearby airport to be air transported to a major regional processing center.
- At the regional processing center, it is combined with other packages from all over the US that are ending up in the same local area.
- It is often placed on another plane and flown to an airport nearer the final destination.
- At the airport, the package is grouped with others into a specific route. It is assigned to a truck to be delivered along a specific route. All of the information is loaded into a portable device the driver carries with him/her.

- The driver organizes the packages in the truck so that he/she can reach each package in sequence as he/she travels the route.
- The driver travels the route in sequence and records into the computer when the package is delivered. The driver might also obtain a signature from the recipient.
- The sender is billed for the service.

Now imagine that Amazon starts this process tens of thousands of times each day.

5.4.1 Hercules Transport: Moving Oranges across the Country

In the following example, we will study just one piece of the logistics system. This example is included in a chapter on integer programming because we will limit the decision variables to integer values of full truckloads. Logistics managers often develop plans based on full truckloads. The cost of the driver, the equipment, and all of the information processing does not vary with the amount loaded in the truck. However, the cost of gasoline will depend on the overall weight of the truck and its shipment.

The Hercules Transport company has won a contract to deliver full truckloads of fresh oranges from California and Florida to six major markets in the Midwest throughout the prime harvest season. These are shipped in refrigerated trucks. The total cost of a fully loaded truck on a long-distance trip is $1.40 per mile. (This assumes the truck and its driver will be contracted to transport something on the return trip as well.) Hercules has purchasing contracts with several major orange groves in both states. The contract allows for a maximum purchase of 180 truckloads from California and a maximum of 80 truckloads from Florida.

Hercules is shipping from California to a number of metropolitan areas in the Midwest. Typically, shipping is consolidated at a few central locations and is received at a few central locations. This simplifies estimating shipping costs. Hercules has a consolidation warehouse in Orange County. It will make deliveries to distribution centers located in large metropolitan areas within each state. These centers are usually located on a major interstate highway not far from a major city.

For example, in Minnesota, this would be outside Minneapolis and for northern Texas this would be near Dallas-Fort Worth. (Major metro areas in this example are Minneapolis, MN; Indianapolis, IN; Kansas City, MO; Oklahoma City, OK; Omaha, NE; Des Moines, IA).

Similarly, Hercules has a warehouse in Orlando in central Florida. Hercules estimated the miles between each origin and destination using MapQuest (www.mapquest.com). The distances between the two sources of oranges and the markets are included in Table 5.4.1. The bottom row includes the seasonal demand for fresh oranges in each market as measured in truckloads.

These distances are converted into costs by multiplying by $1.40 per mile and reported in Table 5.4.2. These values were rounded to the nearest dollar. The total available supply of 260 truckloads is more than enough to meet the total demand of 207 truckloads. Hercules seeks to minimize its total cost of delivery.

		Market						
		1 MN	**2 IN**	**3 MO**	**4 OK**	**5 NE**	**6 IA**	**Supply**
Supply of Oranges	**1 CA**	1929	2076	1583	1329	1554	1686	180
	2 FL	1554	972	1241	1227	1428	1342	80
	Demand	66	34	42	30	20	15	

Table 5.4.1: Travel distances (in miles) between supply locations and demand locations

		Market						
		1 MN	**2 IN**	**3 MO**	**4 OK**	**5 NE**	**6 IA**	**Supply**
Supply of Oranges	**1 CA**	\$2,701	\$2,906	\$2,216	\$1,861	\$2,176	\$2,360	180
	2 FL	\$2,176	\$1,361	\$1,737	\$1,718	\$1,999	\$1,879	80
	Demand	66	34	42	30	20	15	

Table 5.4.2: Transportation costs between supply locations and demand locations

5.4.2 Model Formulation

Hercules Transport must decide on the number of truckloads of oranges to ship from each source to each demand location. The goal is to minimize the total transportation cost. The decisions are constrained by the available supply and the demand in each market. We will use decision variables with double subscripts to formulate the decision. The decision variable $x_{i,j}$ is the number of truckloads shipped from source i to market j. There are 2 sources and 6 markets for a total of 12 decision variables. There are two constraints on the available supply. These are both inequalities.

Orange, CA supply: $1x_{1,1} + 1x_{1,2} + 1x_{1,3} + 1x_{1,4} + 1x_{1,5} + 1x_{1,6} \leq 180$
Orlando, FL supply: $1x_{2,1} + 1x_{2,2} + 1x_{2,3} + 1x_{2,4} + 1x_{2,5} + 1x_{2,6} \leq 80$

There are six constraints that represent Hercules contract that requires it to meet the demand in each market. These are all equalities.

Minnesota demand: $1x_{1,1} + 1x_{2,1} = 66$
Indiana demand: $1x_{1,2} + 1x_{2,2} = 34$
Missouri demand: $1x_{1,3} + 1x_{2,3} = 42$
Oklahoma demand: $1x_{1,4} + 1x_{2,4} = 30$
Nebraska demand: $1x_{1,5} + 1x_{2,5} = 20$
Iowa demand: $1x_{1,6} + 1x_{2,6} = 15$

The complete formulation is listed below. In setting up the formulation, we have carefully aligned each column to represent one decision variable. Notice that each decision variable appears in exactly two constraints, one supply constraint and one demand constraint. In addition, the coefficient before each decision variable is always 1. These observations apply to every transportation decision formulation.

In Figure 5.1 we present the general pattern. Why does this matter? This structure produces an interesting result. Recall, that we have restricted all of the decision variables to integer values. This is designed to match the requirement that we are modeling full truckloads. Due to the structure of this standard transportation model, a linear programming optimal solution to this class of problems will generate decision variable values that are all integers. This will always occur as long as the supply and demand values are also integer. This is important because we would rather not have to restrict the solution procedure to only integers. When we use linear programming, Solver produces a sensitivity analysis report that allows us to easily explore the impact of changes in the model. Solver cannot produce a sensitivity report for integer programming models.

Decision Variables

Let: $x_{1,1}$ = the number of truckloads of oranges from CA to MN
$x_{1,2}$ = the number of truckloads of oranges from CA to IN
$x_{1,3}$ = the number of truckloads of oranges from CA to MO
$x_{1,4}$ = the number of truckloads of oranges from CA to OK
$x_{1,5}$ = the number of truckloads of oranges from CA to NE
$x_{1,6}$ = the number of truckloads of oranges from CA to IA
$x_{2,1}$ = the number of truckloads of oranges from FL to MN
$x_{2,2}$ = the number of truckloads of oranges from FL to IN
$x_{2,3}$ = the number of truckloads of oranges from FL to MO
$x_{2,4}$ = the number of truckloads of oranges from FL to OK
$x_{2,5}$ = the number of truckloads of oranges from FL to NE
$x_{2,6}$ = the number of truckloads of oranges from FL to IA
z = the cost of shipping

Objective Function

Minimize: $z = 2701x_{1,1} + 2906x_{1,2} + 2216x_{1,3} + 1861x_{1,4} + 2176x_{1,5} + 2360x_{1,6} + 2176x_{2,1} + 1361x_{2,2} + 1737x_{2,3} + 1718x_{2,4} + 1999x_{2,5} + 1879x_{2,6}$

Subject to:

Constraints

$$1x_{1,1} + 1x_{1,2} + 1x_{1,3} + 1x_{1,4} + 1x_{1,5} + 1x_{1,6} \le 180$$
$$1x_{2,1} + 1x_{2,2} + 1x_{2,3} + 1x_{2,4} + 1x_{2,5} + 1x_{2,6} \le 80$$
$$1x_{1,1} + 1x_{2,1} = 66$$
$$1x_{1,2} + 1x_{2,2} = 34$$
$$1x_{1,3} + 1x_{2,3} = 42$$
$$1x_{1,4} + 1x_{2,4} = 30$$
$$1x_{1,5} + 1x_{2,5} = 20$$
$$1x_{1,6} + 1x_{2,6} = 15$$

All decision variables are greater than 0.

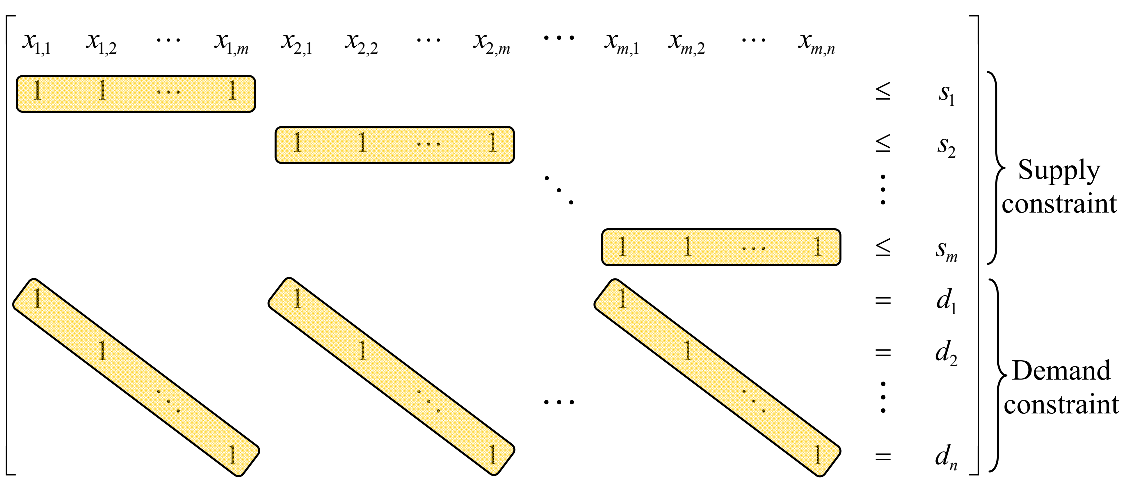


Figure 5.4.1: The general structure of a transportation problem

5.4.3 Optimal Solution and Sensitivity Analysis

Hercules Transportation seeks to minimize its total cost of delivery. The optimal solution has a total cost of \$428,212. The truckloads are reported in Table 5.4.3. In the optimal solution, all oranges shipped to

Missouri, Oklahoma, Nebraska and Iowa come from California. All of the oranges provided to Indiana come from Florida. Only Minnesota receives shipments from both California and Florida. All of the demands are exactly met since they were equality constraints. The optimal solution uses all of the available oranges in Florida but not from California.

Q1. Why is the available supply in California not used up?

		Market							
		1 MN	**2 IN**	**3 MO**	**4 OK**	**5 NE**	**6 IA**	**Total Shipped**	**Supply**
Supply of Oranges	**1 CA**	20	0	42	30	20	15	127	180
	2 FL	46	34	0	0	0	0	80	80
	Demand	66	34	42	30	20	15	207	

Table 5.4.3: Optimal solution for orange transport

Table 5.4.4 reports the Sensitivity Analysis for the optimal solution. Let's consider the two supply constraints. The shadow price for California supplies is 0, because Hercules Transport is not using up all of the California supply. However, the shadow price for Florida is -$525. This means that each additional truckload of oranges obtained in Florida would reduce the total transportation cost by $525. Management has begun exploring additional sources of supply in Florida.

Q2. How much could be saved if the Florida supply increased by15 truckloads?

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$6	Demand 1	66	2701	66	53	20
$I$6	Demand 2	34	1886	34	46	20
$J$6	Demand 3	42	2216	42	53	42
$K$6	Demand 4	30	1861	30	53	30
$L$6	Demand 5	20	2176	20	53	20
$M$6	Demand 6	15	2360	15	53	15
$N$4	Supply CA	127	0	180	1E+30	53
$N$5	Supply FL	80	-525	80	20	46

Table 5.4.4: Sensitivity Analysis for orange transport

The shadow price on Demand 1 is $2,701. This is the exact cost of shipping from California to Minnesota as reported in Table 5.4.2.

Q3. Why does this make sense?

Q4. Which of the other demand constraints have shadow prices that equal a specific transportation cost?

Q5. Which of the other demand constraints have shadow prices that do not equal a specific transportation cost? Explain why this happened.

Section 5.5: Chapter 5 (IP) Homework Questions

1. Products that we purchase from a local store have to be transported from an original location where the product is grown or manufactured to the retail outlet.

   - Fruits and vegetables: fresh or processed.
   - Meat: fresh or processed
   - Fish: fresh or processed
   - Milk
   - Toys
   - Clothing
   - Cars
   - Bottled soda

   a) Which of the above products are transported in specialized vehicles at some point in the transportation network?
   b) Which of the above products are made in a few centralized locations with finished products often transported more than a thousand miles?
   c) Which of these products are likely to be transported across country borders?
   d) Which products are produced usually within 1,000 miles of the final customer? Might it vary by time of year?
   e) Construct a logistics outline of tasks for one of the above products.

2. Hercules transport of oranges in text,
   a) The Allowable Increase for the Florida constraint is 20 truckloads. What is the total reduction in cost of increasing Florida's supply by 20 truckloads?
   b) What is the total reduction in cost of increasing Florida's supply by 30 truckloads? What is the new shadow price?

3. A tailor makes wool sport coats and trousers. He is able to obtain 100 square yards of wool material each month to make the sport coats and trousers, and he can spend 120 hours per month making them. A sport coat brings $100 in profit, uses 3 square yards of material, and takes 8 hrs. to make. A pair of trousers brings $75 in profit, uses 5 square yards of material, and takes 4 hrs. to make. The tailor would like to maximize his profit.

   a) Define the decision variables for an integer programming model of the problem.

   b) Use the decision variables you defined in (a) to define the objective function.

   c) Use the decision variables to formulate any constraints in the problem.

   d) Find the optimal mix of sport coats and trousers the tailor should make each month.

   e) What is the maximum profit he can earn each month?

4. Bob and Sue Penny produce handmade wooden furniture in a workshop on their farm during the winter months. In late December, they purchased 900 board feet of solid cherry wood, and they plan to make kitchen tables, chairs and stools during January, February and March. Each table requires 25 hours to make, each chair requires 15 hours, and each stool requires 8 hours. A table uses 36 board feet of wood, a chair uses 14 board feet, and a stool uses 5 board feet. Between them, Bob and Sue have 650 hours that they can devote to this project. A table earns $650 profit, a chair earns $175, and a stool earns $100. Most people who buy a table also want four chairs, so they decide to make enough chairs so that there are at least four chairs for every table. How many of each piece of furniture should they make in order to maximize their profit?

5. Natty likes making and wearing jewelry. One day she saw an article on Internet talking about the best ways of earning money. After reading it, she decided to turn her skills into money and do some savings in summer holidays. She shared her ideas with her father and he promised to provide her the required budget for a while until she can manage the job on her own. Next week is the start of this summer holiday and Natty has already bought some tools and supplies for her home business.
She is going to make and sell bracelets, earrings, and rings. She is going to use different sizes of beads, clasps, crimps, thread, etc. She wants to work at least 30 hrs a week. The usage amounts from each resources and how much time she is going to spend for making each type of jewelry are given in Table 1.

Resource	Bracelet	Earring	Ring
6 mm faceted iridescent bead	4	2	1
4 mm faceted iridescent bead	2	2	2
4 mm silver beads	4	2	2
silver clasp	1	2	-
crimp	2	2	-
thread (Size D) inches	10.5	-	2
thread (Size B) inches	-	2	-
labor hours	3.5	1.5	1

Table 5.5.1: Usage amounts from each resources

She found a jewelry supplies shop which sells a bit cheaper than the others. The seller made also some discount since she is a student who is trying to do some saving. Table 2 shows the prices of supplies after discount and Table 3 shows the amounts of supplies that Natty purchased.

Resource	Cost ($)
6 mm faceted iridescent bead	0.95
4 mm faceted iridescent bead	0.75
4 mm silver beads	2.75
silver clasp	0.6
crimp	0.52
thread (Size D) inches	1.45
thread (Size B) inches	1.45

Table 5.5.2: Expenses

Resource	Availability (#)
6 mm faceted iridescent bead	35
4 mm faceted iridescent bead	30
4 mm silver beads	40
silver clasp	20
crimp	20
thread (Size D) inches	2,592
thread (Size B) inches	2,304

Table 5.5.3: Available Supplies

Natty is planning to sell each bracelet for $26.50, each earring for $15.95, and each ring for $13.50. How many of each item should she produce to make the most weekly profit?

6. ShiftyGear, Inc. manufactures three types of bicycle, a simple 10-speed, a slightly more elaborate 12-speed, and a deluxe 24-speed mountain bike. The production manager at ShiftyGear, Joe Sprocket, does his production schedule monthly. The company also ships and bills its customers monthly, so Mr. Sprocket wants to complete any bike the company begins to manufacture in the same month when its production began. Table 4 shows the relevant manufacturing details for each of the three bikes that ShiftyGear produces.

There are 40 production workers available to work on the bikes, and the ShiftyGear factory can work on up to six bikes at a time. Next month there are 21 work-days. Therefore, there are 126 production days available to build the bikes.

How many of each type of bike should Mr. Sprocket schedule for production in order to maximize the company's profit?

	SG-10	SG-12	MB-24
Market estimate of maximum to produce	18	12	6
Number of days to build one bike	6	8	12
Production workers needed per bike	1	2	3
Net profit per bike	$80	$100	$135

Table 5.5.4: ShiftyGear manufacturing data

a) Define the decision variables for this problem. Are those decision variables non-negative? Are they integer?

b) Using the decision variables you have defined, define the objective function.

c) How many constraints are there?

d) Define all of the constraints in terms of the decision variables.

e) Use your problem formulation from parts a, b, and d to develop a spreadsheet representation of the problem.

f) Use a spreadsheet solver to find the optimal solution.

7. In your optimal solution to problem 6, the value of one of the decision variables should be 0.

a) What does that mean in the context of the problem?

b) In steps of $5, increase the profitability of that bicycle model until its value is greater than zero. What does that mean in the context of the problem? For what amount of profit does it first happen?

c) When it does happen, does one of the other decision variables leave the optimal solution (i.e., become 0)? If so, which one?

8. In your solution to problem 6, you should have found that one of the decision variables dominates the optimal solution.

a) Which model dominates the optimal solution?

b) In steps of $5, decrease the profitability of that model until the optimal solution changes. For what amount of profit does this first occur?

c) What is the new optimal solution when it does?

9. Returning again to problem 6, suppose that the Market estimate of the number of bikes to produce per month was 12 of each model.

a) What effect does that change have on the optimal solution?

b) Suppose now that the original market estimates apply, but it is the height of the flu season. On average, only 80% of the production workers are available to work on any given day. What is the effect on the optimal solution?

Chapter 5 Summary

What have we learned?

As in the previous two chapters (LP Max and LP Min), Integer Programming is method of modeling a situation in which a decision has to be made to optimize some objective while being constrained by limited resources. The process for solving an **integer programming** problem is the same as other linear programming problems.

1. Formulate the problem.
    - Identify and define the decision variables.
    - Write the objective function.
    - Identify and write the functional constraints.

2. Enter the problem formulation into a spreadsheet.
    - Enter decision variables, objective function, and constraint coefficients.
    - Create formulas for objective function values and constraints' RHS.
    - Set up Solver Parameters and Options.
    - Add integer constraint for decision variable values.
      Note: Some integer programming problems are maximization and others are minimization problems.
    - Solve and generate Answer Report.

3. Interpret the results.
    - Answer Report shows status and amount of slack for constraints
    - Solver cannot create a Sensitivity Report for integer programming problems.

Terms

Integer Programming Integer programming is another example of mathematical programming. It has the added constraint that all decision variables can only take on integer values.

Lattice Point On a two-dimensional plane, a lattice point is any point whose x- and y-coordinates are both integers.

Feasible Region In integer programming problems, the feasible region is the set of lattice points that satisfy all the constraints.

Kernel The kernel is the set of lattice points in the feasible region of an integer programming problem that are candidates for the optimal solution, because they are closest to the boundary of the feasible region. For example, in a two-dimensional maximization problem, the kernel is the set of lattice points that have the largest x_1-coordinate for each x_2-value and the largest x_2-coordinate for each x_1-value.

Chapter 5 (IP) Objectives

You should be able to:

- Identify the objective of the problem
- Identify and define the decision variables
- Write the objective function
- Identify the limited resources involved in the problem
- Write the functional constraints as inequalities
- Enter the problem formulation into Excel
- Set up Solver Parameters and Options
- Interpret the optimal solution in the context of the problem
- Analyze the Answer Report

Chapter 5 Study Guide

1. What is the objective function?
2. What are decision variables?
3. What is different about the decision variables in an integer programming (IP) problem compared to a linear programming (LP) problem?
4. In an LP problem with just two decision variables, the feasible region is part of a plane bounded by lines representing the constraints. Each point in that region satisfies all the constraints and is thus a feasible solution. Compare and contrast the notion of the *feasible region* in an LP problem with that of an IP problem.
5. What are functional constraints?
6. Besides functional constraints, describe two other types of constraints found in IP problems.
7. What information is found in the Answer Report for an IP problem?
8. In an IP problem, is it possible for an integer constraint to be nonbinding? Explain why or why not.
9. To what can the “Final Value” on the Answer Report refer?
10. Define *slack*.
11. In the Answer Report, what information is in the “Cell Value” column?

Section 6.1: Jarvis Selects Projects

Jarvis is considering working on two short-term projects, one with a \$5,000 profit and the other with a \$7,000 profit. Project 1 requires 7 days, and project 2 requires 11 days. To formulate this problem, Jarvis needs to define some decision variables. A binary decision variable is an integer decision variable that has been further restricted to just two values, 0 and 1. Jarvis will use two **binary decision variables**, one for each project. A value of 1 means he works on the project, and a value of 0 means he does not.

The formulation is given below.

Decision Variables

Let: x_1 = the binary decision variable for project 1
x_2 = the binary decision variable for project 2
z = the amount of profit

Objective Function

Maximize: $z = 5{,}000x_1 + 7{,}000x_2$

Subject to:

Constraints

Time available: $7x_1 + 11x_2 \leq A$, where A is the number of days Jarvis is available for the projects

Binary Decision Variables: x_1 and x_2 equal 0 or 1

Q1. What are the four possible feasible solutions for this problem?

Q2. Why do these possibilities make sense?

We will consider several possible values for A. Suppose A = 5 days. Clearly, there is not enough time to complete either project, so the optimal solution is $x_1 = x_2 = 0$.

Q3. For what other values of A is this the optimal solution?

Q4. What is the smallest value of A that changes the optimal solution? What is the new optimal solution? What other values of A generate the same optimal solution?

Q5. What is the smallest value of A that causes the optimal solution to change again? What is the optimal solution for that value of A? What other values of A produce that same optimal solution?

Q6. What is the smallest value of A that changes the optimal solution again? What is that new optimal solution? What other values of A produce that same optimal solution?

Q7. Explain why the optimal solution cannot change again.

In standard *linear* programming, a nonbinding constraint has two ramifications. To see this, let's consider a less-than-or-equal constraint with a right-hand-side value of 10 days and a slack of 3 days.

In a *linear* programming example,

a. There would be no value in increasing this resource above 10 days (assuming it stays within the allowable increase for the constraint).

b. Reducing this resource by up to 3 days and eliminating the slack would also not impact the optimal solution.

As a result, linear programming reports a $0 shadow price of a nonbinding constraint. In *linear* programming the decision variables are continuous. Thus, if a limited resource were restricting the ability to improve on the optimal solution, it would have to be completely utilized and be binding.

With regard to Jarvis's decision, if *A* were 10, the optimal solution clearly would be to do project 1, as there is not enough time to do project 2. The constraint will not be binding: it will have three days of slack. However, there would be value in increasing the number of days available from 10 days to 11 days. If that were done, Jarvis could now complete project 2 and earn $7,000 instead of only $5,000. Perhaps he could reschedule his time to provide an extra day to work on the project. As in integer programming problems, Solver does not generate Sensitivity Reports for binary integer programming problems.

Q8. What is the marginal value of the extra day of work?

Section 6.2: Flipping Houses—A Detailed Example

Dream Homes, Inc. is a company that buys houses that are in need of repair. The houses are renovated, updated, and then sold for a profit. This process is referred to as *house flipping*. Larry Dale, a retired realtor, owns Dream Homes, Inc. and wants to provide a summer job for each of his five grandchildren, all of whom are in college. Mr. Dale would pay each grandchild by dividing the profits.

Each of Mr. Dale's five grandchildren—Ani, Benita, Cameron, Donte, and Edwin—has specific skills appropriate for house restoration. Ani is skilled in plumbing and carpentry, Benita in plumbing and landscaping, Cameron in painting and carpentry, Donte in painting and landscaping, and Edwin in landscaping and carpentry. All of the grandchildren are capable of cleaning.

Mr. Dale has identified ten available houses on the market. Each house has a potential profit margin. This profit margin is based on the difference between the cost of acquisition, which includes the purchase cost plus the cost of necessary materials plus the expense of outside contractor assistance, and the potential resale value of the house. The goal is to earn the highest possible total profit.

The grandchildren are all college students, so they have just 12 weeks in which to work. They are focused on the goal of acquiring the most lucrative houses so they can earn the most profit. The grandchildren first need to select which houses to flip in order to yield the highest profit. They have all agreed to work 8 hours a day, 6 days a week. Each house requires the appropriate skill set of each grandchild. Which house, or set of houses, will allow the grandchildren to earn the largest profit over the course of the summer?

6.2.1: Formulating the Flipping Houses Problem

Larry Dale decides the maximum amount that he can invest in this summer venture for his grandchildren's benefit is $500,000. This includes the costs to purchase the homes and the materials, and to pay all contractors' fees. Since he plans on investing in more than one house, the question becomes, which of the ten available houses should Larry invest in to maximize the profit for his grandchildren?

The formulation of this problem utilizes *binary decision variables*. A binary decision variable takes on the value one to indicate that a particular house will be purchased and the value zero to indicate that particular house will not be purchased.

Let the decision variables $x_1, x_2, x_3, \dots, x_{10}$ represent the binary values of house 1, house 2, house 3, … , house 10, and let $P_1, P_2, P_3, \dots, P_{10}$ represent the potential profit for house 1, house 2, house 3, … , house 10.

The objective function then is $z = \sum_{i=1}^{10} P_i \cdot x_i$.

We seek to maximize z subject to constraints related to total cost and available labor.

- There is a total cost constraint.
- There is a constraint on the number of hours of labor available for each type of work (e.g., plumbing).
- There is a constraint on the total number of hours of labor.

The total cost restriction takes into consideration the cost to purchase the home, the necessary material costs, and the costs to any outside contractors. This value cannot exceed $500,000. Therefore, we let:

H_i = the cost to purchase house *i*,
M_i = the cost of materials for house *i*, and
C_i = the contractors' costs for house *i*.

Then, the total cost constraint is:

$$\sum_{i=1}^{10}\left(H_i + M_i + C_i\right)\cdot x_i \le 500,000.$$

To better understand the total labor restriction constraint, we must analyze the availability and skills of each grandchild. Each grandchild has committed to working a maximum of 8 hours a day, 6 days a week, for 12 weeks during summer vacation. This amounts to each grandchild being able to work a maximum of 576 hours.

Recall that Ani is skilled at plumbing and carpentry, Benita is skilled at plumbing and landscaping, Cameron is skilled at painting and carpentry, Donte is skilled at painting and landscaping, and Edwin is skilled at landscaping and carpentry. Additionally, all the grandchildren are able to clean. Since there are only two grandchildren skilled at plumbing and each is available for a maximum of 576 hours, Mr. Dale must ensure that the required plumbing hours are less than 1,152 hours. Similarly, the total painting hours cannot exceed 1,152 hours. There are three grandchildren who are skilled at landscaping as well as carpentry, so he can assign a maximum of 1,728 hours to each of those trades. Since all the grandchildren are able to clean, he can allot a total of 2,880 hours to cleaning.

Given all of these factors, we let:

p_i = the number of plumbing hours required for house *i*,
q_i = the number of painting hours required for house *i*,
r_i = the number of landscaping hours required for house *i*,
s_i = the number of carpentry hours required for house *i*,
t_i = the number of cleaning hours required for house *i*.

Then, we have the following labor constraints:

$$\sum_{i=1}^{10} p_i \cdot x_i \le 1,152 \text{ (plumbing labor constraint),}$$

$$\sum_{i=1}^{10} q_i \cdot x_i \le 1,152 \text{ (painting labor constraint),}$$

$$\sum_{i=1}^{10} r_i \cdot x_i \le 1,728 \text{ (landscaping labor constraint), and}$$

$$\sum_{i=1}^{10} s_i \cdot x_i \le 1,728 \text{ (carpentry labor constraint).}$$

There is also an expression for the total amount of time available for cleaning:

$$\sum_{i=1}^{10} t_i \cdot x_i \text{ (cleaning time available).}$$

The total cleaning hours is not really a constraint, but it is included in the next constraint that restricts the total hours worked. Each of the five grandchildren can work a maximum of 576 hours, and thus the total amount of labor spread over all areas cannot exceed 576 · 5 = 2,880 hours. This constraint on the total available hours of labor can be expressed as:

$$\sum_{i=1}^{10}\left(p_i + q_i + r_i + s_i + t_i\right)\cdot x_i \le 2{,}880.$$

6.2.2: Solving the Problem

The problem formulation from the previous section is captured in the spreadsheet shown in Figure 6.2.1. Use the spreadsheet to answer the following questions.

Q1. What is the largest profit the grandchildren can make during the summer flipping houses?

Q2. Which houses will be purchased to produce this profit?

Q3. How much money will each person receive if the profit is divided equally among the grandchildren?

The optimal solution identifies two houses to be purchased and flipped to maximize profits. One of these houses has the largest margin for profit. But the second home in the optimal solution is fifth out of the possible ten homes when ranked based on profit margin.

Q4. Why do you suspect this has happened?

In binary programming, decision variables change in units of one. That means they consume resources in discrete increments. To explore this phenomenon, open the Excel file and change the values of two decision variables. Change the one under House 1 to zero, and place a one under House 7, the second most profitable house. House 1 requires a total of 1,162 hours, and House 7 requires 1,570, an increase of 408 hours. Notice the total hours worked now exceeds the total hours available. That is why House 7 is not included in the optimal solution.

As a result we have identified the fact that the total available hours is restricting the group's ability to make a profit, even though the constraint is not binding. We wish to explore this restriction to see whether or not it might be advantageous to hire a cleaning crew.

Q5. To explore this issue, increase the available hours in 100-hour increments, up to a maximum of 800 more hours.

Q6. Determine in each case whether or not the optimal solution changes.

Q7. If it does, what is the new optimal set of houses to purchase?

Q8. Does the optimal solution ever include more than two houses?

Q9. What is the marginal increase in revenue?

Q10. How much per hour would you pay the cleaning company?

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	House Flipping	Binary Problem With Budget Restriction												
2	Maximization Problem													
3		House 1	House 2	House 3	House 4	House 5	House 6	House 7	House 8	House 9	House 10			
4	Total Value	$125,000	$135,000	$178,000	$110,000	$108,000	$124,000	$244,000	$192,000	$130,000	$275,000			
5														
6	Cost (H_i)	$65,000	$100,000	$125,000	$70,000	$35,000	$99,000	$140,000	$115,000	$88,000	$129,000			
7	Material Costs (M_i)	$5,000	$3,000	$3,750	$5,500	$7,500	$2,000	$8,000	$6,000	$7,000	$16,000			
8	Contractors Cost (C_i)	$6,000	$3,000	$8,000	$1,000	$17,500	$0	$10,000	$9,300	$0	$22,000			Budget
9	Total Cost	$76,000	$106,000	$136,750	$76,500	$60,000	$101,000	$158,000	$130,300	$95,000	$167,000	$243,000	<=	$500,000
10	**Net Profit**	**$49,000**	**$29,000**	**$41,250**	**$33,500**	**$48,000**	**$23,000**	**$86,000**	**$61,700**	**$35,000**	**$108,000**			
11														
12	Decision Variable	H1	H2	H3	H4	H5	H6	H7	H8	H9	H10			
13	Dec. Var. Values	1	0	0	0	0	0	0	0	0	1			
14	Objective Function	**$157,000**												
15	Labor (400-800/House)													Available Hours
16	Plumbing	238	211	264	145	211	100	400	422	185	304	541	<=	1152
17	Painting	150	125	115	80	130	50	250	160	100	200	350	<=	1152
18	Landscaping	210	175	161	112	182	70	350	224	140	280	490	<=	1728
19	Carpentry	264	330	143	242	165	180	220	385	330	396	660	<=	1728
20	Cleaning/Help	300	280	310	410	335	200	350	325	390	325	625		
21	**Total Labor Hours**	**1162**	**1121**	**993**	**989**	**1023**	**600**	**1570**	**1516**	**1145**	**1505**	2666	<=	2880
22														
23	12	weeks			Plumbing	Painting	Landscape	Carpentry	Cleaning	Total Hrs				
24	6	Days/week		a	1			1	1	576				
25	8	hours		b	1		1		1	576				
26	5	Students		c		1		1	1	576				
27				d		1	1		1	576				
28				e			1	1	1	576				
29					1152	1152	1728	1728	2880	2880				

Figure 6.2.1: A flipping houses spreadsheet with the optimal solution

Section 6.3: Sam Johnson Makes a Hard Decision

Sam Johnson lives in Brooklyn, New York. He is deciding which colleges to apply to this fall. He has taken the SAT, and his total score is 1650. He wants to find colleges that are more than 50 miles from home, but no more than 300 miles from home. He also wants to find colleges with acceptance rates of at least 50%. Sam has lived in a large city all of his life and enjoys life in the city, so he would like to find a college in or near an urban setting. He also prefers a medium-sized school.

Sam decides to do some Internet searches to learn more about potential colleges that satisfy his criteria. Along with the web sites for individual school, he finds these resources useful:

- http://collegesearch.collegeboard.com/search/index.jsp
- http://www.uscollegesearch.org/
- http://www.collegeview.com/collegesearch/index.jsp
- http://cnsearch.collegenet.com/cgi-bin/CN/index

Table 6.3.1 contains Sam's short list of 11 schools that fit his criteria. His problem is that his mom and dad will pay the application fee for only five colleges at most. How should Sam decide where to apply?

School	Location	Distance from Home (miles)	Tuition ($)	Average Debt at Graduation ($)	Acceptance Rate (%)	Enrollment (# of students)
Western Connecticut State University	Danbury, CT	55	15,344	NR	57	6,001
Drexel University	Philadelphia, PA	97	28,780	NR	76	20,821
Rutgers: Camden Regional Campus	Camden, NJ	80	18,263	18,645	53	49,760
College of Saint Rose	Albany, NY	141	20,620	24,732	68	5,000
Johnson & Wales University	Providence, RI	155	21,717	19,890	81	16,095
Worcester State College	Worcester, MA	157	11,619	13,742	56	5,470
Loyola College in Maryland	Baltimore, MD	169	34,250	16,073	64	6,131
University of Massachusetts Boston	Boston, MA	190	19,977	14,805	63	13,433
Suffolk University	Boston, MA	190	24,250	NR	69	5,196
University of Massachusetts Lowell	Lowell, MA	194	19,714	14,833	70	11,635
Syracuse University	Syracuse, NY	197	31,686	24,000	51	19,082

Table 6.3.1: Sam's list of 11 colleges meeting his criteria with their data

6.3.1: Understanding the Problem

Sam decides that the most important criteria to him are size, distance from home, acceptance rate, tuition, and average debt at graduation. However, he is not able to learn the average debt at graduation for several of the colleges on his list, so he decides not to use that criterion in his selection process.

Now Sam wants to find the colleges to apply to based on their acceptance rates and an attractiveness score he will determine. He will calculate the attractiveness of each college based on its distance from his home and its tuition, as well as some personal preferences based on what he has learned from his Internet searches about the quality of life on each campus.

To calculate the attractiveness score for each college, Sam will rely on some techniques used in the Multi-Criteria Decision Making process from Chapter 1. He will rescale scores by assigning a score between zero and one for each criterion. Then he will weight scores by multiplying the criterion scores by their assigned weight. Finally he will add these three weighted scores to obtain the attractiveness score for each school. He wants to give equal weight to the objective criteria (distance from home, tuition, and size) and a slightly higher weight to his personal preferences. He decides to weight distance, tuition, and size at 20% each and his personal preferences at 40%. Table 6.3.2 shows how he will assign the scores for distance and tuition.

Tuition (weight = 0.2)		Distance (weight = 0.2)		Size (weight = 0.2)	
Range ($)	Score	Range (miles)	Score	Range (students)	Score
< 10,000	1	50 – 90	0	< 5,000	0
10,000 – 14,999	0.67	91 – 130	0.33	5,000 – 9,999	0.8
15,000 – 19,999	0.33	131 – 170	0.67	10,000 – 19,999	1
≥ 20,000	0	≥ 171	1	≥ 20,000	0.2

Table 6.3.2: Sam's scheme for assigning scores

The attractiveness score for each school will be computed as follows:

(0.2)(distance score) + (0.2)(tuition score) + (0.2)(size score) + (0.4)(personal preference score).

Finally, Sam wants to be fairly certain that he gets accepted into at least two of the schools to which he is applying. To ensure this happens, he introduces a constraint that will force at least two of the schools in his portfolio will have acceptance rates of at least 70%. It is possible (although we will not do it here) for Sam to set up additional constraints to ensure that he is applying to a wide range of schools when considering any criterion.

The idea of finding a solution set that balances a number of variables is called **portfolio planning**. The most common example of this is planning a stock portfolio. In financial investment decisions, portfolio planning is designed to balance risks against rewards. A balanced portfolio will contain investments from a wide range of industries and geographic regions. It will include high-risk investment with high potential for profit as well as low-risk investment with smaller profit returns.

For his college application portfolio, Sam is seeking balance in acceptance rates. He wants to apply to schools where he will be challenged, so he has a constraint requiring the average of all the acceptance rates at the schools where he will apply to be at least 60%. However he also wants to be confident that he will get accepted to at least two schools, so he creates another constraint requiring at least two schools to have acceptance rates of at least 70%.

In a similar way, Sam could balance his preferences for distance from home, size, and cost of the colleges in his portfolio.

Below are the constraints on his portfolio of colleges that he decides to use:

- At least *two* colleges with acceptance rates of at least 70% should be selected.
- The average acceptance rate of all the selected colleges should be at least 60%.
- He will apply to exactly *five* colleges.

6.3.2: Formulating the Problem

To formulate this problem, Sam must define some decision variables. He decides to use the ones listed in Table 6.3.3. Notice that the table contains *binary decision variables* and one **binary indicator coefficient**.

A binary decision variable is an integer decision variable that has been further restricted to just two values, zero and one. Sam will use 11 binary decision variables, one for each school he is considering. Sam's problem formulation will maximize his objective function, subject to constraints. When the problem is solved, the binary decision variables for those colleges that will appear in Sam's portfolio will be assigned a value of one. The binary decision variables for those colleges that are not in Sam's portfolio will be assigned a value of zero.

A binary indicator coefficient is similarly restricted to two values, zero or one. Think about the process of checking each of the 11 colleges in Sam's list against the list of the three constraints appearing at the end of the previous section. For each college, its data must be checked to see if it meets each of the constraints.

For example, the first constraint is that at least two of the colleges selected have acceptance rates greater than 70%. So, for each college, you must ask the question, "Is the acceptance rate greater than 70%?" This is a yes-or-no question, and this is where binary indicator coefficients come into play. If the answer is yes, the binary indicator will be assigned a value of one, but if the answer is no, the binary indicator will be assigned a value of zero.

In Sam's formulation, ${}_bR_i$ represents whether school i satisfies the first constraint (i.e., whether its acceptance rate is greater than 70%). Referring to Table 6.3.1, the first school on the list is Western Connecticut State University, and its acceptance rate (A_1) is 57%. So it does not meet the first constraint. Therefore, the binary indicator is assigned the value zero: ${}_bR_1 = 0$. Similarly, ${}_bR_2 = 0$ because Rutgers's acceptance rate is also less than 70%. However, the third school in the list, Drexel University, has an acceptance rate of 76%. Therefore, ${}_bR_3 = 1$.

Variable	Description
X_i	Binary decision variable for school i, i = 0, 1, 2, … , 11
R_i	Acceptance rate for school i
S_i	Size of school i; 1 = small, 2 = medium, 3 = large
T_i	Tuition ($) for school i
H_i	Distance (miles) from home to school i
A_i	Attractiveness score for school i
${}_bR_i$	The binary indicator for R_i; 1 if the acceptance rate of school i satisfies the restriction and 0 if it does not

Table 6.3.3: The variables and indicator Sam will use

To complete the formulation, Sam decides to define an objective function that takes into account the acceptance rates and attractiveness scores (which are based on tuition, distance from home, size, and personal preferences) for the selected schools.

Maximize:

$$z = \sum \left(X_i \cdot R_i \cdot A_i \right).$$

This objective function sums 11 products, one for each school. Each product consists of three factors: the binary decision variable X_i representing whether a particular school is selected, that school's acceptance rate R_i, and its attractiveness score A_i. The summation of these products represents the total value of Sam's portfolio. Since exactly five schools will be selected, six of the values of X_i will be zero, and those resulting products will also be zero. The sum of the products $R_i \cdot A_i$ needs to be maximized to determine which schools will be selected.

Finally, Sam's three constraints will be:

$\sum \left(X_i \cdot {}_bR_i \right) \geq 2$ because at least *two* schools with acceptance rates greater than 70% should be selected,

$\sum X_i = 5$ because he will apply to exactly *five* schools, and

$\frac{\sum X_i \cdot R_i}{5} \geq 0.6$ because the average acceptance rate of the selected schools should be greater than 60%.

In the first constraint, the binary indicator coefficient (${}_bR_i$) for each school is multiplied by the binary decision variable X_i for that school. Since both factors in this constraint are binary, every one of the products also binary. If the particular school does not meet the constraint, then the left hand side of the constraint is zero. If the school does meet the constraint, then it will be one. Therefore, the sum of those 11 products is the number of schools in the portfolio that meet the particular constraint (e.g., acceptance rate greater than 70%).

In the second constraint, the sum of all of the binary decision variables X_i is just the number of schools in the portfolio. In the last constraint, the average acceptance rate is the sum of the acceptance rates at the selected colleges ($\sum X_i \cdot R_i$) divided by the number of colleges selected for the portfolio.

6.3.3: Solving the Problem

The process of checking data against constraints and assigning values to the binary variables would be very tedious and time-consuming if done by hand. Fortunately spreadsheet solvers allow the user to define variables as binary, just as they allow the definition of integer variables. Figure 6.3.4 shows Sam's spreadsheet formulation after he has set up and solved the problem.

Choosing a College **Binary Problem**

Variable	X_i	T_i	E_i				A_i		R_i	$_bR_i$			
School	Distance from home (miles)	Tuition ($)	Size (# of students)	Tuition score (20%)	Distance score (20%)	Size score (20%)	Attractive score (40%)	Total preference score	Acceptance rate (%)	Acceptance rate indicator (1 if >70%, 0 otherwise)	Objective function coefficient (acceptance rate * total preference score)	Decision variable value	Objective function value
Western Connectic	58	15,344	6,001	0.33	0	0.8	0.14	0.28	0.57	0	0.161	0	0
Rutgers Camden	78	18,263	49,760	0.33	0	0.2	0.61	0.35	0.53	0	0.186	0	0
Drexel U	97	28,780	20,821	0	0.33	0.2	0.71	0.39	0.76	1	0.296	0	0
College of Saint Ro	141	20,620	5,000	0	0.67	0.8	0.63	0.55	0.68	0	0.372	1	0.37
Johnson & Wales U	155	21,717	16,095	0	0.67	1	0.33	0.46	0.81	1	0.376	1	0.38
Worcester State C	157	11,619	5,470	0.67	0.67	0.8	0.44	0.6	0.56	0	0.338	0	0
Loyola U in MD	169	34,250	6,131	0	0.67	0.8	0.4	0.45	0.64	0	0.291	0	0
U Mass Boston	189	19,977	13,433	0.33	1	1	0.81	0.79	0.63	0	0.499	1	0.5
Syracuse U	197	31,686	19,082	0	1	0.2	0.6	0.48	0.51	0	0.245	0	0
Suffolk University	191	24,250	5,196	0	1	0.8	0.72	0.65	0.69	0	0.448	1	0.45
U Mass Lowell	194	19,714	11,635	0.33	1	1	0.53	0.68	0.7	1	0.474	1	0.47

Constraint	LHS	Test	RHS
Number of schools	5	=	5
Number whose acceptance rate is >70%	2	≥	2
Average acceptance rate	0.7	≥	0.6

Objective Function
2.17

Figure 6.3.4: Sam's spreadsheet showing the optimal portfolio of schools to which to apply

Use Sam's solver spreadsheet to answer the following questions.

Q1. A coefficient for the objective function was computed for each college. Which columns in the spreadsheet were used to do this?

Q2. Why are most of the values in the column labeled "Objective function value" 0.00?

Q3. Which schools are included in Sam's portfolio?

Q4. Did the five selected schools have the largest objective function coefficients? Why/why not?

Q5. Which of the selected schools met the acceptance rate constraint?

Q6. What is the average (mean) acceptance rate of the five selected colleges?

Q7. What is the range of the distance from home of the five selected colleges?

Q8. What is the median tuition of the five selected colleges?

Q9. What is the average (mean) size of the five selected colleges?

Section 6.5: Chapter 6 (Binary Programming) Homework Questions

1. The City Council in Monroe, Michigan is considering four proposed new recreational facilities: a swimming pool, a tennis center, athletic fields (football/soccer, baseball/softball), and a gymnasium. The Council wants to construct the facilities that will maximize the expected daily use, but there are budgetary and land restrictions. The expected daily use, cost, and land requirements for each of the proposed facilities are given in Table 1.

Facility	Expected Use (people/day)	Cost ($)	Land Required (acres)
Swimming pool	500	350,000	4
Tennis center	150	100,000	2
Athletic fields	750	250,000	7
Gymnasium	400	500,000	3

Table 6.4.1: Information on proposed recreational facilities

The Council has budgeted $900,000 for the construction of new recreational facilities. There are 12 acres of land available, but only one site that is adequate for the swimming pool or gymnasium. Thus, only one or the other of these two facilities can be built.

a) Define a set of binary decision variables for this problem.
b) Use the decision variables you defined to define the objective function.
c) Formulate the constraints in terms of the decision variables.
d) Enter the problem formulation into a spreadsheet.
e) Use solver to obtain the optimal solution.
f) Which of the proposed facilities should the Council build?

2. The Research Triangle Electronics Company is considering eight new research and development projects. The company cannot conduct all eight projects, due to limitations on their R & D budget and the number of research scientists available. Table 2 contains the resource requirements and estimated profit for each of the projects. In addition, not more than two of projects 4, 5, and 6 can be undertaken, because they require many of the same research scientists. Which projects should be selected in order to maximize estimated profit?

Project	Cost ($1,000s)	Research Scientists Required	Estimated Profit (Millions of $)
1	650	7	8.2
2	1,200	6	9.5
3	350	8	3.7
4	450	9	1.1
5	1,000	10	2.3
6	850	8	2.2
7	750	7	8.2
8	700	4	5.8

Table 6.4.2: Information about eight possible research projects

3. TopTen Recording Studios is considering funding recording projects with four different artists over the next three years. The studio has allocated $12 million per year to cover the expenses of the new projects. The artists, necessary expenditures per year, and the expected profit from their projects are given in Table 3. Which projects should be selected to maximize the expected total profit?

Artist	Expenditures (Millions of $/Year)			Profit
	Year 1	Year 2	Year 3	(Millions of $)
Rambling Lou	3	2	6	32
Rita Rivera	5	5	1	19
Nightrider	6	5	10	33
SoozieQT	3	8	1	23
Available Funds/Year	12	12	12	

Table 6.4.3: Expenditures per year and expected profit from 4 recording projects

a) Define a set of binary decision variables to fit this problem.
b) Use your decision variables to define the objective function.
c) Formulate the annual budget constraints.
d) Use a spreadsheet solver to obtain the optimal solution.
e) Which artists' projects should Top Ten Recording select?

Chapter 6 Summary

What have we learned?

This is the last in a series of four chapters on mathematical programming. As in the previous three chapters (LP Max, LP Min, and IP), Binary Integer Programming (BIP) is method of modeling a situation in which a decision has to be made to optimize some objective while being constrained by limited resources.

The process for solving a binary integer programming problem is the same as other linear programming problems.

1. Formulate the problem.
    - Identify and define the decision variables.
    - Write the objective function.
    - Identify and write the functional constraints.

2. Enter the problem formulation into a spreadsheet.
    - Enter decision variables, objective function, and constraint coefficients.
    - Create formulas for objective function values and constraints' RHS.
    - Set up Solver Parameters and Options.
    - Add binary constraint for decision variable values.
      Note: Some integer programming problems are maximization and others are minimization problems.
    - Solve and generate Answer Report.

3. Interpret the results.
    - Answer Report shows status and amount of slack for constraints
    - Solver cannot create a Sensitivity Report for binary integer programming problems.

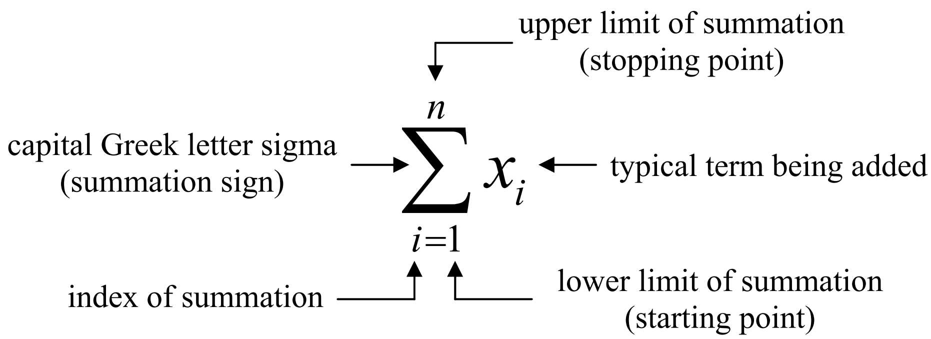
Terms

Binary Decision Variable A decision variable that can take on only two possible values, zero or one.

Binary Indicator Coefficient A coefficient that takes the value of one if the quantity meets a given condition or zero if it does not.

Portfolio Planning The process of ensuring that a solution set balances a number of criteria.

Summation Notation



$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n$$

Chapter 6 (Binary Programming) Objectives

You should be able to:

- Identify the objective of the problem
- Identify and define the decision variables
- Write the objective function, including using summation notation
- Identify the limited resources involved in the problem
- Write the functional constraints as inequalities, including using summation notation
- Use binary indicator coefficients in writing constraints
- Enter the problem formulation into Excel
- Set up Solver Parameters and Options
- Interpret the optimal solution in the context of the problem
- Analyze the Answer Report

Chapter 6 Study Guide

1. What is the objective function?

2. What are decision variables?

3. What is different about the decision variables in a binary integer programming (BIP) problem compared to an integer programming (IP) problem?

4. What are functional constraints?

5. Besides functional constraints, describe two other types of constraints found in BIP problems.

6. Consider a BIP problem whose decision variables are of the form x_i, and one of the functional constraints is $\sum_{i=1}^{12} x_i \leq 7$.
   a. Explain who to determine the number of decision variables in the problem?

   b. Interpret the meaning of the constraint.

7. What information is found in the Answer Report for a BIP problem?

8. In a BIP problem, is it possible for a binary constraint to be nonbinding? Explain why or why not.

9. To what can the “Final Value” on the Answer Report refer?

10. Define *slack*.

11. In the Answer Report, what information is in the “Cell Value” column?

Section 7.1: Coach Bass's Problem

Bill Bass, coach of the Jefferson High School girls' swimming team, has four swimmers he always assigns to compete in the 400-yard medley relay. In this event, there are four 100-yard legs that each must be swum by a different competitor using a different stroke.

Q1. What are the four different strokes in the medley relay?

Coach Bass knows the best times for each of his swimmers for each leg of the relay. He wonders what way of assigning the four swimmers to the four legs of the relay would be the best, based on their best times.

Q2. Other than using best times, what other criteria could Coach Bass use to assign the swimmers?

7.1.1 Assignment Problems

Coach Bass's problem is an example of an **assignment problem**. An assignment problem arises whenever a one type of thing, such as a person, must be assigned to another type of thing, such as a task. For example, Coach Bass has to assign four swimmers to the four legs of the medley relay. To solve an assignment problem, a **"cost" matrix**:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & c_{m3} & \cdots & c_{mn} \end{bmatrix}$$

is usually defined based on the parameters of the given problem. For example, if m workers must be assigned to n tasks, c_{ij} could be the time needed by worker i to perform task j.

Q3. In the case of the medley relay team, what would c_{ij} represent?

Q4. What is Coach Bass's objective in the medley relay problem?

Q5. In terms of the cost matrix, what must Coach Bass do?

Q6. Can you think of anything in an actual situation similar to this one that would complicate the problem?

7.1.2 Formulating Coach Bass's Problem

Coach Bass must assign each of the four swimmers to one of the four 100-yard legs in the 400-yard medley relay: butterfly, backstroke, breaststroke, and freestyle. Coach Bass has decided to use the best times for each of his swimmers for each of the legs of the relay to determine which swimmer to use for which leg of the relay. He wants to assign the four swimmers so as to complete the four legs of the relay in the minimum total time. He wonders what assignment would be the best, based on their best times. Table 7.1.1 contains those best times for the four swimmers.

(seconds)		Event			
		100-yd fly	100-yd back	100-yd breast	100-yd free
Swimmer	Schmidt	59.59	59.83	72.83	52.61

	Reid	60.45	59.56	74.14	53.31
	Sanchez	61.84	64.63	73.69	53.70
	Lamartina	62.37	59.13	74.36	54.77

Table 7.1.1: Best times for four swimmers in the medley relay legs

Q7. Coach Boss first considered using the best swimmer in each event. Why does this strategy not work?

We can set up a spreadsheet and use Solver to tackle this problem. First, we need an LP formulation of the problem. To do that, we will use a binary decision variable to represent the possibility that each of the four swimmers could be assigned to any one of the four legs. So, for $1 \le i \le 4$ and $1 \le j \le 4$, let x_{ij} = the binary decision variable representing whether swimmer i is assigned to leg j.

Q8. How many binary decision variables are there in this problem formulation?

Next, we need to define an objective function in terms of the decision variables. Coach Bass's objective is to minimize the time for his relay team to swim the event. Thus, for each of the four swimmers, we will need her best time in each of the four legs. So, for $1 \le i \le 4$ and $1 \le j \le 4$, let t_{ij} = the best time for swimmer i in leg j.

Q9. How are all of the t_{ij} values different from all of the x_{ij} values?

Next, using x_{ij} and t_{ij} we will represent the total time for the medley relay event. For each possible pair of values for i and j, we will multiply x_{ij} by t_{ij}. Most of the time, x_{ij} will be equal to zero and the corresponding product will be zero. There are 16 x_{ij} but only four of the x_{ij} values are going to equal 1. This represents the four swimmers who are assigned to swim the four legs of the relay. In those cases, the product will be the best time of the swimmer for the leg she is assigned to swim. Finally, adding all of the products, $x_{ij} \cdot t_{ij}$, gives us the total of those four best times. Thus, our objective function, which we would like to minimize, is

$$z = \sum_{i=1}^{4} \left(\sum_{j=1}^{4} x_{ij} \cdot t_{ij} \right).$$

Finally, we need to add the constraints to the formulation. The constraints apply to the assignment of swimmers and can be stated simply as:

- Every leg of the relay must have exactly one swimmer assigned to it.
- Every swimmer must be assigned to exactly one leg of the relay.

Because each leg must have exactly one swimmer assigned to it, there are four different leg constraints.

For j = 1, 2, 3, and 4, $\sum_{i=1}^{4} x_{ij} = 1$.

Because every swimmer must be assigned to exactly one event, there are also four different swimmer constraints.

For i = 1, 2, 3, and 4, $\sum_{j=1}^{4} x_{ij} = 1$.

Putting the various parts of the formulation all together, we have the following.

Decision Variables
For $1 \le i \le 4$ and $1 \le j \le 4$,
Let: i refer to swimmers and j refer to legs
x_{ij} = the binary decision variable showing whether swimmer i is assigned to leg j
t_{ij} = the best time for swimmer i in leg j
z = the total of the best times of the four swimmers assigned to swim the relay

Objective Function

Maximize: $$z = \sum_{i=1}^{4}\left(\sum_{j=1}^{4} x_{ij} \cdot t_{ij}\right)$$

Subject to:
Constraints

for j = 1, 2, 3, and 4, $$\sum_{i=1}^{4} x_{ij} = 1$$

for i = 1, 2, 3, and 4, $$\sum_{j=1}^{4} x_{ij} = 1$$

7.1.3 Putting the Formulation into a Spreadsheet

Figure 7.1.1 shows a spreadsheet formulation of the medley relay team problem. Notice that the spreadsheet contains two matrices. One matrix has the best times in each leg for each swimmer (t_{ij}). Another matrix represents the decision variables. It records the assignment of swimmers to legs (x_{ij}).

Notice also that in the best times matrix, all of the times that were one minute or more have been converted to seconds. For example, 1:04.63 was converted to 64.63. At the beginning of the solution process, no swimmer has yet been assigned to any leg of the relay. Thus, each cell in this assignment matrix contains a zero.

The numbers to the right of the assignment matrix are the row sums. They represent the four swimmer constraints. Since each swimmer must be assigned to swim exactly one leg, each swimmer's row of decision variables must contain one 1 and three 0s. Therefore, in a valid assignment, each row sum must equal 1. Similarly, the numbers below the assignment matrix are the column sums. They represent the four leg constraints. Each leg must have exactly one swimmer assigned to it. As a result in a valid assignment, each column will contain one 1 and three 0s. Thus, each column sum must also equal 1. The only other constraint is that each of the decision variables must be binary.

The objective function is the sum of the best times of the four swimmers who are assigned to the medley relay. It is computed in the cell directly following the Best Times Matrix by multiplying each cell in the Assignment Matrix by the corresponding cell of the Best Times Matrix and adding all of the products. For example, the cell containing Schmidt's best time in the fly (59.59) is multiplied by the cell in the Assignment Matrix opposite Schmidt's name and below "fly". This can easily be done using the Sum Product function of your spreadsheet. The current value of the objective function is 0, because no swimmer has yet been assigned to swim any leg of the medley relay. At this point, we do not have a valid assignment. None of the rows or columns sum to 1.

	A	B	C	D	E	F	G	H	I
1	Chapter 7: Assignments								
2	7.1 Medley Relay Team								
3	Relay Time Minimization								
4									
5	**Swimmers' Times**								
6			Butterfly (sec)	Backstroke (sec)	Breaststroke (sec)	Freestyle (sec)			
7		Schmidt	59.59	59.83	72.83	52.61			
8		Reid	60.45	59.56	74.14	53.31			
9		Sanchez	61.84	64.63	73.69	53.7			
10		Lamartina	62.37	59.13	74.36	54.77		**Total time (sec)**	
11									
12	**Assignments**								
13			Butterfly	Backstroke	Breaststroke	Freestyle			
14		Schmidt					0	≤	1
15		Reid					0	≤	1
16		Sanchez					0	≤	1
17		Lamartina					0	≤	1
18			0	0	0	0			
19			=	=	=	=			
20			1	1	1	1			

Figure 7.1.1: Spreadsheet formulation of the Medley Relay Problem

7.1.4 Using Solver to Solve Coach Bass's Problem

As in previous chapters, the spreadsheet solver must be set up to solve the problem by:

- specifying the cell in which the objective function is displayed,
- specifying the cells containing the values of the decision variables,
- entering all of the constraints, including the binary constraint on the decision variables, and
- choosing "Assume linear model" and "Assume non-negativity" from the solver options.

Figure 7.1.2 shows the spreadsheet after the Solver has found a solution.

	A	B	C	D	E	F	G	H	I
1	Chapter 7: Assignments								
2	7.1 Medley Relay Team								
3	Relay Time Minimization								
4									
5	**Swimmers' Times**								
6			Butterfly (sec)	Backstroke (sec)	Breaststroke (sec)	Freestyle (sec)			
7		Schmidt	59.59	59.83	72.83	52.61			
8		Reid	60.45	59.56	74.14	53.31			
9		Sanchez	61.84	64.63	73.69	53.7			
10		Lamartina	62.37	59.13	74.36	54.77		**Total time (sec)**	
11								245.72	
12	**Assignments**								
13			Butterfly	Backstroke	Breaststroke	Freestyle			
14		Schmidt	1	0	0	0	1	≤	1
15		Reid	0	0	0	1	1	≤	1
16		Sanchez	0	0	1	0	1	≤	1
17		Lamartina	0	1	0	0	1	≤	1
18			1	1	1	1			
19			=	=	=	=			
20			1	1	1	1			

Figure 7.1.2: Spreadsheet showing a solution to Coach Bass's Problem

Q10. What is the optimal assignment of swimmers to legs of the medley relay?

Q11. In this optimal assignment, which swimmers were the fastest for the swim strokes they were assigned? Which swimmers were not the fastest for the strokes assigned to them? Explain why this can occur in the optimal solution.

Q12. What is the minimum of the total of the best times for each of the swimmers in the legs to which they have been assigned?

7.1.5 A Complication: More Swimmers

In the previous section, we saw how to solve a "balanced" assignment problem. Coach Bass's problem was balanced. He had the same number of swimmers to assign as he had legs of the relay. What if the problem were out of balance? For example, suppose that Coach Bass has a large and very competitive swim team. There are eight girls in competition for the four legs of the medley relay. Table 7.1.2 contains the best times for each of the eight girls in each of the four events.

		Leg			
		100-yd fly	100-yd back	100-yd breast	100-yd free
Swimmer	Schmidt	59.59	59.10	72.83	52.61
	Reid	60.45	59.56	74.14	53.31
	Sanchez	61.84	64.63	73.69	53.7
	Lamartina	62.37	59.13	74.36	54.77
	Wu	60.33	64.30	72.74	54.05
	Greene	62.41	59.03	72.19	56.61
	Kleinfeld	62.43	67.63	74.05	55.55
	Lepinski	59.44	65.06	72.74	52.49

Table 7.1.2: Best times for eight swimmers in the medley relay legs

Much as we did in the first problem, we can set up a spreadsheet solver to tackle the problem. First, we need an LP formulation of the problem. We will use a binary decision variable to represent the possibility that each of the eight swimmers could be assigned to any one of the four legs. So, for $1 \leq i \leq 8$ and $1 \leq j \leq 4$, let x_{ij} = the binary decision variable representing whether swimmer i is assigned to leg j.

Q13. How many binary decision variables are there in this problem formulation?

Next, we need to define an objective function in terms of the decision variables. Coach Bass's objective is still to minimize the time for his relay team to swim the event. Thus, for each of the eight swimmers, we will need the best time in each of the four legs. So, we let t_{ij} = the best time for swimmer i in leg j, for $1 \leq i \leq 8$ and $1 \leq j \leq 4$.

Next, using x_{ij} and t_{ij} we will represent the total time for the medley relay event. For each pair of values for i and j, we will multiply x_{ij} and t_{ij}. Finally, adding all of the products, ($x_{ij} \cdot t_{ij}$), gives us the total of those four best times. Thus, our objective function is

$$z = \sum_{i=1}^{8} \left(\sum_{j=1}^{4} x_{ij} \cdot t_{ij} \right).$$

Finally, we need to add the constraints to the formulation. One set of constraints is the same as before: Every leg must have exactly one swimmer assigned to it.

However, the other set of constraints is a bit different. It is no longer true that every swimmer must be assigned to exactly one leg, because four of the swimmers will not be assigned to any leg. Thus, the second set of constraints is: No swimmer can be assigned to more than one leg.

Because each leg must have exactly one swimmer assigned to it, there are four different leg constraints.

For $j = 1, 2, 3$, and 4, $\sum_{i=1}^{8} x_{ij} = 1$.

However, because no swimmer can be assigned to more than one event, there are also eight swimmer constraints.

For $i = 1, 2, 3, 4, 5, 6, 7$, and 8, $\sum_{j=1}^{4} x_{ij} \leq 1$.

Putting the various parts of the formulation all together, we have the following:

Decision Variables

For $1 \leq i \leq 8$ and $1 \leq j \leq 4$,

Let: i refer to swimmers and j refer to legs

x_{ij} = the binary decision variable showing whether swimmer i is assigned to leg j

t_{ij} = the best time for swimmer i in leg j

z = the total of the best times of the four swimmers assigned to swim the relay

Objective Function

Maximize: $z = \sum_{i=1}^{8} \left(\sum_{j=1}^{4} x_{ij} \cdot t_{ij} \right)$

Subject to:

<u>Constraints</u>

for j = 1, 2, 3, and 4, $\sum_{i=1}^{8} x_{ij} = 1$

for i = 1, 2, 3, 4, 5, 6, 7, and 8, $\sum_{j=1}^{4} x_{ij} \leq 1$

Figure 7.1.3 shows a spreadsheet formulation of the medley relay team problem. Notice that, just as before, the spreadsheet contains two matrices. One matrix has the best times for each event for each swimmer (t_{ij}). Another matrix shows the decision variables, the assignment of swimmers to events (x_{ij}). Each cell in the assignment matrix contains a zero because at the beginning of the solution process, no swimmer has been assigned to any event.

Q14. If Sanchez is assigned to swim the freestyle leg, what value will appear in the cell under "free" and next to "Sanchez"?

Q15. What value will appear in every other cell in that row of the assignment matrix?

Q16. What value will appear in every other cell in that column of the assignment matrix?

The objective is to minimize the total of the best times of the four swimmers who will be assigned to the medley relay. The objective function has been stored in the cell labeled "Total Time."

Q17. Why is there a zero in the cell labeled "Total Time" in the spreadsheet depicted in Figure 7.1.3?

	A	B	C	D	E	F	G	H	I
1	Chapter 7: Assignments								
2	7.1 Medley Relay Team								
3	Relay Time Minimization								
4									
5	**Swimmers' Times**								
6			Butterfly (sec)	Backstroke (sec)	Breaststroke (sec)	Freestyle (sec)			
7		Schmidt	59.59	59.83	72.83	52.61			
8		Reid	60.45	59.56	74.14	53.31			
9		Sanchez	61.84	64.63	73.69	53.7			
10		Lamartina	62.37	59.13	74.36	54.77			
11		Wu	60.33	64.3	72.74	54.05			
12		Greene	62.41	64.34	66.19	56.61			
13		Kleinfeld	62.43	67.63	74.05	55.55			
14		Lepinski	59.44	65.06	72.74	52.49		**Total Time (sec)**	
15								0	
16	**Assignments**								
17			Butterfly	Backstroke	Breaststroke	Freestyle			
18		Schmidt					0	≤	1
19		Reid					0	≤	1
20		Sanchez					0	≤	1
21		Lamartina					0	≤	1
22		Wu					0	≤	1
23		Greene					0	≤	1
24		Kleinfeld					0	≤	1
25		Lepinski					0	≤	1
26			0	0	0	0			
27			=	=	=	=			
28			1	1	1	1			

Figure 7.1.3: Spreadsheet formulation of the medley relay team with eight swimmers.

Notice that the cell directly below each leg column of the assignment matrix contains a 0. The cell below the 0 contains an = sign, and the cell below that contains a 1. Notice also that the row of zeros is labeled "assigned" and the row of ones is labeled "required."

Q18. What does that mean in the context of the problem?

Similarly, notice that the cell directly to the right of each row of the assignment matrix contains a 0. The cell to the right of the 0 contains a ≤ sign. Lastly, the cell to the right of that one contains a 1. Also notice that the column of zeros is labeled "assigned" and the column of ones is labeled "capacity."

Q19. What does that mean in the context of the problem?

Q20. Why are these eight cells labeled ≤, while the other four were labeled =?

Figure 7.1.4 shows the previous spreadsheet after the Solver has found a solution for the problem of assigning eight swimmers to the four legs of the medley relay.

Q21. What is the optimal assignment of swimmers to legs of the relay?

Q22. Are there any swimmers in the optimal solution who are not the fastest swimmer for a specific leg? Are they all at least the second fastest swimmer in the event they have been assigned?

Q23. What is the minimum total time?

Q24. What is that time when converted to minutes and seconds?

	A	B	C	D	E	F	G	H	I
1	Chapter 7: Assignments								
2	7.1 Medley Relay Team								
3	Relay Time Minimization								
4									
5	**Swimmers' Times**								
6			Butterfly (sec)	Backstroke (sec)	Breaststroke (sec)	Freestyle (sec)			
7		Schmidt	59.59	59.1	72.83	52.61			
8		Reid	60.45	59.56	74.14	53.31			
9		Sanchez	61.84	64.63	73.69	53.7			
10		Lamartina	62.37	59.13	74.36	54.77			
11		Wu	60.33	64.3	72.74	54.05			
12		Greene	62.41	59.03	72.19	56.61			
13		Kleinfeld	62.43	67.63	74.05	55.55			
14		Lepinski	59.44	65.06	72.74	52.49		**Total Time (sec)**	
15								243.37	
16	**Assignments**								
17			Butterfly	Backstroke	Breaststroke	Freestyle			
18		Schmidt	0	0	0	1	1	≤	1
19		Reid	0	0	0	0	0	≤	1
20		Sanchez	0	0	0	0	0	≤	1
21		Lamartina	0	1	0	0	1	≤	1
22		Wu	0	0	0	0	0	≤	1
23		Greene	0	0	1	0	1	≤	1
24		Kleinfeld	0	0	0	0	0	≤	1
25		Lepinski	1	0	0	0	1	≤	1
26			1	1	1	1			
27			=	=	=	=			
28			1	1	1	1			

Figure 7.1.4: Optimal assignment of swimmers to the medley relay

Section 7.2: Homecoming Events at State U.

Amy Higgins, Chair of the Homecoming Committee at State University is to select catering services for six special events the Committee has planned for Homecoming Weekend in November. Food for each of these events must be brought in by a catering service. The six events are an alumni brunch, a parents' brunch, a luncheon for the booster club, a postgame party for season ticketholders, a lettermen's dinner and a fund-raising dinner for major contributors.

The Committee is committed to using local catering services. Ms. Higgins can choose from seven different catering services that have submitted bids. Table 7.2.1 shows the seven catering services and their bids, in $1000s, for each of the six events. For example, Big Bash bid $12,500 to cater the alumni brunch and bid $30,400 for the contributor's dinner.

		Event					
		Alumni Brunch	Parent Brunch	Booster Luncheon	Postgame Party	Lettermen Dinner	Contributor Dinner
Catering Service	Big Bash	12.5	11.3	14.5	20.5	25.2	30.4
	Bons Temps	14.9	12.7	15.9	17.6	22.8	32.5
	As You Wish	12.9	13.9	17.5	21.9	23.2	34.9
	Delightful	11.6	12	14	18.5	26	34.5
	Extra Special	11.1	12.1	13.9	17.9	22.9	29.5
	Henri's	12.5	13.9	15.9	22.5	24.8	35.8
	Campus	13.5	13.5	16.5	21.9	25.7	33.7

Table 7.2.1: Bids on Six Events from Seven Catering Services

The Bons Temps, As You Wish, and Campus catering services can each handle two events. However, the others are smaller businesses and can each handle only one event. The Committee is confident that each of the catering services would do a good job. Ms. Higgins decides to choose the catering services so that the total cost is as low as possible. How should she select the catering services in order to minimize the total cost of the six events?

7.2.1 Formulating the Problem

To formulate and solve the problem, Ms. Higgins contacts the Operations Research Department at State University. That Department assigns a team of graduate students to tackle the problem. They know that they must begin by defining the decision variables and the objective function. The team uses a 0-1 decision variable for each caterer and for each event. If the value is 0, it means that a particular caterer was not chosen for a given event. If the value is 1, it means that the particular caterer was chosen for the given event. For example, if Campus was not selected for the Alumni Brunch, the value of that decision variable would be 0. If Campus was selected for that event, the value of the decision variable would be 1.

Q1. For each of the catering services, there must be a separate decision variable for each event. How many decision variables are needed for each catering service? How many are needed altogether?

To formulate the problem, we need to define some of our variables, coefficients, and objective function:

For $1 \le i \le 7$ and for $1 \le j \le 6$, let $x_{ij} = \begin{cases} 0, \text{ if caterer } i \text{ is not selected for event } j. \\ 1, \text{ if cater } i \text{ is selected for event } j. \end{cases}$

For $1 \le i \le 7$ and for $1 \le j \le 6$, let b_{ij} = the amount that caterer i bids to provide event j.

Minimize $$z = \sum_{i=1}^{7}\left(\sum_{j=1}^{6} x_{ij} \cdot b_{ij}\right).$$

Now the team knows that they need to pay attention to constraints. The first constraint is that every one of the six events must be assigned to exactly one catering service:

For each $1 \le j \le 6$, $$\sum_{i=1}^{7} x_{ij} = 1 .$$

The next constraint is that some of the caterers can handle two different events, but others can handle only one event. If we designate the caterers by number in order, then Big Bash is 1, Bons Temps is 2, As You Wish is 3, Delightful is 4, Extra Special is 5, Henri's is 6, and Campus is 7. Then,

for i = 1, 4, 5, and 6, $\sum_{j=1}^{6} x_{ij} \le 1$, and

for i= 2, 3, 7, $$\sum_{j=1}^{6} x_{ij} \le 2 .$$

Q2. In all, how many constraint inequalities are there?

Figure 7.2.1 shows a spreadsheet representation of the problem, along with the dialog windows that were used to set up the Solver options.

Q3. Which cells in the spreadsheet contain the decision variables?

Q4. How were the decision variables designated as 0-1 (binary) variables?

Q5. Which cell contains the value of the objective value? How was that cell defined?

Q6. For the cells that contain the constraints that each event must be assigned to exactly one caterer, which specific cells contain the constant value, 1? How were the other cells defined?

Q7. In rows 18 through 24, which columns are constant? Which cells are defined in terms of the decision variables? How are they defined?

Q8. What is the optimal assignment of caterers to events?

Q9. What is the total cost of that optimal assignment?

Q10. Are there any caterers in the optimal solution who are not the lowest bidder for a specific event? Are they all at least the second lowest priced caterer in the event they have been assigned?

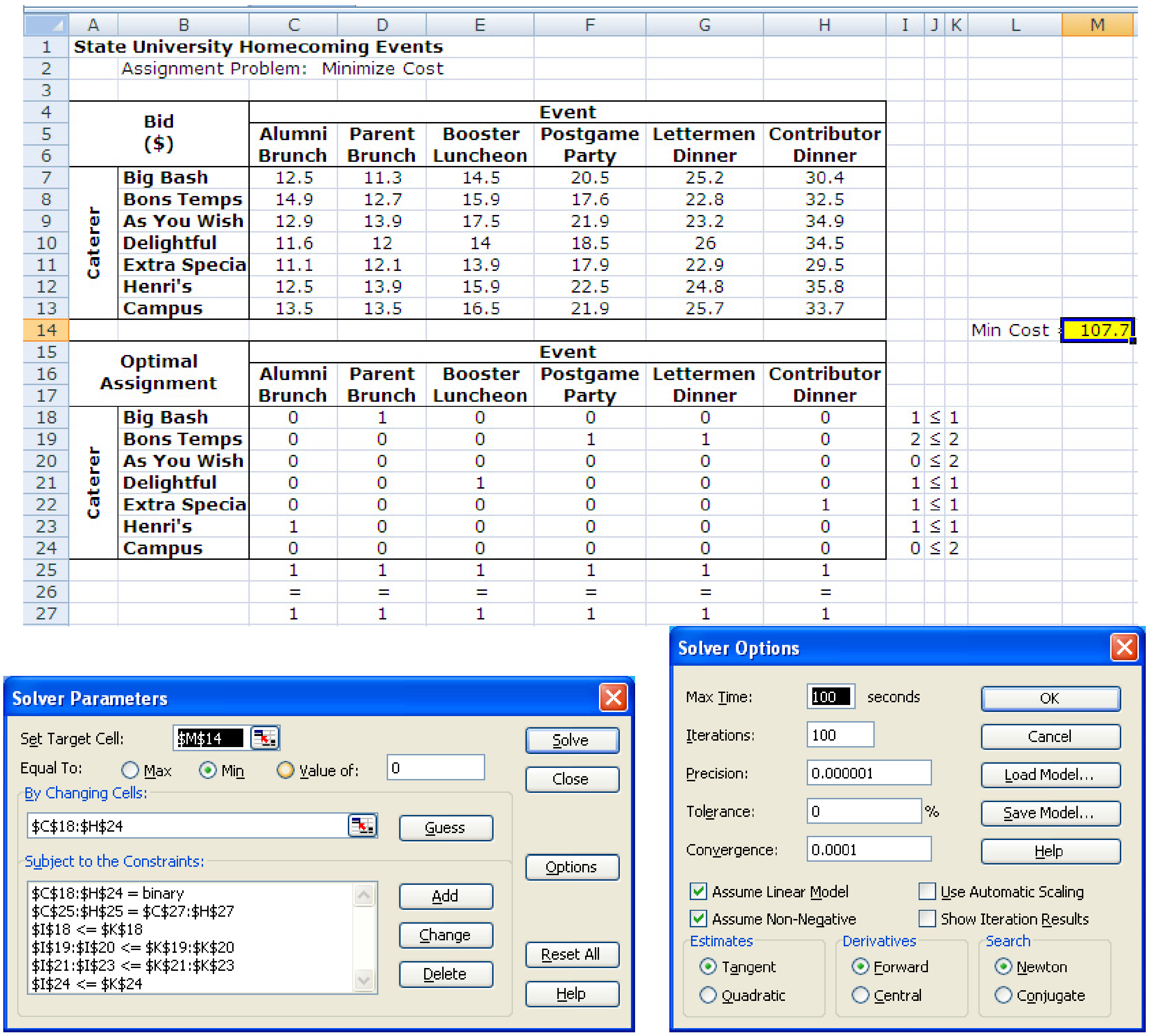


Figure 7.2.1: Spreadsheet Solver Set-Up for the Homecoming Problem

Section 7.3: Ms. Newman Assigns Students to Teams

Ms. Cynthia Newman wants to assign her 20 eighth-grade students to four teams of five students so that they can work on a science project. She wants to be fair about the team assignments, so that no team has an advantage over any other team. She knows that each team must have at least one effective leader to keep the team organized and on-task. She is going to require a written report of the project, so she decides to require at least two students with good writing skills on each team. The project will require some analysis of data that will have to be collected, so she decides to require that each team have two analysts and two data collectors. Finally, she is also going to require each team to make an oral presentation to the rest of the class, so each team is going to need at least two students who are good presenters.

Ms. Newman has identified the roles that she believes each of her students might satisfy based on their skills. For example, she thinks that Don McAllister, one of her students, could play the role of leader, analyst or data collector. She has identified at least two possible roles for each of her students. Table 7.3.1 shows the possible roles for each of her students. In the table, Ms. Newman has used "1" to indicate that she believes that a particular student can fill a particular role. Otherwise, she has used "0". So, in this sense, these cell values are binary indicators.

		Skill				
		Leader	Writer	Analyst	Presenter	Data Collector
Student	1	1	0	1	0	1
	2	0	1	0	1	0
	3	0	1	0	0	1
	4	1	0	1	1	0
	5	0	1	0	0	1
	6	0	1	0	1	1
	7	1	0	1	0	1
	8	0	1	0	1	1
	9	0	1	0	0	1
	10	1	0	1	1	0
	11	0	1	0	1	1
	12	0	0	1	0	1
	13	0	1	0	1	0
	14	1	0	1	0	1
	15	0	1	0	1	0
	16	0	0	1	0	1
	17	1	1	0	1	0
	18	0	1	0	1	0
	19	0	1	0	0	1
	20	1	0	1	1	1

Table 7.3.1: Roles Ms. Newman has identified for her students

Q1. Which roles does Ms. Newman believe that student number 17 could fill?

Q2. Which students does Ms. Newman believe could fill the role of "leader"?

Q3. Notice that for each role except for "leader", Ms. Newman is going to require that each team have at least two students who can fill that role. Why might it make sense to only require one leader on each team?

Mrs. Newman also does not want to give any team a significant academic advantage. Of course, she has the grade point averages (GPAs) for each of her students, but she wonders what the best way to use that information would be. She knows that almost certainly the average GPA of each of the four teams is going to be different. She also knows that the average GPA for the entire class is 2.94. Table 7.3.2 contains the GPA for each of her students.

Student	GPA
1	3.13
2	3.43
3	3.70
4	2.84
5	3.81
6	2.70
7	3.75
8	2.35
9	3.44
10	2.42
11	2.04
12	2.46
13	2.47
14	2.39
15	2.92
16	2.24
17	2.14
18	3.82
19	3.20
20	3.64

Table 7.3.2: Class GPAs

Ms. Newman's younger brother, Fred, is a graduate student in operations research. He suggests that she might use a technique called maximizing the minimum in order to solve her problem. Fred explains that she could form teams in such a way that the minimum average of four teams' GPA would be as large as possible. Of course, the solution would still have to satisfy her other constraints.

Assume, for example, she created four teams' with average GPAs that were 2.75, 2.91, 3.03, and 3.07. The minimum average GPA for these four teams is 2.75. It is significantly lower than the highest average. Maximizing the minimum average is one way to establish fairness with respect to the academic records of the four teams. Ms. Newman decides to use Fred's suggestion**.**

7.3.1 Formulating the Problem

Fred helped his sister formulate her team problem. He recognizes it as an assignment problem with a minimum team GPA objective and five constraints. The decision variables are all binary. A zero represents the decision not to assign a particular student to a particular team, and a one represents assigning that particular student to that team. Therefore, each student will have four different decision variables, one for each team. When a solution is found, for each student, three out of the four associated decision variables will equal zero; the other will equal one.

Q4. What is the meaning of the values of the four decision variables in the context of the problem?

Since there are 20 students in Ms. Newman's class, there are a total of 20 · 4 = 80 decision variables. Now Fred is ready to begin the formulation. First, he defines those decision variables.

For $1 \le i \le 20$ and $1 \le j \le 4$, let $s_{ij} = \begin{cases} 0, \text{ if student } i \text{ is } \textit{not} \text{ assigned to team } j \text{, and} \\ 1, \text{ if student } i \text{ is assigned to team } j. \end{cases}$

Next, he includes the students' GPAs.

For $1 \le i \le 20$, let g_i = the GPA of student i.

Now, he creates a formula to compute the average GPA for each team.

For $1 \le j \le 4$, let $A_j = \dfrac{\sum_{i=1}^{20} s_{ij} \cdot g_i}{5}$.

Recall that the objective here is to have a fair distribution of the students' GPAs as they are assigned to teams. To ensure fairness, Fred has suggested maximizing the minimum average GPA of the four teams. So, the objective function is

$z = \min (A_1, A_2, A_3, A_4)$,

And the objective is to maximize z.

maximize $z = \max [\min (A_1, A_2, A_3, A_4)]$,

Ms. Newman with the help of Fred establishes the basic assignment constraints for students and teams. First, every student must be assigned to exactly one team.

For $1 \le i \le 20$, $\sum_{j=1}^{4} s_{ij} = 1$.

Next, every team must consist of exactly five students.

For $1 \le j \le 4$, $\sum_{i=1}^{20} s_{ij} = 5$.

In addition, each team must have students with the following skills: at least one leader, two writers, two analysts, two presenters and two data collectors. In order to formulate these constraints, Fred used a matrix, R, similar to Table 7.2.1. If student i has been identified as capable of handling role k, then the value of r_{ik} is 1. Otherwise, the value of r_{ik} is 0. Now for each team, there is a constraint for each possible role.

Q5. How many more constraints is that?

For example, for the role of leader, Ms. Newman must check to see if, among the five students assigned to team 1, there is at least one whom she has identified as a possible leader. Then she must do the same for each of the other four roles. Finally, she must repeat the entire process for each of the other three teams. To formulate all of this:

For $1 \le j \le 4$, for $k = 1$, $\sum_{i=1}^{20} s_{ij} \cdot r_{i1} \le 1$, (leader constraint) for $k = 2$, $\sum_{i=1}^{20} s_{ij} \cdot r_{i2} \le 2$, (writer constraint)

for $k = 3$, $\sum_{i=1}^{20} s_{ij} \cdot r_{i3} \le 2$, (analyst constraint)

for $k = 4$, $\sum_{i=1}^{20} s_{ij} \cdot r_{i4} \le 2$, and (presenter constraint)

for $k = 5$, $\sum_{i=1}^{20} s_{ij} \cdot r_{i5} \le 2$. (data collector constraint)

7.3.2 Interpreting the Spreadsheet Solution

Figure 7.3.1 shows this formulation in a spreadsheet format.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1																		
2											TEAMS							
3							ROLES				1	2	3	4				
4					Leader	Writer	Analyst	Presenter	Data Collector	GPA	Decision Variables							
5				1	1		1		1	3.13	0	0	0	1		1	=	1
6				2		1		1		3.43	0	1	0	0		1	=	1
7				3		1			1	3.70	0	1	0	0		1	=	1
8				4	1		1	1		2.84	0	1	0	0		1	=	1
9				5		1			1	3.81	0	0	1	0		1	=	1
10				6		1		1	1	2.70	1	0	0	0		1	=	1
11			S	7	1		1		1	3.75	1	0	0	0		1	=	1
12			T	8		1		1	1	2.35	1	0	0	0		1	=	1
13			U	9		1			1	3.44	1	0	0	0		1	=	1
14			D	10	1		1	1		2.42	0	0	1	0		1	=	1
15			E	11		1		1	1	2.04	0	0	1	0		1	=	1
16			N	12			1		1	2.46	0	0	1	0		1	=	1
17			T	13		1		1		2.47	0	1	0	0		1	=	1
18			S	14	1		1		1	2.39	1	0	0	0		1	=	1
19				15		1		1		2.92	0	0	0	1		1	=	1
20				16			1		1	2.24	0	1	0	0		1	=	1
21				17	1	1		1		2.14	0	0	0	1		1	=	1
22				18		1		1		3.82	0	0	1	0		1	=	1
23				19		1			1	3.20	0	0	0	1		1	=	1
24				20	1		1	1	1	3.64	0	0	0	1		1	=	1
25										2.94	2.93	2.94	2.91	3.01	2.91			
26											A1	A2	A3	A4	min A1:A4			
27					2	3	2	2	5	0.02	5	5	5	5				
28					1	3	2	3	2	0.03	=	=	=	=				
29					1	3	2	3	3	0.00	5	5	5	5				
30					3	3	2	3	3	0.10								
31					>=	>=	>=	>=	>=	>=								
32					1	2	2	2	2	0			2.91					
33													2.91	Optimal				

Figure 7.3.1: Spreadsheet formulation of Ms. Newman's Team Assignment Problem

Q6. Which cells in the spreadsheet contain the values of the decision variables?

Q7. Cell J25 contains the value 2.94. How was that value obtained?

Q8. What do the values in cells K25 through N25 represent?

Q9. In which cell(s) in the spreadsheet is the value of the objective function? How was that cell defined?

Q10. Which cells in the spreadsheet correspond to the constraint that every student must be assigned to exactly one team? How were the left hand sides of those constraints defined?

Q11. What do the values in cells E27 thru I27, E28 thru I28, E29 thru I29, E30 thru I30 represent? How were those values obtained?

Q12. How were the values in cells J27 thru J30 obtained? What do those values tell you?

Q13. The values in the cells of the spreadsheet in Figure 7.3.1 represent an optimal solution. What is that optimal solution? What is the smallest team average GPA that satisfies all of the constraints?

Q14. For the optimal solution, what is the range of the team GPAs. Do you think that this solution meets Ms. Newman's fairness objective? Explain.

7.3.3 Students Who Cannot Work Together on a Team

After looking at the spreadsheet solution to her team assignment problem, Ms. Newman realizes that she did not account for some important information. She knows from past experience that students 7 and 8, who were assigned to the same team, cannot work with each other. She also knows that one other pair of students is incapable of working together, students 17 and 18. She tells her brother Fred about this, to see if there is any way to account for this additional complication. Fred assures her that, although it will add constraints to the formulation, the problem will still be solvable. Figure 7.3.2 contains the spreadsheet after Fred altered it to include the new constraints.

Here is how he handled the "Can't work together" constraints. Since students 7 and 8 cannot work together, we do not want to assign them to the same team. To ensure that this does not happen, Fred realized that the sum of cells K11 and K12 , L11 and L12, M11 and M12, and N11 and N12 each must be less than or equal to 1. Row 34 contains those sums in columns K through N. Similarly for students 17 and 18, the sum of cells K21 and K22, L21 and L22, M21 and M22, and N211 and N22 each must be less than or equal to 1. Row 33 contains those sums in columns K through N. Notice that in row 35, columns K through L contain "<=" and row 36, columns K through N are filled with 1s. Thus, rows 33, 35, and 36 contain the restriction that students 17 and 18 cannot be on the same team. Each value in row 33 from columns K thru N must be less than 1 as recorded in row 36. In the same way, rows 34, 35, and 36 contain the restriction that students 7 and 8 cannot be on the same team.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1																		
2											TEAMS							
3							ROLES				1	2	3	4				
4					Leader	Writer	Analyst	Presenter	Data Collector	GPA	Decision Variables							
5				1	1		1		1	3.13	0	0	0	1		1	=	1
6				2		1		1		3.43	1	0	0	0		1	=	1
7				3		1			1	3.70	0	0	0	1		1	=	1
8				4	1		1	1		2.84	0	0	0	1		1	=	1
9				5		1			1	3.81	0	1	0	0		1	=	1
10				6		1		1	1	2.70	1	0	0	0		1	=	1
11	S	Cannot Work		7	1		1		1	3.75	1	0	0	0		1	=	1
12	T	together		8		1		1	1	2.35	0	0	0	1		1	=	1
13	U			9		1			1	3.44	0	0	1	0		1	=	1
14	D			10	1		1	1		2.42	0	1	0	0		1	=	1
15	E			11		1		1	1	2.04	0	1	0	0		1	=	1
16	N			12			1		1	2.46	0	1	0	0		1	=	1
17	T			13		1		1		2.47	1	0	0	0		1	=	1
18	S			14	1		1		1	2.39	0	0	1	0		1	=	1
19				15		1		1		2.92	0	0	1	0		1	=	1
20				16			1		1	2.24	1	0	0	0		1	=	1
21		Cannot Work		17	1	1		1		2.14	0	0	1	0		1	=	1
22		together		18		1		1		3.82	0	1	0	0		1	=	1
23				19		1			1	3.20	0	0	0	1		1	=	1
24				20	1		1	1	1	3.64	0	0	1	0		1	=	1
25								Average GPA		2.94	2.92	2.91	2.91	3.04	2.91			
26											A1	A2	A3	A4	min A1:A4			
27					1	3	2	3	3	0.01	5	5	5	5				
28	2.91				1	3	2	3	3	0.00	=	=	=	=				
29	2.91	Optimal			3	3	2	3	3	0.00	5	5	5	5				
30					2	3	2	2	4	0.14								
31					>=	>=	>=	>=	>=	>=								
32					1	2	2	2	2	0								
33											0	1	1	0				
34											1	0	0	1				
35											<=	<=	<=	<=				
36											1	1	1	1				

Figure 7.3.2: Spreadsheet Formulation Adjusted to Handle Two New Sets of Constraints

Q15. How many constraints were added? Why so many?

Q16. What does the value 0 mean in cell K33? What does the value 1 mean in cell K34?

Q17. Compare the spreadsheets in Figures 7.3.1 and 7.3.2. What effect(s) did these two new sets of constraints have on the optimal solution? What effect did they have on the resulting Team GPAs?

Section 7.4: Chapter 7 (Assignment Problems) Homework Questions

1. **Housework for Kids**. Joanne Mankowski has three sons: Brad, Mike, and Paul. She is getting tired of doing the housework by herself, and she wants her sons' cooperation in keeping the house clean. She offered them payment if they share the housework on the weekend. She determined three types of tasks that are doable for her sons: washing the dishes after dinner, vacuuming the family room, and dusting the furniture in the living room. The kids told Joanne their preferred payment amount for each task secretly. Those amounts are represented in Table 7.4.1. Assign each son to a task so that the assignment generates the least cost to their mom.

($)		Bid		
		Dish Washing	Vacuuming	Dusting
Son	Brad	6	11	7
	Mike	9	13	9
	Paul	7	14	10

Table 7.4.1: Preferred payment amount of each son

2. **Flight Attendants**. Triangle Airlines is assigning six new flight attendants to fly on the six types of aircraft flown by the airline. Each of the new attendants has been trained on each type of aircraft, but the number of training hours the new attendants have on the different aircraft varies. The airline wants to assign the attendants based on their number of hours of training on each on each aircraft type. Table 7.4.2 provides the number of training hours on each aircraft type for each attendant. How should the airline assign the attendants if it wants to assign them based on their training experiences? (Hint: Is this a maximization or minimization problem?)

(hours of training)		Aircraft					
		CRJ	DC-9	A320	747	757	767
Attendant	Albert	4	4	2	4	2	8
	Jack	4	4	4	4	4	4
	Mary	4	2	2	4	8	4
	Katie	2	2	4	4	4	8
	Dave	2	2	4	6	6	4
	Matthew	4	2	2	6	6	4

Table 7.4.2: Training experience of the new attendants

3. **Leaders for Projects**. Mr. Summit has four projects, one each in marketing, product development, logistics, and finance. He has chosen four employees with good leadership skills, Ahmad, John, Julia, and Subhash. Now, it is time to assign the right person to the right project. First, he developed a test of 20 questions for each project and asked the four employees to answer all of the questions. He wants to assign the leaders to the tasks so that the total number of mistakes on the test will be the minimum possible. The number of mistakes each employee made on the test is displayed in Table 7.4.3.

(# of mistakes)	Project			
	Marketing	Product Development	Logistics	Finance

Employee	Ahmad	1	2	3	3
	John	4	4	3	4
	Julia	1	2	1	3
	Subhash	4	2	2	2

Table 7.4.3: Number of mistakes made in the test

4. **Industrial Training.** Industrial Training Consultants is offering four types of courses in August and there are five instructors who are specialized in the subjects. The assignment will be done based on past student evaluations of the five instructors. The student evaluation scores appear in Table 7.4.4. How shall instructors be assigned to courses so that the total of the student evaluation scores is a maximum?

(% positive)		Course			
		Lean Manufacturing	Six Sigma	Logistic Management	Simulation
Student	Randolph	93	96	86	87
	Angela	90	94	92	89
	Anthony	91	87	84	88
	Deborah	92	88	90	85
	Myles	95	97	94	88

Table 7.4.4: Student evaluations

5. **Renovaçión Home Improvement Store**. The Renovaçión Home Improvement Store will assign an employee to each of the five departments: Appliances, Flooring, Outdoor Living, Kitchen, and Tools and Hardware. There are seven employees available who have past experience in all of these five departments. The average daily sales of each employee are shown in Table 7.4.5.
   a. Assign employees to departments so that the average daily sales of the five employees assigned are as great as possible.

($)		Department				
		Appliances	Flooring	Plumbing	Doors	Lighting
Employee	Joshua	1,555	525	370	275	560
	Adan	1,250	450	285	250	540
	Ha	850	500	320	330	550
	Tyson	1,675	490	375	350	580
	Valley	1,125	510	365	345	190
	Lacole	950	500	195	335	350
	Haemon	1,050	300	345	200	545

Table 7.4.5: Average daily sales of employees by department

   b. In the optimal solution, which individuals were the best in their selected category? Which individuals were not the best? Explain why the optimal solution did not pick the best in each.

6. **VogueTech Computer**. The VogueTech Customer Service provides 24-hour online technical support. There are three 8-hour shifts and thirteen representatives. Five representatives will be

assigned to the morning shift, five will be assigned to the afternoon shift, and 3 will be assigned to the night shift. The manager wants to assign them according to their preferences. Table 7.4.6 shows the shift preferences of each representative. "5" indicates the most preferred and "1" indicates the least preferred. Assign them according to their preferences.

(rank)		Shift		
		6 a.m. – 2 p.m.	2 p.m. – 10 p.m.	10 p.m. – 6 a.m.
Representative	1	3	5	1
	2	2	4	2
	3	4	1	4
	4	4	4	4
	5	3	1	4
	6	5	5	4
	7	3	4	3
	8	5	5	2
	9	2	1	1
	10	2	2	4
	11	1	4	1
	12	5	4	5
	13	5	1	5

Table 7.4.6: The preferences of each representative

a. Experience shows that representatives 3 and 4 from exercise 6 do not work well together. Therefore, management decides not to assign them to the same shift. How does this affect the optimal solution?

7. **Disaster Kits.** Counselor Cynthia Walker at Foster High School assigns students to community service work as apart of their graduation requirements. She recently received a notice about a project from the Community Help organization to pack kits for disaster relief. A section of the country needs assistance after a hurricane struck the area. Six types of kit are needed: Emergency Food Packs, Children's, Personal Care, Food Support, Layette, and Household. Only one student can be assigned to pack each of the six different types of kit, but ten students have signed up. All of these students have previously packed relief kits for this organization. The organization would like to pack 60 of each type kit.

As a time saving method, the Community Help organization calculates and records the packing rate for each volunteer, in order to assign the most efficient volunteer to the right task. The packing rate is the number of kits packed per hour. The organization would like to assign the volunteers so that 60 of each type kit are packed in the least time.

(minutes)		Kit					
		Emergency Food Packs	Children's	Personal Care	Food Support	Layette	Household
Student	Abdullah	3	6	6	6	4	3
	Susan	12	18	7	9	3	9

	Jeff	14	12	6	10	1	3
	Briana	5	13	6	13	2	4
	Naomi	6	14	9	15	1	9
	Brenden	14	9	2	12	3	9
	Carlos	12	17	11	5	2	1
	LaQuita	4	16	2	14	7	6
	Matthew	4	14	9	6	1	3
	Erika	8	17	4	14	4	3

Table 7.4.7: Packing rates for different kinds of kits

a. How can the Community Help organization determine which student to assign to each task?

b. Which student should be assigned to each task in order to pack the kits in the least amount of time?

8. **School Bus Route Assignment.** The school district in Livonia, Michigan makes annual contracts with school bus companies. There are three companies who are bidding for nine routes in the Livonia School District. First, the companies announce their bids for the routes they are interested in. Then, the city decides which company to assign to each route. The announced bids of each company are shown in Table 7.4.8. A blank cell in the table indicates that the company did not offer a bid for the route. None of the companies that are bidding can be assigned to more than three routes. Help the Livonia Schools assign companies to all routes with a minimum total cost.

($)		Route								
		1	2	3	4	5	6	7	8	9
Company	Never Late	11,000		6,200		8,200	9,850	7,000	9,250	4,900
	Snail's Pace	11,150	8,700		12,950	8,800		6,600	8,400	4,900
	On Time		8,300	6,250	12,600	8,150	9,750	6,500		4,300

Table 7.4.8: Bids on routes of school bus companies

a. Identify the lowest and the second lowest bids on each route.

b. None of the companies can be assigned to more than 3 routes. Assign the companies to the routes manually without violating the 3-route restriction.

c. Formulate the problem.

d. Solve the problem using a spreadsheet solver. Make sure the changing cells include only those routes for which a company actually bid.

10. As you experienced in problem 1 picking the changing cells and writing the equations are tedious. An easier way is to put very high numbers into the blank cells as given in Table 7.4.9.

($)	Route

		1	2	3	4	5	6	7	8	9
Company	1	11,000	100,000	6,200	100,000	8,200	9,850	7,000	9,250	4,900
	2	11,150	8,700	100,000	12,950	8,800	100,000	6,600	8,400	4,900
	3	100,000	8,300	6,250	12,600	8,150	9,750	6,500	100,000	4,300

Table 7.4.9: Modified bids of school bus companies

a. Why do you think you can find the same solution using the values above?

b. Formulate the problem.

c. Solve it again using Solver. You can use the SUMPRUDUCT function to write the objective function.

11. Four more school bus companies are added to the bidding; now there are seven companies that are bidding for nine routes in the Livonia School District. The announced bids are shown in Table 7.4.10. Now each company can be assigned at most two routes. Use the idea from problem 2 to simplify the modeling.

($)		Route								
		1	2	3	4	5	6	7	8	9
Company	1	11,000		5,750		8,200	9,600	5,700	7,600	4,450
	2	10,450	8,600		12,950	8,800		6,600	7,900	4,400
	3		8,350	5,700	12,580	8,150	9,650	5,600		4,600
	4	10,300	8,400	5,675	13,000	8,150	9,750	5,550	7,600	
	5	10,200	8,300	5,600	12,500		9,900		7,575	4,600
	6	10,400	8,500	5,850		8,050	9,500	5,650	7,500	4,300
	7			5,750	12,600	8,750	9,550	5,500	7,700	4,375

Table 7.4.10: Bids of School Bus Companies

a. What will the new assignment be?

b. The city wants to see the affect of assigning each company three routes at most instead of only two. What will the impact be? What if they assign each company four routes at most?

12. Referring to the Homecoming example in section 7.2, what, if anything, would happen to the optimal solution if all of the caterers were able to handle two events? Suppose Campus has been used regularly and has always done an outstanding job. If the committee wanted to assign at least one event to Campus, how, if at all, would doing so change the optimal solution?

Chapter 7 Summary

What have we learned?

We have learned that assignment problems are special cases of binary integer programming problems. The mathematical formulation of these problems has many requirements.
Matrices are used extensively.

- The binary decision variables are arranged in a compact matrix X. If agent i is assigned to perform task j, then x_{ij} will equal one, otherwise it will equal zero.

- The "cost" of agent i performing task j is element c_{ij} in the cost matrix C.

- The overall "cost" is calculated by taking the sum of all the products of corresponding elements of the decision variable matrix and the cost matrix.

The spreadsheet formulation of assignment problems is somewhat different than in previous chapters.

- The SUMPRODUCT formula needs to be used for the objective function.

- Constraints typically involve row and column totals from the decision variable matrix.

Terms

Assignment Problem	An assignment problem arises whenever a number of agents must be paired with a number of tasks
Agent	In an assignment problem, the agents are the ones able to perform the tasks
Task	In an assignment problem, the tasks are those things needing to be accomplished
Cost	The "cost" depends on the context and units of the problem, but it represents the amount of the objective quantity required for an agent to perform a task
Cost Matrix	For an assignment problem with m agents and n tasks, the cost matrix C will be an $m \times n$ matrix, and element c_{ij} will represent the cost of agent i performing task j
Binary Indicator Coefficients	A coefficient that takes the value of one if the quantity meets a given condition or zero if it does not

Notation

If $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \dots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & c_{m3} & \dots & c_{mn} \end{bmatrix}$ and $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \dots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}$, then

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij} \cdot x_{ij}\right) = \begin{bmatrix} c_{11} \cdot x_{11} & c_{12} \cdot x_{12} & c_{13} \cdot x_{13} & \dots & c_{1n} \cdot x_{1n} \\ c_{21} \cdot x_{21} & c_{22} \cdot x_{22} & c_{23} \cdot x_{23} & \dots & c_{2n} \cdot x_{2n} \\ c_{31} \cdot x_{31} & c_{32} \cdot x_{32} & c_{33} \cdot x_{33} & \dots & c_{3n} \cdot x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1} \cdot x_{m1} & c_{m2} \cdot x_{m2} & c_{m3} \cdot x_{m3} & \dots & c_{mn} \cdot x_{mn} \end{bmatrix} = \text{SUMPRODUCT}\ (C, X).$$

Chapter 7 (Assignment Problems) Objectives

You should be able to:

- Identify the objective of the problem
- Identify and define the binary decision variables
- Formulate the problem using a compact matrix
  - Each row represents an agent
  - Each column represents a task
- Write the objective function, including using double summation notation
- Identify the constraints involved in the problem
  - Standard assignment constraints
    - row totals ≤ 1 indicating each agent can be assigned to at most one task
    - column totals = 1 indicating that each task must be completed by exactly one agent
  - Extra constraints (e.g., modifications to totals, binary indicator coefficients)
- Write the functional constraints as equations or inequalities, including using summation notation
- Utilize binary indicator coefficients
- Enter the problem formulation into Excel
- Use SUMPRODUCT formula in Excel
- Set up Solver Parameters and Options including the constraints that decision variables are binary
- Interpret the optimal solution in the context of the problem
- Analyze the Answer Report

Chapter 7 Study Guide

1. Explain how Excel would execute the command "=SUMPRODUCT(A1:B3,D1:E3)."

2. If an assignment problem has nine agents to assign to seven tasks, why is it convenient to use matrices to formulate the problem?

3. Assuming that each agent can perform at most one task and each task must be performed, what is the relationship between the number of agents and the number of tasks in an assignment problem? Why?

4. If an assignment problem has six agents, four tasks, decision variable matrix X, and cost matrix A, use double summation notation to write the objective function.

5. Consider the problem where four swimmers are being assigned to the four different legs of a 400-yard medley relay. What is wrong with the logic that the swimmer with the fastest time for each stroke be assigned to swim that leg of the relay?

6. Consider an assignment problem with more agents than tasks. Explain why it is typical that the constraints based on row (agent) totals are inequalities while the constraints based on column (task) totals are equations.

7. Explain the "maximize the minimum" technique that was used in Problem 7.3 (assigning students to teams).

8. Do all assignment problems have a unique solution? Explain why or why not.

Section 8.0: Introduction

This chapter focuses on different types of location problems. The chapter is split into three sections.

In the Section 8.1, there are two examples in which employees of a hot dog stand company need to choose the optimal location for their hot dog stands along a city road. In each example, the hot dog stand employees want to minimize their customers' walking distances.

Section 8.2 is split into three examples surrounding the smoothie industry. First, a strawberry freezer warehouse needs to be located in a city along interstate 40 in North Carolina so that the number of truckloads from the warehouse will be minimized. Second, the location for a smoothie store in a city needs be determined so that customers' walking distances will be minimized. In these two examples, an algorithm will be employed to find the optimal location. Third, a smoothie stand needs to be located somewhere on a fairground to maximize the number of customers at the stand. In this example, geometric constructions will be used to find the optimal location.

In Section 8.3, disaster response agencies need to be placed in central Oklahoma. The governor of Oklahoma wants to minimize the number of agencies built while still ensuring that the agencies are within a reasonable distance to a number of cities. To solve this problem, binary programming will be used.

Throughout the chapter, different methods will be used to find the optimal location. However, in each case, the best location for that particular system will be found.

Section 8.1: Stadium Hot Dog Stands

Stadium Hot Dog Stands is a small company in North Carolina. Employees of Stadium Hot Dogs Stands operate mobile hot dog stands throughout the state. They can choose the location of their stand based on where they think they will get the most business. Often, two or more employees will work together to find locations for their stands to minimize customers' travel.

8.1.1 Locating Two Hot Dog Stands on Main Street

Lodge and Alyssa run two mobile Stadium Hot Dog Stands on Main Street in downtown Raleigh, NC. Most of their customers are employees of nearby companies in the downtown area.

Lodge and Alyssa noticed that customers on Main Street tend to visit the hot dog stand that is closest to their office. Therefore, they decided to work together to maximize both of their profits while minimizing their customers' average walking distance.

Assume the following:

- The segment of Main Street that Lodge and Alyssa serve is approximately 1 mile in length,
- Customers are equally distributed throughout the mile, and
- Customers will walk to the stand that is closest to their office or company.

Q1. Using Figure 8.1.1 as a depiction of Main Street, where do you think Lodge and Alyssa should position their stands along the street to minimize their customers' average walking distance?

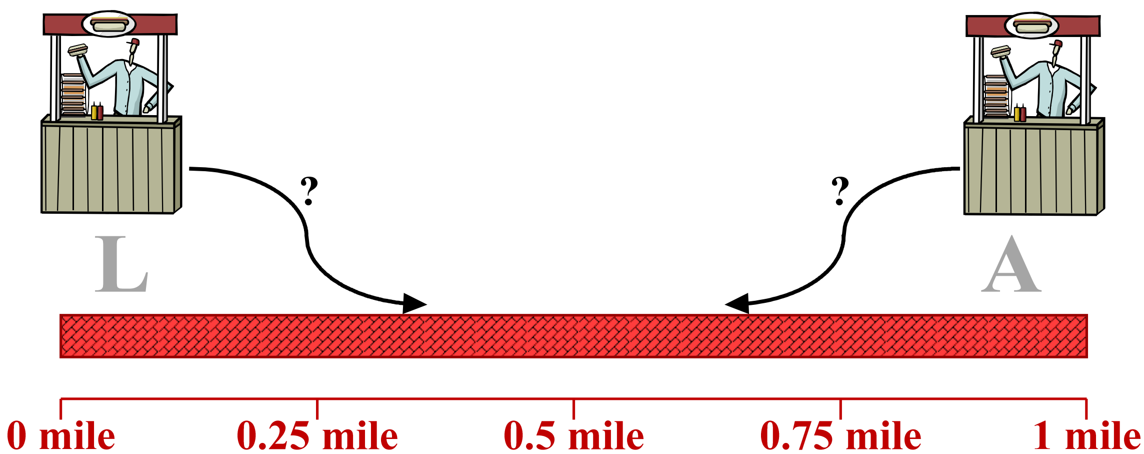


Figure 8.1.1: Possible locations for Lodge's and Alyssa's hot dog stands

In this context, the goal is minimize customers' average walking distance. Therefore, the **cost** refers to customers' average walking distance. Cost does not have to refer to money.

Q2. What are some other examples of cost?

Suppose Lodge and Alyssa position their stands next to each other at the halfway point of the line. Notice that they each would have an equal market share of 0.5 miles of customers to serve. The maximum distance a customer would need to travel is 0.5 miles, and the minimum distance would be 0 miles. Thus, the average distance that a customer would travel to one of the stands is

$$\frac{0+0.5}{2} = 0.25 \text{ miles.}$$

Q3. Suppose Lodge positions his stand at the 0-mile mark, and Alyssa positions her stand at the 1-mile mark.

a. What will be the maximum distance a customer would need to travel? What would be the minimum?

b. What will be the customers' average walking distance?

Q4. Suppose Lodge positions his stand at the 0.25-mile mark, and Alyssa positions her stand at the 0.75-mile mark.

a. What will be the maximum distance a customer would need to travel? What would be the minimum?

b. What will be the customers' average walking distance?

Q5. Based on your responses to the previous two scenarios, what are the optimal locations for Lodge's and Alyssa's hot dog stands? Explain your reasoning.

In this example, certain assumptions were made, such as the street being one mile long, customers being equally distributed throughout the street, and customers walking to the stand nearest to their office. It is important to consider assumptions such as these. Each time a real situation is modeled, it is limited by the assumptions made. However, if these limitations are understood, the model can be used to understand, change, manage, and control that situation.

Q6. What other assumptions were made in order to solve this problem?

Q7. Suppose the customers were not equally distributed throughout the street. How would this change the problem?

Lodge and Alyssa determined the optimal location for their hot dog stands based on their model of the situation. They used the customers' average walking distance to calculate the cost of the **system**, which is a collection of organized pieces. Fortunately, the system was not influenced by where the customers were located because the customers were assumed to be equally distributed. This will not always be the case. In the following section, customers' locations need to be considered when finding the optimal location.

8.1.1 Locating One Hot Dog Stand on Park Street

Suppose another employee of Stadium Hot Dog Stands, Grayson, wants to set up his hot dog stand on Park Street in Greensboro, NC. Based on previous experience, he knows that two people from Park Street Bank will come to his stand every day and one person from Green Dry Cleaning will come to his stand every day. Park Street Bank is 300 meters from Green Dry Cleaning.

Q8. Using Figure 8.1.2 as a depiction of Park Street, where do you think Grayson should position his stand along the street to minimize their customers' average walking distance?

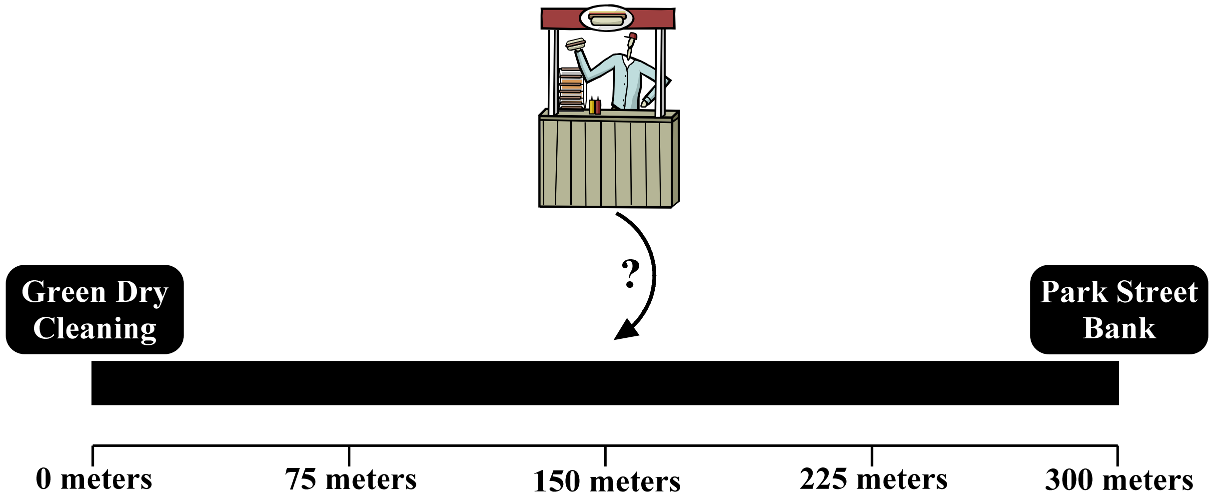


Figure 8.1.2: Distance between Green Dry Cleaning and Park Street Bank

Grayson considers locating his stand at the geographical center of the street at the 150-meter mark. In this case, the one customer at Green Dry Cleaning would travel 150 meters, one-way, to get to the stand, and the two customers from Park Street Bank would each travel 150 meters, one-way.

In this example, *the cost of the system is the one-way total distance traveled*. Thus, if Grayson sets up his stand at the 150-meter mark, the cost would be 450 trip meters, as shown in Table 8.1.1.

Proposed Stand Location	**Green Dry Cleaning Distance (One-way)**	**Park Street Bank Distance (One-Way)**	**Total Distance**
150-meter mark	(150 meters)(1 trip)	(150 meters)(2 trips)	450 trip meters

Table 8.1.1: Total trip meters with hot dog stand at the 150-meter mark

Grayson thinks a different location could lead to a shorter total distance. To find the best location, he begins listing possible locations for his stand.

Q9. Complete Table 8.1.2 to find the optimal location of Grayson's hot dog stand.

Proposed Stand Location	**Green Dry Cleaning Distance (One-way)**	**Park Street Bank Distance (One-Way)**	**Total Distance**
150-meter mark	(150 meters)(1 trip) = 150	(150 meters)(2 trips) = 300	450 trip meters
200-meter mark	(200 meters)(1 trip) = 200	(100 meters)(2 trips) = 200	400 trip meters
100-meter mark	(100 meters)(1 trip) = 100	(200 meters)(2 trips) = 400	500 trip meters
250-meter mark			
50-meter mark			
275-meter mark			
290-meter mark			
300-meter mark			

Table 8.1.2: Total trip meters with hot dog stand at various meter marks

Q10. What is the optimal location for Grayson's hot dog stand? What is the total distance traveled at this location?

Q11. Was the optimal location for Grayson's hot dog stand what you expected? Why or why not?

Just as in the previous example, certain assumptions were made. For example, Grayson assumed that there would be two customers from Park Street Bank and one customer from Green Dry Cleaning each day.

Q12. What other assumptions were made in order to solve this problem?

In this example, Grayson determined the optimal location for his hot dog stand based on the model of the situation. He needed to consider where his customers were located in order to find the best location. He used his customers' locations to calculate the total cost of the system.

This is an example of a **single facility location problem**. In this problem, the goal was to determine the location of one new facility (a hot dog stand) with respect to existing facilities (Park Street Bank and Green Dry Cleaning). The objective of the single facility location problem is to determine the location that minimizes the sum of all the distances traveled. That is, Grayson wanted to find the location that minimized the sum of all the distances between his hot dog stand and his customers. Note that in the first problem, Lodge and Alyssa *both* needed to find a location for this hot dog stands. Therefore, this was not an example of a single facility location problem.

The following section gives another example of a single facility location problem. However, there will be eight, rather than two, existing facilities.

Section 8.2: The Smoothie Industry

Sal's Strawberries is a local strawberries supplier. He supplies strawberries to customers throughout North Carolina. Some of customers include stores, restaurants, and smoothie shops. For example, Ellie's Eco-Smoothies uses strawberries supplied by Sal's Strawberries. Sal only supplies his customers with strawberries grown in North Carolina, which is why Ellie prefers to purchase strawberries from him. Because strawberries are a seasonal food and Sal has customers like Ellie that need strawberries year round, Sal freezes strawberries to meet off-season demand.

In section 8.2.1, Sal needs to determine the location for his freezer warehouse that will best serve his customers, such as Ellie. In this example, Sal only considers locations in certain cities and along a certain highway. Because he only considers locations along one highway, this is a *one-dimensional problem*, as were the problems in section 8.1.

In section 8.2.2, Ellie wants to open a smoothie and juice store in the downtown Raleigh area, and she would like to find the best location to meet her customers' needs. Ellie has more options than Sal because she can place her store anywhere in the downtown Raleigh area. However, this makes for a different type of problem than the previous ones. In this case, Ellie must consider a two-dimensional city map, and therefore this is a *two-dimensional problem*.

In both one-dimensional and two-dimensional facility location problems, the median location is used to calculate where a new facility should be located given the weights of the existing facilities. An example of a *weight* is the number of trips a student makes to the snack machine from their classroom each day. If the student makes 2 trips to the snack machine per day, then the weight is 2. The ideas of median location and weight will be explored in this section.

8.2.1 Sal's Strawberries

Sal wants to locate a new freezer warehouse to store frozen strawberries. He is looking at plots of land along I-40 in North Carolina to better serve his customers in the following cities: Asheville, Hickory, Statesville, Winston-Salem, Greensboro, Durham, Raleigh, and Wilmington. In order to have access to a large labor pool to staff the warehouse, Sal decides to purchase land in or just outside of one of the aforementioned cities. Because Sal restricts himself to a finite number of locations along one highway, this is considered a one-dimensional location problem.

Figure 8.2.1 gives a map of North Carolina showing the potential cities along I-40 where Sal will purchase land for his new freezer warehouse. The illustration also shows the number of highway miles (i.e., mile markers) from the beginning of I-40 at the state's western border to each city. The weekly demand in truckloads to each city is listed below the city name. Table 8.2.1 gives the same information in a table format.

Sal wants to locate the strawberry freezer warehouse so that he minimizes the distance he needs travels on I-40 in order to serve his customers. He needs to find the best location for the warehouse among the eight customers' cities located on I-40 between Asheville and Wilmington. Due to limitations in the truck and freezer time, Sal can only go to one customer for each truckload. That is, he will only go from the warehouse to a customer and not visit other customers along the way.

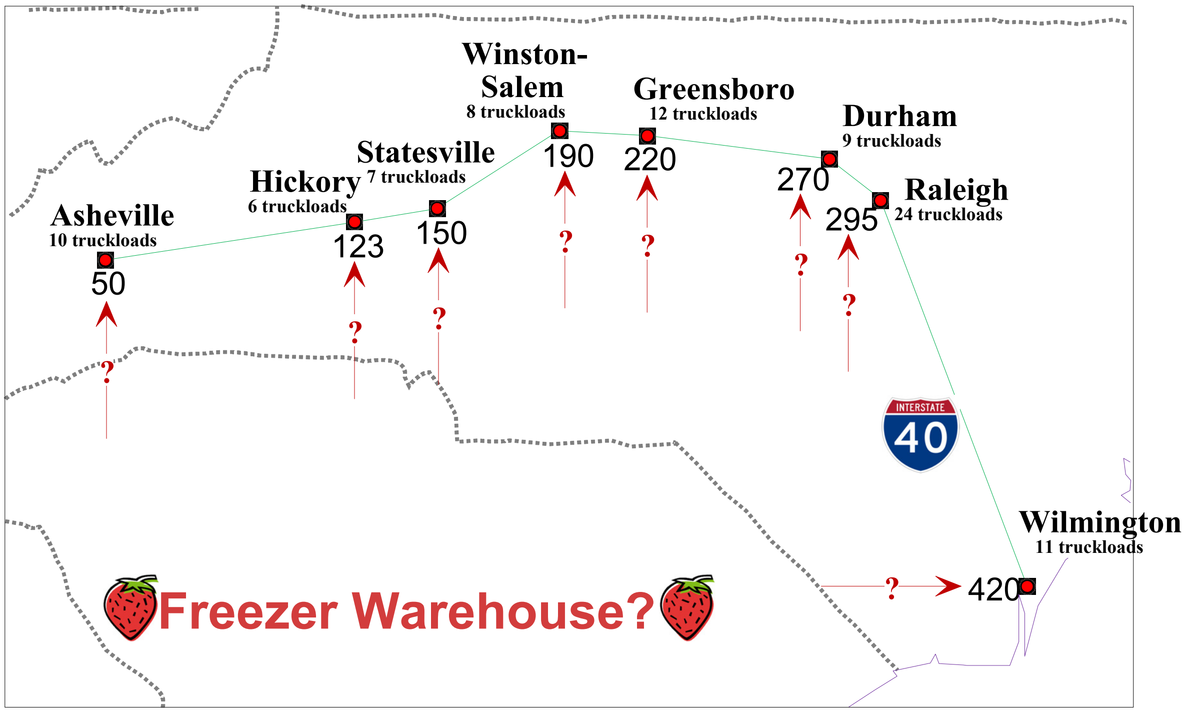


Figure 8.2.1: Mile-marker and number of truckloads for each city along I-40 in North Carolina

City	I-40 mile-marker (distance from the state's western border to each city)	Number of truckloads of strawberries per week
Asheville	50	10
Hickory	123	6
Statesville	150	7
Winston-Salem	190	8
Greensboro	220	12
Durham	270	9
Raleigh	295	24
Wilmington	420	11

Table 8.2.1: Mile-marker and number of truckloads for each city along I-40 in North Carolina

Q1. Using Figure 8.2.1 and/or Table 8.2.1, where do you think Sal should position his warehouse along the highway to minimize the number of truckloads to each city per week?

In the previous section, Grayson determined the optimal location of his hot dog stand by calculating the total distance at several locations and finding which location gave the shortest distance for his customers to travel. If Sal were to use the same method, he would need to calculate the total cost for each possible city location. That is, Sal would have to calculate the total number of truckloads to each city from each city.

When calculating the total cost of the system in this example, there are some things to keep in mind:

- The distance from a fixed point is measured in absolute value.
- Distances will be measured in reference to the city where the freezer warehouse is located.
- Weight represents the weekly demand, in truckloads, of the customers in each city.

The term **weight** does not refer to the physical weight of the trucks or the strawberries. Rather, it is a representation of an entity's contribution to the overall system; in this case, Sal must consider the contribution of the number of truckloads to each city per week to the system.

Weight and Cost

Note that *weight* and *cost* refer to different pieces of the system. Weight refers to something contributing to the system, such as number of customers or number of truckloads, while cost refers to something that needs to be minimized, such as money, distance, or time.

Suppose the warehouse was located in Asheville. The cost for the city of Hickory in this case would be |123 – 50| × 6 = 438 miles. Similarly, the cost for Statesville is |150 – 50| × 7 = 700 miles. Continuing in this fashion, Table 8.2.2 provides the total cost if the warehouse were located in Asheville.

City	Distance from Asheville	Number of truckloads per week	Cost
Asheville	|50 – 50| = 0	10	0(10) = 0 miles
Hickory	|123 – 50| = 73	6	73(6) = 438 miles
Statesville	|150 – 50| = 100	7	100(7) = 700 miles
Winston-Salem	|190 – 50| = 140	8	140(8) = 1120 miles
Greensboro	|220 – 50| = 170	12	170(12) = 2040 miles
Durham	|270 – 50| = 220	9	220(9) = 1980 miles
Raleigh	|295 – 50| = 245	24	245(24) = 5880 miles
Wilmington	|420 – 50| = 370	11	370(11) = 4070 miles
		Total Cost:	**16228 miles**

Table 8.2.2: Cost of the system if the warehouse were located in Asheville

Q2. Why was absolute value used to find the distances between cities?

Q3. Determine the total cost of the system if the warehouse were located in Hickory.

While it is possible to continue in this way to find the best location for the warehouse, it is not desirable. Fortunately, there is an algorithm to determine the optimal location in a system such as this.

The first step of the algorithm is the find the *total weight of the system*. This is done by adding all of the weights:

$$10 + 6 + 7 + 8 + 12 + 9 + 24 + 11 = 87 \text{ truckloads.}$$

The second step is to calculate the **median weight** of the system. This is found by dividing the total weight in half. For Sal's strawberry freezer warehouse, half of the total weight is

$$\frac{10+6+7+8+12+9+24+11}{2} = 43.5 \text{ truckloads}.$$

Third, list the cities in sequential highway order and calculate the cumulative sum of the weights, as shown in Table 8.2.3.

City	Weight (# of Truckloads per week)	Cumulative Weight (# of Truckloads per week)
Asheville	10	10
Hickory	6	16
Statesville	7	23
Winston-Salem	8	31
Greensboro	12	43
Durham	9	52
Raleigh	24	76
Wilmington	11	87

Table 8.2.3: The cumulative weights of the possible strawberry freezer warehouse cities

The last step of the algorithm is to find the **median location**, which will be the optimal location for the freezer warehouse. To do so, find the location at which the cumulative weight first exceeds the median weight. In this example, the median location is the city along I-40 that first exceeds 43.5 truckloads.

Note: when calculating cumulative weights, you can list the locations west-to-east, or east-to-west, but you cannot change the order of locations. Tables 8.2.4 and 8.2.5 illustrate that Durham is the first city to exceed 43.5 truckloads, regardless of starting from the left or right of I-40. Therefore, Durham is the optimal location for Sal's freezer warehouse.

City	Weight (# of Truckloads per week)	Cumulative Weight (# of Truckloads per week)
Asheville	10	10
Hickory	6	16
Statesville	7	23
Winston-Salem	8	31
Greensboro	12	43
Durham	**9**	**52**
Raleigh	24	76
Wilmington	11	87

Table 8.2.4: The cumulative weights of the possible strawberry freezer warehouse cities, listed west-to-east along the highway

City	Weight (# of Truckloads per week)	Cumulative Weight (# of Truckloads per week)
Wilmington	11	11
Raleigh	24	35
Durham	**9**	**44**
Greensboro	12	56
Winston-Salem	8	64
Statesville	7	71
Hickory	6	77
Asheville	10	87

Table 8.2.5: The cumulative weights of the possible strawberry freezer warehouse cities, listed east-to-west along the highway

Q4. Did you expect Durham to be the optimal location of the freezer warehouse? Why or why not?

Median Weight and Median Location

The terms *median weight* and *median location* can be confusing. The term *median* typically is used in statistics to mean the middle value in an order set of values. However, rather than referring to the middle value, the term *median weight* refers to the middle weight and *median location* refers to the middle location.

Because Durham is the optimal location for the warehouse, it represents the location where no more than one-half of the total weight of the system is to the right and to the left of the freezer warehouse's location. You can check your work by finding the cumulative weight of the cities to the left and right of the strawberry freezer warehouse (do not include the weight for the city in which the warehouse is located).

Q5. Check that the total weight to the left of Durham is less than the median weight.

Q6. Check that the total weight to the right of Durham is less than the median weight.

Now that the optimal location for Sal's strawberry freezer warehouse has been determined, the *minimum total cost of the system* can be calculated. Earlier, the total cost was found if the warehouse were located in Asheville (see Table 8.2.2). The minimum total cost of the system is found in the same way, but now using Durham as the location for the warehouse.

City	Distance from Durham	Number of truckloads per week	Cost
Asheville	|50 – 270| = 220	10	220(10) = 2200 miles
Hickory	|123 – 270| = 147	6	147(6) = 882 miles
Statesville	|150 – 270| = 120	7	120(7) = 840 miles
Winston-Salem	|190 – 270| = 80	8	80(8) = 640 miles
Greensboro	|220 – 270| = 50	12	50(12) = 600 miles
Durham	|270 – 270| = 0	9	0(9) = 0 miles
Raleigh	|295 – 270| = 25	24	25(24) = 600 miles
Wilmington	|420 – 270| = 150	11	150(11) = 1650 miles
		Total Cost:	**7412 miles**

Table 8.2.6: Total cost of system given that Durham is the optimal location.

Therefore, Sal's delivery truck(s) will drive a minimum total of 7,412 miles per week in order to deliver frozen strawberries to his customers.

Q7. Suppose another city was added on Interstate 40 between Asheville and Wilmington. What pieces of the solution change? How would they change?

Q8. Suppose another city was added in North Carolina that is not located along I-40, such as Charlotte. How would this effect the system?

Q9. Besides distance traveled, are there any other types of costs that influence Sal's system?

Q10. List the steps followed to find the optimal location of the strawberry freezer warehouse.

8.2.2 Ellie's Eco-Smoothies Store

All of the previous examples measured distance in only one dimension. For example, Lodge, Alyssa, and Grayson focused on moving their hot dog stands only left or right along the same street. Similarly, the location of the strawberry freezer warehouse used only one dimension—traveling west or east on I-40. If the location problems are expanded to include two dimensions, measuring distance becomes more difficult. In this section, finding the optimal location in a two-dimensional system will be explored.

Recently, Ellie decided to open a smoothie and juice store in the downtown Raleigh area. Patrons that work in ten local companies are expected to make daily trips to her store. Ellie needs to determine where she should open her new smoothie store to minimize the total distance customers must travel from their respective offices to Ellie's store. Thus, for Ellie's smoothie store location, the *cost* of the system is the customer's one-way distance to her store and the *weight* of the system is the number of visits per day from a particular office.

Because Ellie's store will be located in a city, the streets can be viewed as a grid or a coordinate plane. This can be seen in Figure 8.2.2, which shows the locations of the ten companies from which Ellie will draw her customers. Table 8.2.7 gives the same information as well as the weights of each of the offices.

Q11. Based on the information in Figure 8.2.2 and Table 8.2.7, where do you think Ellie should open her new smoothie store in order to minimize the total distance customers must travel?

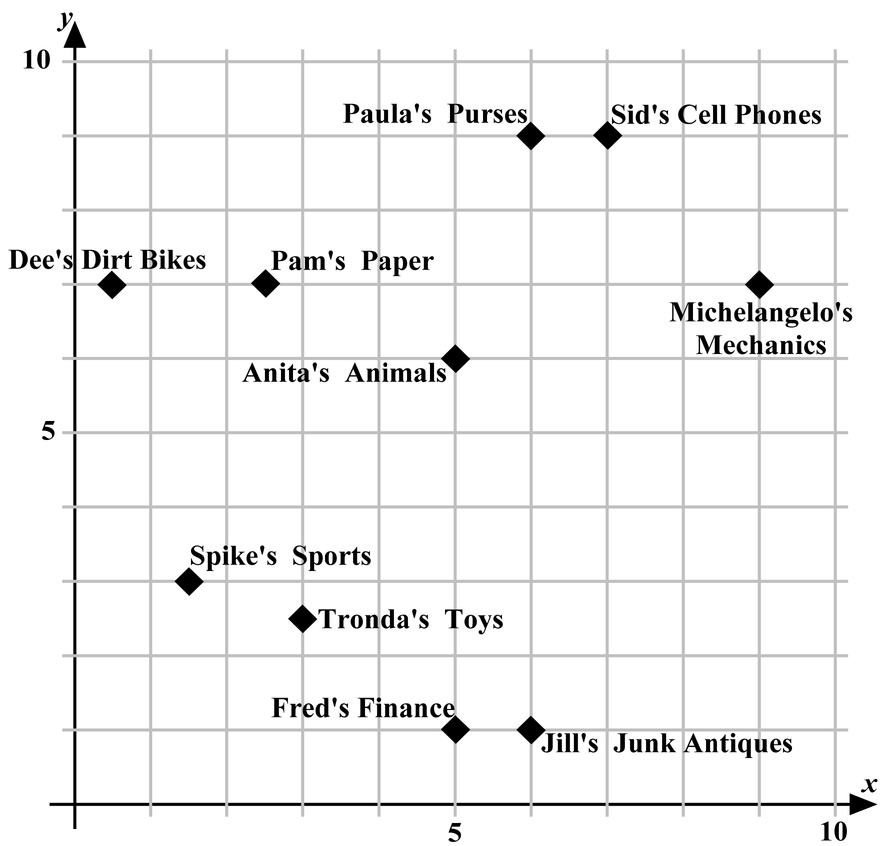


Figure 8.2.2: Locations of the offices from which Ellie will draw her customers

Office	Location (*x*, *y*)	Weight (# of visits per day)
Anita's Animal	(5, 6)	5
Dee's Dirt Bikes	(0.5, 7)	8
Fred's Finance	(5, 1)	16
Jill's Junk Antiques	(6, 1)	9
Michelangelo's Mechanics	(9, 7)	21
Pam's Paper	(2.5, 7)	19
Paula's Purses	(6, 9)	12
Sid's Cell Phones	(7, 9)	20
Spike's Sports	(1.5, 3)	3
Tronda's Toys	(3, 2.5)	17

Table 8.2.7: Locations and weights of the offices from which Ellie will draw her customers

Ellie's smoothie store location problem is very similar to Sal's strawberry freezer warehouse location problem except that Ellie must take into account the rectilinear distance of her customers. Another difference to consider is that Sal's optimal location has to be at one of the eight cities along I-40. Ellie does not have this restriction; the optimal location for her store can be anywhere in the downtown Raleigh area. Despite these differences, the weight of the system will still have the same influence and the method of finding the optimal location does not change.

Recall that the North Carolina I-40 problem was started by identifying the total weight and the median weight of the system. The same must be done in this two-dimensional problem.

First, calculate the *total weight of the system*. For Ellie's smoothie store, the total weight is 5 + 8 + 16 + 9 + 21 + 19 + 12 + 20 + 3 + 17 = 130 daily store visits.

Second, calculate the *median weight of the system* by finding half of the system's total weight:

$$\frac{5+8+16+9+21+19+12+20+3+17}{2} = 65 \text{ daily store visits.}$$

Next, use the median weight of 65 daily store visits to find the optimal location for Ellie's store. In the one-dimensional problem, the next step was to start at a fixed endpoint and calculate the cumulative sum of the weights of each location until the median weight was just surpassed. The two-dimensional problem continues in the same way, except that two separate cumulative sums are needed—one along the x-axis and one along the y-axis. Starting at the origin (0, 0), the cumulative sum of the weights is found as the offices are listed in sequential order of the x-coordinate, while moving from left to right. Table 8.2.8 gives this information.

Note: because Anita's Animals and Fred's Finance have the same x-coordinate, it does not matter which office is listed first; the outcome of the optimal location will be the same. The same principle applies to Paula's Purses and Jill's Junk Antiques.

Office	*x*-coordinate	Weight (visits)	Cumulative Weight (visits)
Dee's Dirt Bikes	0.5	8	8
Spike's Sports	1.5	3	11
Pam's Paper	2.5	19	30
Tronda's Toys	3	17	47
Anita's Animals	5	5	52
Fred's Finance	**5**	**16**	**68**
Paula's Purses	6	12	80
Jill's Junk Antiques	6	9	89
Sid's Cell Phones	7	20	109

Michelangelo's Mechanics	9	21	130

Table 8.2.8: *x*-Coordinate Optimal Location

The cumulative sum just exceeds the median weight when $x = 5$. This will represent the *x*-coordinate for the optimal location of Ellie's store. It represents the location on the *x*-plane where no more than 65 daily store visits occur to the right and left.

Q12. Do the same for the *y*-axis, starting at (0, 0), while moving from bottom to top. Complete Table 8.2.9 (the first office is given).

Office	***y*-coordinate**	**Weight (visits)**	**Cumulative Weight (visits)**
Fred's Finance	1	16	16

Table 8.2.9: *y*-Coordinate Optimal Location

Q13. At what point on the *y*-axis does the cumulative sum exceed the median weight?

This value will represent the *y*-coordinate for the optimal location of Ellie's store meaning the location on the *y*-plane where no more than 65 daily store visits occur above and below. The *x*- and *y*-coordinates can now be used to find the median location.

Q14. Where will Ellie locate her new store? Explain.

The last step is to calculate the minimum total cost of the system. So far, only weight has been considered. Now, the distances between the offices and Ellie's store need to be used.

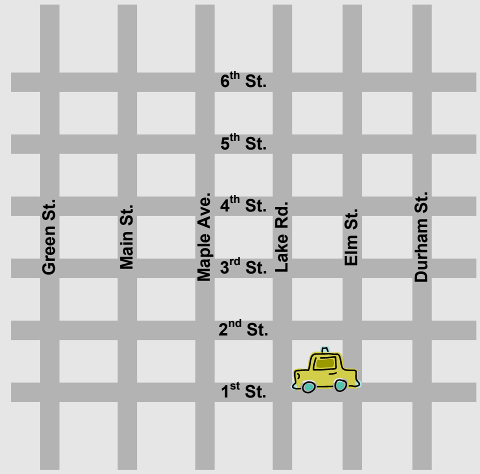
To find the minimum total cost of the system, the distance between points must be calculated. Given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, there are a number of different ways you can calculate the distance between points P_1 and P_2, depending on the context of the problem. In the context of this problem, rectilinear distance will be utilized. **Rectilinear distance** is defined as the distance between two points measured along axes at right angles.

Rectilinear Distance

To help think about rectilinear distance, consider a taxicab driving in a city. A taxicab would travel a rectilinear distance along the streets in the downtown area of a city between two points P_1 and P_2. The taxicab cannot just drive along the straight-line path between the two points; it must drive along the streets of the city. This assumes that streets are parallel or perpendicular to one another.

Rectilinear distance is given by the equation:

$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$



Rectilinear distance will be a necessary component of solving this two-dimensional location problem. Since rectilinear distance is the distance between two points measured along axes at right angles, it can be thought of as the number of city blocks (or taxicab distance) from point *A* to point *B*.

Ellie needs to minimize the total rectilinear distance that all customers must travel from their respective offices to the store. Hopefully, you found that the optimal location for Ellie's smoothie store is (5, 7), as seen in Figure 8.2.3. With this information, the total minimum cost for Ellie's new location can now be determined.

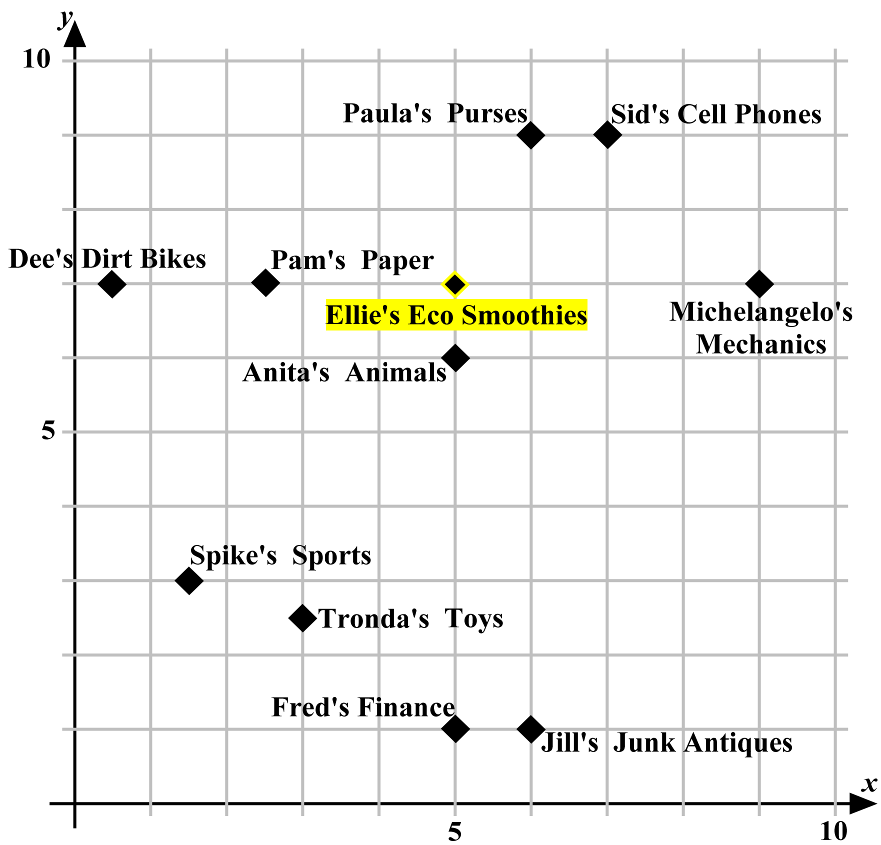


Figure 8.2.3: Location of Ellie's smoothie store in relation her customers

First, find the rectilinear distance from each office to Ellie's store. Geometrically, the rectilinear distance represents the shortest path along the city streets that one must travel to get to Ellie's new store. Recall that straight-line distances between two points cannot be used because that would mean that customers would need to walk through certain obstructions that may be impassable. Therefore, rectilinear distances are found by either using the formula given above or by counting the city blocks on the graph.

Consider, for example, Dee's Dirt Bikes. The location of Dee's Dirt Bikes is (0.5, 7). In reference to Ellie's new store at (5, 7), a customer would need to travel 4.5 blocks east to get there. Similarly, a customer from Sid's Cell Phones at (7, 9) would need to travel 2 blocks west and 2 blocks south, for a total of 4 blocks, to arrive at Ellie's new store. The values can be found by counting the city blocks on the graph, as shown in Figure 8.2.4.

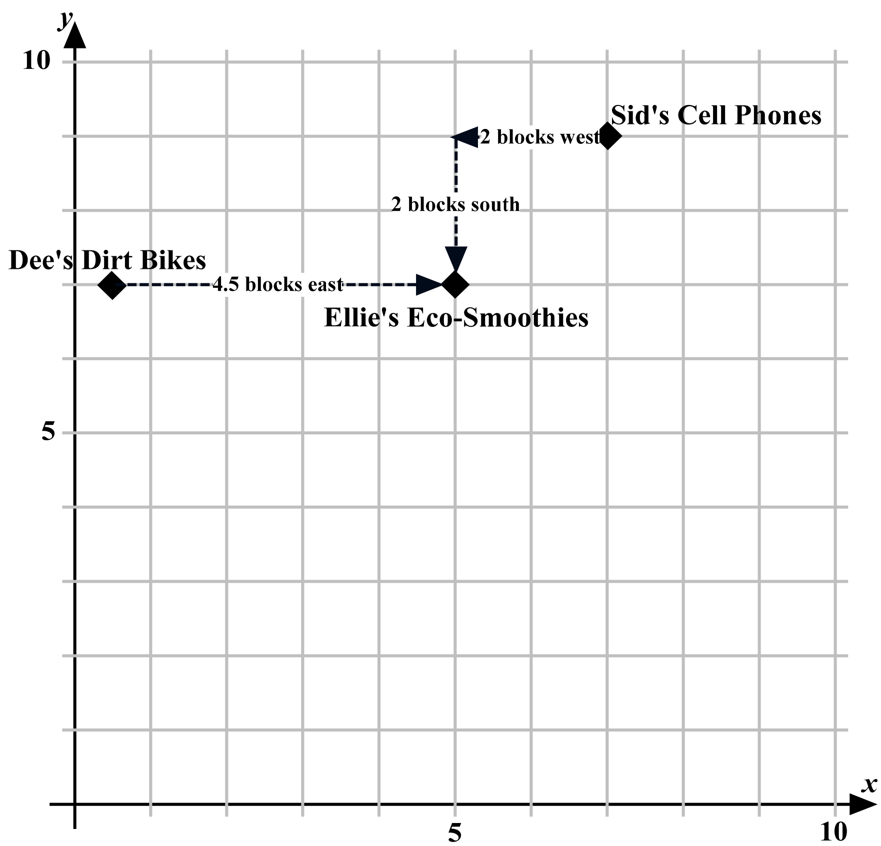


Figure 8.2.4: Distance from Dee's Dirt Bikes and Sid's Cell Phones to Ellie's store

Second, after the rectilinear distances are found, multiply each distance by the respective weight of each company. This product is the individual cost (in terms of distance traveled) for each office.

For example, since the weight for Dee's Dirt Bikes is 8 and its distance to Ellie's store is 4.5, the cost for Dee's employees is 4.5 × 8 = 36. The weight of Sid's Cell Phones is 20 and its distance to Ellie's store is 4, so the cost for Sid's employees is 4 × 20 = 80.

Algebraically, the rectilinear distance can be found by using the formula presented previously:

$$d\left(P_1, P_2\right) = \left|x_1 - x_2\right| + \left|y_1 - y_2\right|.$$

For example, the rectilinear distance from Ellie's new store at (5, 7) to Paula's Purses at (6, 9) can be calculated as:

$$\begin{aligned} d(\text{Ellie's, Paula's}) &= |5-6| + |7-9| \\ &= |-1| + |-2| \\ &= 1+2 \\ &= 3 \end{aligned}$$

In addition, the weight of Paula's Purses is 12 because the employees at Paula's Purses visit Ellie's store 12 times each day. Thus, the cost for Paula's Purses is 3 × 12 = 36.

Finally, once all of the stores' costs have been identified, the sum of the costs can be found. This sum represents the total cost of the system in terms of the distance travelled by all employees. The total cost of the system has been minimized using the notion of median location.

Q15. Calculate the distance from each company to Ellie's new store (either by using the rectilinear distance equation or by counting blocks). Then use these distances and the respective weights to calculate the total cost of this system. Table 8.2.10 will help you organize your information.

Office	**Coordinates**		**Distance from Optimal Location (city blocks)**	**Weight (visits)**	**Cost (city blocks)**
	x	*y*			
Anita's Animal	5	6		5	
Dee's Dirt Bikes	0.5	7		8	
Fred's Finance	5	1		16	
Jill's Junk Antiques	6	1		9	
Michelangelo's Mechanics	9	7		21	
Pam's Paper	2.5	7		19	
Paula's Purses	6	9		12	
Sid's Cell Phones	7	9		20	
Spike's Sports	1.5	3		3	
Tronda's Toys	3	2.5		17	
				Minimum Total Cost	

Table 8.2.10: Total cost of system given that (5, 7) is the optimal location

Q16. Is there a location that would give a smaller total cost? Why or why not?

Q17. Suppose a store is already located at the point (5, 7). Where should Ellie locate her smoothie store instead? Calculate the total cost of this new location.

Q18. List possible scenarios that would change the median location of Ellie's smoothie store.

Q19. List possible scenarios that would change the minimum total cost but not the median location.

Q20. List the steps followed to find the optimal location of Ellie's smoothie store.

Q21. Explain the differences between one-dimensional and two-dimensional location problems.

Thus far, you have seen one-dimensional location problems (Lodge, Alyssa, and Grayson's hot dog stands and Sal's freezer warehouse) as well as two-dimensional location problems (Ellie's smoothie store). Ellie needed to locate her smoothie store in a city. She considered the city as a coordinate plane, which allowed her to use the algorithm to find the optimal x- and y-coordinates for her store. In the next section, you will continue to explore two-dimensional location problems by determining the best location for Ellie to place her smoothie stand at the fair. However, a coordinate plane cannot be used to represent the city streets. Instead, geometry will be used to find the optimal location.

8.2.3 Ellie's Smoothie Stand

Ellie would like to open a smoothie stand at the Raleigh Fair. She looks at the map of the fairgrounds (Figure 8.2.5) and notes that a coordinate plane could not be used to represent the map because there are no city streets to act as a grid. Instead, the smoothie stand could be located anywhere on the fairgrounds.

Ellie notes the three most popular rides on the map: roller coaster, swings and bumper cars. She wants to maximize the number of people who visit her smoothie stand. Thus, she will locate her stand so that it is the same distance from the three most popular rides.

Q22. Looking at the map in Figure 8.2.5, where do you think the Ellie should locate her smoothie stand so that it has easy access to the roller coaster, swings and bumper cars? Explain your reasoning.

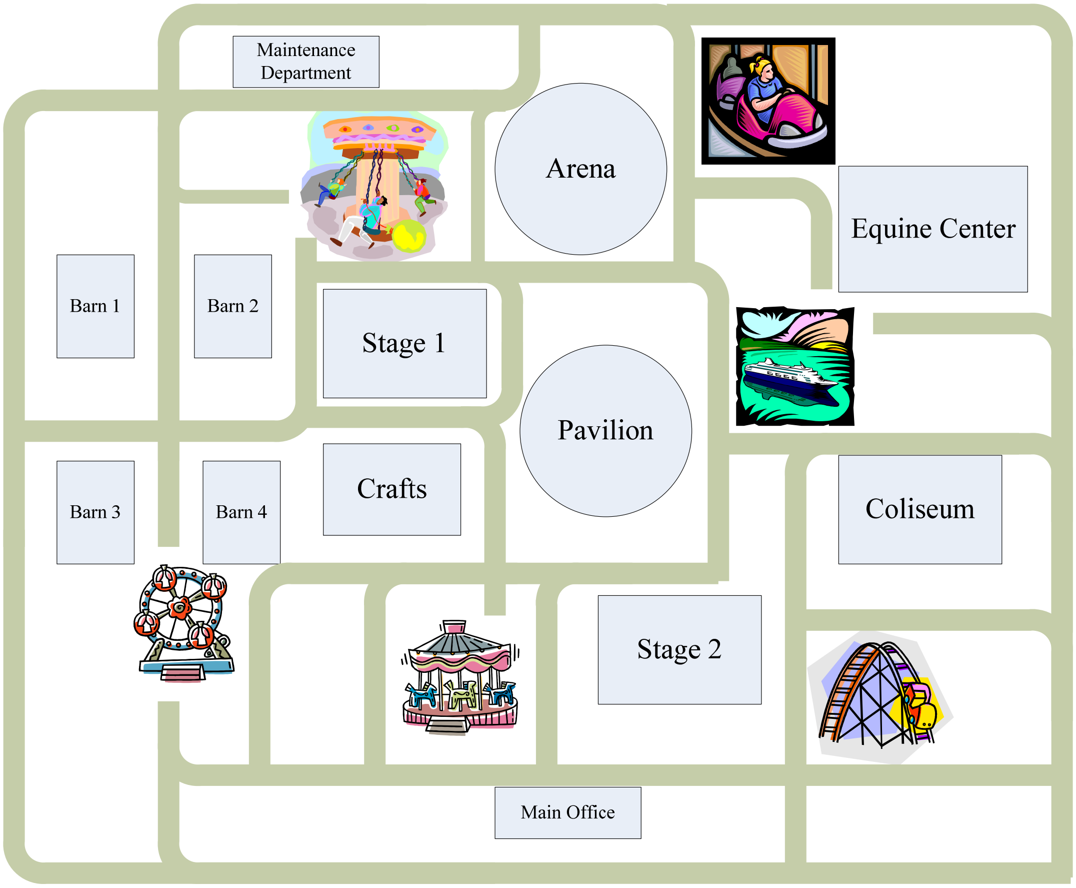


Figure 8.2.5: A map of the Raleigh Fairgrounds

Ellie wants to find the optimal location for the smoothie stand. She decides that the stand should be at the point that is the same distance to each of the most popular rides. In the previous sections, the location was determined by considering the weights of the existing facilities. Now, Ellie assumes that the three rides will provide an equal number of customers. She no longer needs to consider the weights. Therefore, she can simply to find the point that will be equidistant from each ride.

Recall from geometry that the circumcenter of a triangle is equidistant from each vertex of the triangle and that it is the point of intersection of the perpendicular bisectors of the sides of the triangle. A **perpendicular bisector** is a line that intersects a side of a triangle at its midpoint, forming two 90° angles. All of the points on the perpendicular bisector are equidistant from the endpoints, as seen in Figure 8.2.6 (i.e., point P is the same distance from point A as it is from point B). Therefore, since the circumcenter is the intersection of the perpendicular bisectors, it is will be equidistant from the vertices of the triangle.

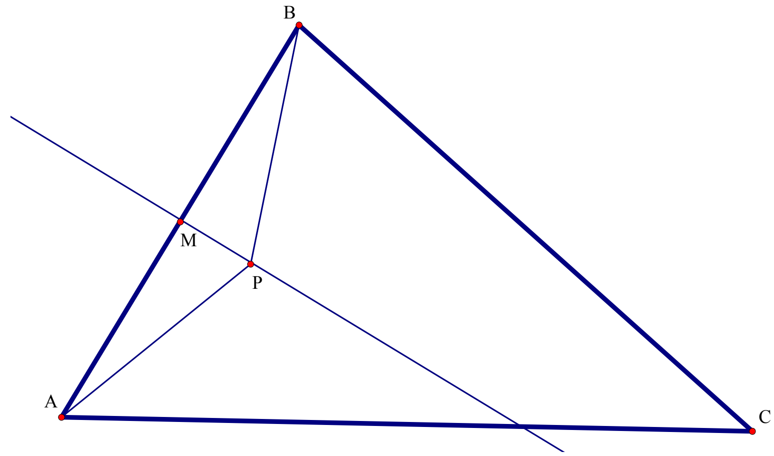


Figure 8.2.6: The perpendicular bisector of segment $\overline{AB}$

To find the circumcenter of a triangle, Ellie will perform a *geometric construction* using a compass and straightedge.

Q23. Use the map provided in Figure 8.2.5 and follow the steps given below to find the location equidistant from the three rides (i.e., the circumcenter).

Step 1. Use a straightedge to draw a triangle connecting the three rides (the roller coaster, the swings, and the bumper cars). Label the roller coaster *R*, the swings *S*, and the bumper cars *B*.

Step 2. Construct the perpendicular bisectors of each side of the triangle.

a. Place the compass point at the roller coaster (vertex *R*) and open the compass so that it is greater than half the length from the roller coaster to the swings (segment $\overline{RS}$, with the swings being vertex *S*).
b. Draw a large arc that crosses segment $\overline{RS}$, but be sure to make the arc appear on both sides of the segment.
c. While keeping the compass setting (i.e. do not adjust the compass), move the compass point from the roller coaster to the swings and draw another arc on $\overline{RS}$. This arc should cross the one drawn in step (b) twice, on either side of $\overline{RS}$.
d. Use the straightedge to connect the points of intersection of the two arcs. The point where this segment intersects $\overline{RS}$ represents the perpendicular bisector of $\overline{RS}$.

Step 3. Repeat Step 2 for the line segments connecting the bumper cars (vertex *B*) to the roller coaster ($\overline{BR}$) and the bumper cars to the swings ($\overline{BS}$).

Step 4. Locate the circumcenter, which is the intersection of the three perpendicular bisectors.

Q24. The point nearest the circumcenter represents the location for the new disaster response agency. Where should Ellie place the new smoothie stand?

Q25. Do you think the location is in a place where it is practical to locate a smoothie stand? Why or why not?

Q26. How was the method used in this section different from the methods used in the previous sections? How was it the same?

This section began by looking at a one-dimensional single facility location problem. Here, Sal needed to locate his strawberry freezer warehouse along I-40. He used an algorithm to find the median location. Then, a two-dimensional single facility location problem was explored. Ellie found a location for her smoothie store in a city. She used the same algorithm as Sal but extended it to consider two dimensions (the x- and y-axes of a coordinate plane). The final section looked at another two-dimensional single facility location problem, but this time the algorithm could not be employed. Instead, Ellie could locate her smoothie stand anywhere on the fairgrounds. Therefore, she used a geometric construction to find the optimal location for her smoothie stand.

In this next section, disaster response agencies need to be located in one or more of six cities. To determine the best locations for the agencies, binary programming will be employed.

Section 8.3: Disaster Response Agency

The United States experiences approximately 1,000 tornadoes each year. Some areas of the country consistently encounter strong and dangerous tornadoes each year, such as the region between the Appalachian and Rocky Mountains. This region has been given the nickname of *Tornado Alley*. Tornado Alley consists of Texas, Kansas, Oklahoma, Nebraska, and some of the neighboring states.

This region is relatively flat, which allows in the cold, dry air from Canada as well as the warm, moist air from the Gulf of Mexico, as shown in Figure 8.3.1. This instable weather can cause severe storms to form.

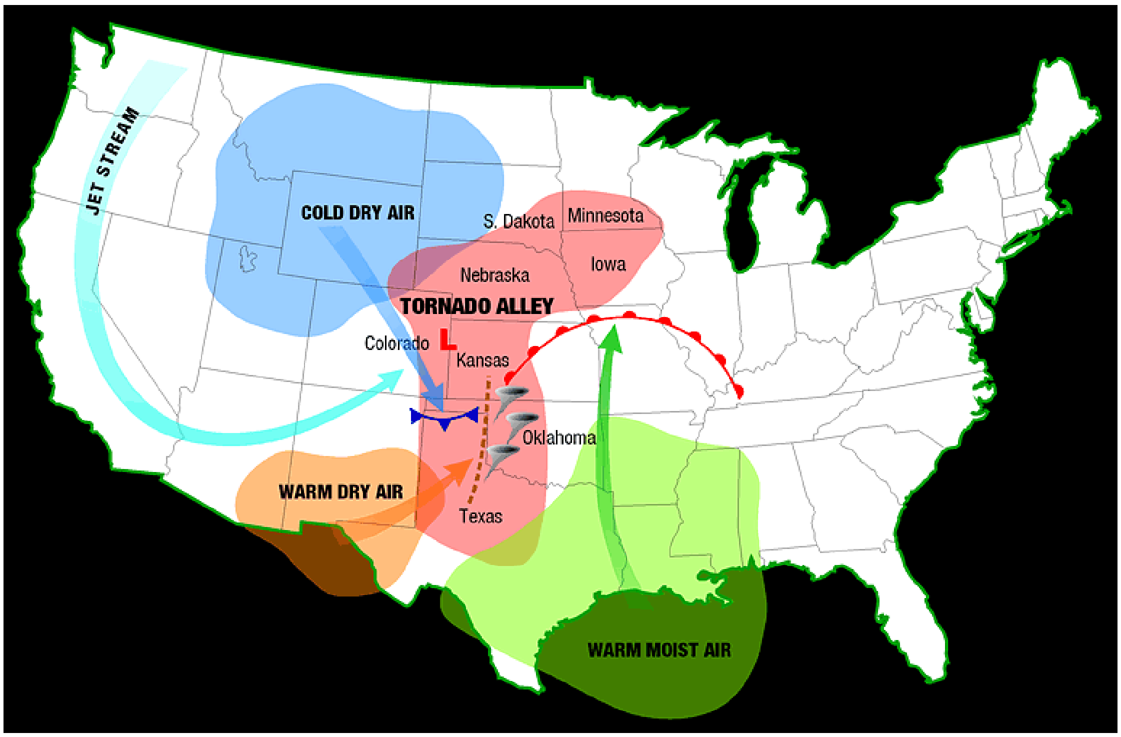


Figure 8.3.1: The weather patterns causing tornadoes in Tornado Alley

The 2008 tornado season was one of the most active in recent history. In fact, the number of reported tornadoes in March 2008 far outpaced the reports during the same timeframe in each year since 2005. There were 126 deaths from tornadoes in the United States, making it the worst fatality rate in a decade. After this deadly tornado year, government officials were interested in locating more disaster response agencies in tornado-prone areas of the United States. The new facilities house EMS and specially trained disaster personnel who can respond to the loss of life and property inherent in a tornado's violent path.

In particular, the officials are considering six cities in central Oklahoma: Edmond, Midwest City, Moore, Mustang, Norman, and Oklahoma City. These are some of the most tornado-ravaged cities in the United States. The governor would like to determine in which cities to construct new disaster response agencies (no such facilities currently exist).

That is, the governor needs to determine how many agencies need to be constructed and in which of the six cities the agencies should be located.

Due to budgetary constraints, the governor wants to build the minimum number of facilities needed to ensure that at least one facility is within a 25-minute drive of each city. The time in minutes required to drive between the six cities is shown in Table 8.3.1. In addition, the governor does not want agencies in both Moore and Oklahoma City.

Therefore, the governor wants to:

- build the minimum number of facilities.
- have at least one facility within a 25-minute drive of each city.
- not have agencies in both Moore and Oklahoma City.

Q1. Why may the governor choose to not have agencies in both Moore and Oklahoma City?

A binary programming problem will be formulated to inform the governor of how many agencies need to be constructed and in which of the six cities they should be located. A decision of "yes" will mean that an agency will be built in that city; a decision of "no" means that an agency will not be built in that city.

Cities	**To**					
From	Edmond	Midwest City	Moore	Mustang	Norman	Oklahoma City
Edmond	0	27	31	36	42	20
Midwest City	27	0	22	33	30	12
Moore	31	22	0	29	17	16
Mustang	36	33	29	0	40	24
Norman	42	30	17	40	0	28
Oklahoma City	20	12	16	24	28	0

Table 8.3.1: Driving time in minutes between six cities in central Oklahoma

Q2. Based on the information in Table 8.3.1 and the governor's constraints, where do you think the new disaster response agencies should be located? Explain your reasoning.

Q3. The governor will use binary programming to make his decision. Define the decision variables for this problem.

Q4. Since the goal is to build the minimum number of facilities, what is the equation for the objective function?

Q5. In constructing the constraints for this binary programming problem, it is helpful to think about which locations can reach each city in 25 minutes or less. Using the data from Table 8.3.1, fill in the cities that are within 25 minutes of each given city in Table 8.3.2.

City	**Cities within 25 minutes**
Edmond (x_1)	
Midwest City (x_2)	
Moore (x_3)	
Mustang (x_4)	
Norman (x_5)	
Oklahoma City (x_6)	

Table 8.3.2: Cities within 25 minutes driving time

The information in Table 8.3.2 can be used to create constraints.

For example, consider Edmond. According to Table 8.3.1, Oklahoma City is the only city that is within a 25-minute drive of Edmond. And, of course, Edmond is within a 25-minute drive of itself. Thus, the first constraint must result in at least one agency constructed in either Edmond or Oklahoma City or both to satisfy the 25-minute driving distance requirement for Edmond.

Therefore, the constraint for Edmond is $x_1 + x_6 \geq 1$.

Q6. Use similar logic to identify the remaining constraints for this problem.

Q7. Formulate the 0-1 integer programming problem in Excel. Use Excel Solver to find the optimal solution.

Q8. Where should the disaster response agencies be located?

Q9. Are these locations reasonable? Why or why not?

Q10. Suppose the governor does not want more than one agency in Oklahoma City, Moore, or Norman. Where should the disaster response agencies be located in this case?

Q11. Suppose the governor now wants to include Yukon to the list of cities that require access to a disaster response agency. The time, in minutes, to drive to each city is given in Table 8.3.3. Determine where the disaster response agencies should be located. (Assume that the governor does not more an agency to be in both Oklahoma City and Moore, as in the original problem).

Cities	**To**						
From	Edmond	Midwest City	Moore	Mustang	Norman	Oklahoma City	Yukon
Edmond	0	27	31	36	42	20	29
Midwest City	27	0	22	33	30	12	31
Moore	31	22	0	29	17	16	34
Mustang	36	33	29	0	40	24	18
Norman	42	30	17	40	0	28	47
Oklahoma City	20	12	16	24	28	0	24
Yukon	29	31	34	18	47	24	0

Table 8.3.3: Driving time in minutes between seven cities in central Oklahoma

In this section, the governor of Oklahoma wanted to find the best locations for disaster response agencies to help those in need after tornadoes damage the area. He decides that the agencies should be located in one or more of six cities in central Oklahoma. He wants to minimize the number of agencies while still maintaining a 25-minute drive to any of the cities from an agency. To solve this location problem, the governor utilized binary programming. He found that the optimal locations for the agencies should be in Norman and Oklahoma City. However, these cities may change based on other possible considerations, such as adding another city to the list or not allowing agencies to be in certain cities.

Section 8.4: Chapter 8 (Location Problems) Homework Questions

Problem 1

USC University Hospital (H_1), Cedars Sinai Medical Center (H_2), Olympia Medical Center (H_3), and California Hospital (H_4) are all located in the Los Angeles area. They are cooperating to establish a centralized blood-bank facility that will serve all four hospitals. The new facility is to be located such that distance traveled is minimized. The hospitals are located as follows: $H_1 = (5, 10)$, $H_2 = (7, 6)$, $H_3 = (4, 2)$, and $H_4 = (16, 3)$. The number of deliveries to be made per year between the blood-bank facility and each hospital is estimated to be 450, 1,200, 300, and 1,500, respectively. Assuming rectilinear travel, determine the optimum location of the blood-bank facility.

a) Draw a graph and plot the locations of the four hospitals. Label each hospital with its respective weight.
b) What is the median location?
c) What is the *x*-coordinate for the optimal location of the blood-bank facility?
d) What is the *y*-coordinate for the optimal location of the blood-bank facility?
e) Calculate the distance from each hospital to the blood-bank facility and using these distances and the respective weights, calculate the total travel cost of this system.
f) What assumption must be made concerning the deliveries of blood?
g) Is it reasonable to expect delivery trips will always be to only one hospital when a delivery is made?

Problem 2

Mrs. Williams has to pick up several items for her daughter's sweet sixteen party at various locations. She wants to locate a parking space downtown such that she can get to all the stores in a minimum amount of time without reparking. Each block is square, 100 feet on a side. Streets running north to south are numbered consecutively. Those running east to west are lettered consecutively. The bakery is at 6^{th} and E; she must walk half as fast as normal from the bakery so that she won't drop the birthday cake. At 10^{th} and D is the grocery store. The dress shop is at 12^{th} and G. Mrs. Williams picks up her daughter, Cecilia, from the hair salon at 10^{th} and G and they walk twice as fast as normal back to the car so that the wind doesn't mess up her hair. It is assumed she must stay on the sidewalks that enclose each block- distance used crossing streets is considered negligible. It is also assumed that she must return to the car after visiting each store before visiting the next one.

a) Construct a grid to represent this problem situation.
b) Determine the location of the parking space that satisfies her objective taking note of all assumptions you make in formulating your decision.
c) What is the distance from each shop to the parking space?
d) How did the walking speeds influence your solution?

Problem 3

Find the populations of Wichita, Tulsa, and Oklahoma City recorded in the 2006 U.S. Population Estimate. Considering the population data, do you think the location is optimal for serving the distribution of residents in these three cities? Why or why not? Choose a city that you think serves as a better location for the new disaster response agency. Justify your reasoning.

Problem 4

Officials would also like the people of Bartlesville, Oklahoma (shown on the map in figure 8.5.2) to be served by the new disaster response agency due to a recent increase in tornado activity in and around this city. Using a geometrical approach, determine the geographical location for the agency considering that 4 cities (Oklahoma City, Tulsa, Bartlesville, and Wichita) now require coverage. Do you think the location is in a place where it is practical to locate a disaster response agency? Why or why not? What type of metric distance would you use to calculate the distance from the new agency to each city? Rectilinear or Euclidean? Why?

Problem 5
Refer back to the scenario posed in Section 8.6. Suppose the minimum driving time was set at 30 minutes instead of 25. How does that change the problem formulation? Does it have an effect on the optimal solution?

Problem 6
Referring back again to the scenario posed in Section 8.6, take away the constraint limiting the locations of agencies in both Moore and Oklahoma City and find a solution. Is this solution unique? Interpret your findings in the context of the problem.

Problem 7
In metropolitan areas it is common for high schools to share football stadiums since each school may not have enough land adjacent to it to construct a large football stadium complex. Figure 8.6.1 shows a representation of the downtown Atlanta area. Each block or shape contains one high school and the surrounding area in which students live who attend that high school. The nodes represent candidate locations for new football stadiums that will be constructed in the city in 3 years. Each football stadium can serve all schools that are adjacent to it. For example, potential football stadium 1 could serve high schools 8, 9, and 11. Your objective is to determine the minimum number of football stadiums that need to be constructed to cover all high schools and what high schools do they serve.

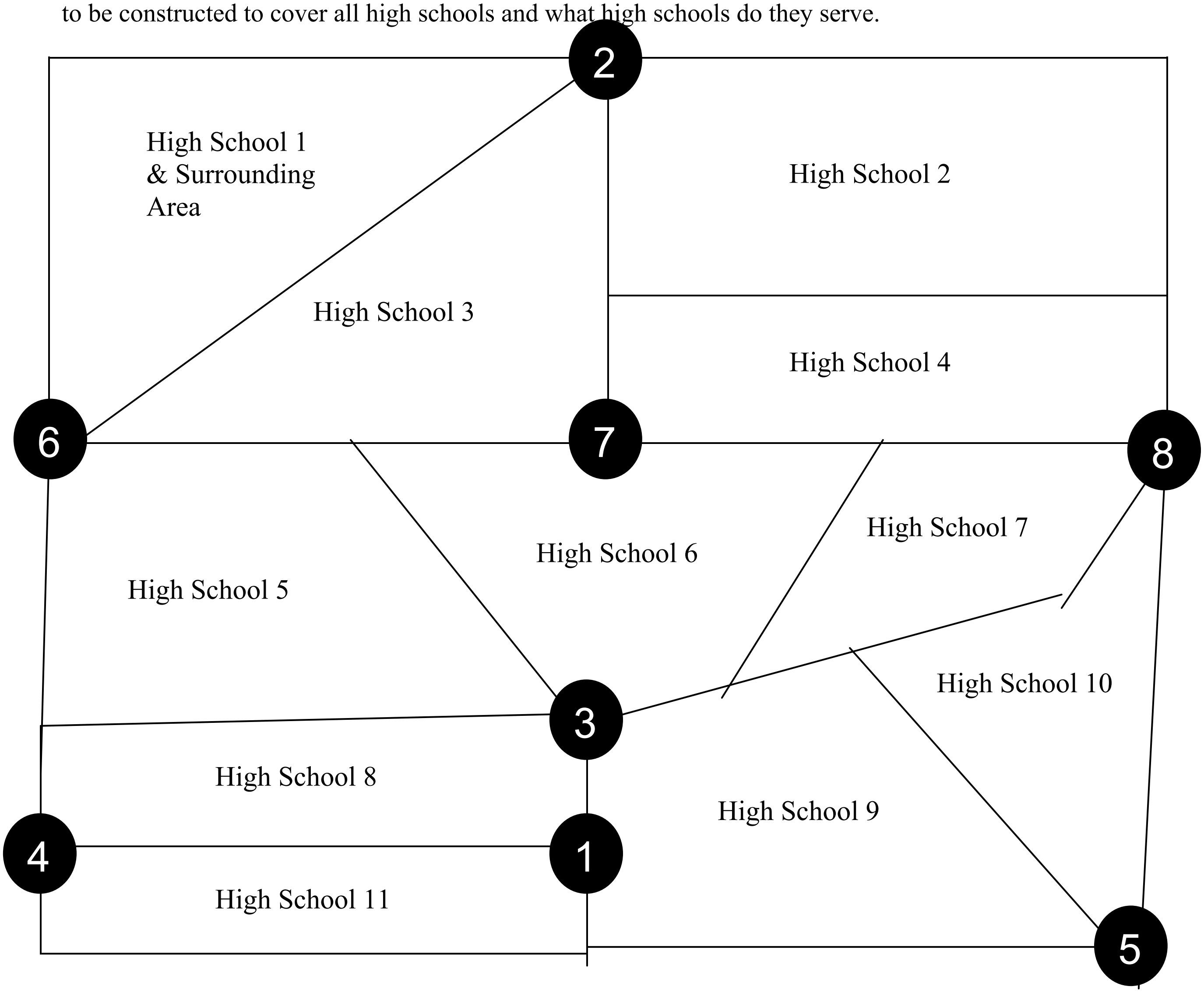


Figure 8.6.1: Atlanta, GA High School Districts

a) What kind of mathematical programming problem is this?
b) Define the decision variables and objective function.
c) What are the constraints? Complete Table 8.6.3 to assist you.

High School	Adjacent candidate football stadium locations
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

Table 8.6.3: Football stadiums adjacent to high schools

d) Formulate and solve the problem using the Excel and Solver.
e) What is special about high schools 8 and 9?
f) Some of the constraints created using Table 8.6.3 are redundant, meaning that they can be removed from the problem without changing the solution. Which constraints are redundant? Remove them and reformulate the problem in Excel. Did you get the same solution?

Problem 8

Ramona Ferraro is an up and coming purse designer that will sell high-end purses to clients located in 4 regions of the United States. Retail price for each purse will be between $400 and $500. Ramona is considering renting an existing manufacturing facility in one of three different cities: Cincinnati, Chicago, or St. Louis. The cost of producing a purse in a city and shipping it to a region along with each region's annual demand is given in Table 8.6.4.

	Cost to Deliver by Region ($)			
City	(1) Northeast	(2) South	(3) Midwest	(4) West
(1) Cincinnati	206	225	182	317
(2) Chicago	225	250	170	328
(3) St. Louis	245	221	175	311
Region Demand	4,000	2,000	5,000	1,500

Table 8.6.4: Variable cost and annual demand

Assume each potential manufacturing facility has sufficient capacity to produce the purses for all regions. The annual cost to operate each potential facility is shown in Table 8.6.5.

City	Cost ($)
Cincinnati	100,000
Chicago	115,000
St. Louis	90,000

Table 8.6.5: Annual fixed costs

Determine where Ramona should rent manufacturing space to minimize the total annual cost of meeting her clients' demand.

a) What is the difference between fixed cost and variable cost? Provide examples of each.
b) What would be the total cost if Ramona decided to manufacture the purses in Cincinnati?
c) What would be the total cost if Ramona decided to manufacture the purses in Chicago?
d) What would be the total cost if Ramona decided to manufacture the purses in St. Louis?
e) In which city do you think she should manufacture the purses?
f) How do you think this problem could be solved using integer programming with 0-1 decision variables?

Problem 9

Flexxon is the world's largest oil, gas, and energy company. With the rising costs of crude oil and increasing pressure from the United States government to invest in various forms of renewable energy, Flexxon has decided to increase its funding of wind power projects. Wind power is converted to useable electricity through the use of a wind turbine that is placed on a small plot of land and "harvests" the wind from surrounding farm lands. Flexxon is interested in building a set of wind turbines in Texas, Colorado, North Dakota, and/or California. Using the data provided, determine where Flexxon should purchase land for their wind power projects (can be in more than one location) in order to maximize their savings from replacing oil powered electricity with wind generated electricity. Assume that Flexxon has $10 million available for investment.

	Cost of Land & Turbine Equipment	**Expected Savings from Relying on Wind vs. Oil**
(1) California	$6 million	$9 million
(2) Colorado	$3 million	$5 million
(3) Texas	$5 million	$6 million
(4) North Dakota	$2 million	$4 million

Table 8.6.6: Wind Power Project Data

a) Define the decision variables.
b) What is equation for the total expected savings Flexxon will incur?
c) What is the equation showing how much money they need to fund the potential projects?
d) Formulate the 0-1 integer programming problem in Excel.
e) Use the Excel Solver to find the optimal solution.
f) How much of the initial $10 million budget do they have left over?
g) What if Flexxon decided that they wanted to have wind power projects in North Dakota or Colorado, but not in both? How does the integer programming problem change?

Problem 10

Officials are interested in locating a new disaster response agency in a segment of Tornado Alley near Wichita, Kansas; Tulsa, Oklahoma; and Oklahoma City, Oklahoma.

Figure 8.6.2: A map showing a segment of Tornado Alley

a) Looking at the map in Figure 8.6.2, where do you think the officials should build the new disaster response agency so that it can easily access Wichita, Tulsa, and Oklahoma City? Explain your reasoning.

b) Using the map, find the location equidistant from the three cities (Wichita, Tulsa, and Oklahoma City).

c) The city nearest the circumcenter represents the location for the new disaster response agency. In what city should the government place the new disaster response agency?

d) Search the Web to find the latitude and longitude coordinates of Wichita, Tulsa, and Oklahoma City.

e) Find the latitude and longitude of the city that is closest to the circumcenter location you found.

f) Do you think the location is in a place where it is practical to locate a disaster response agency? Why or why not?

Chapter 8 Summary

What have we learned?

Terms

Cost A value (such as money, distance, or time) that is to be minimized

Circumcenter The point of intersection of the perpendicular bisectors of the sides of the triangle

Median Location The first existing facility location where the cumulative weight of the existing facilities up to that point is at least half of the total weight of all existing facilities; the median location splits the total cost in half

Median Weight Half of the total weight of the system

Perpendicular Bisector The line that is perpendicular to a segment at its midpoint

Rectilinear Distance The distance between two points (x_1, y_1) and (x_2, y_2), measured along axes at right angles, found using the following formula:
$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$

Single Facility Location Problem A type of location problem in which the location for only one facility needs to be found

System A collection of organized components, forming a structured whole

Weight An entity that contributes to the overall system in such a way that it could imbalance or skew the system (such as number of customers or number of truckloads)

Chapter 8 (Location Problems) Objectives

You should be able to:

- Given a contextual location problem (1-D or 2-D), define the cost and weight of a system
- Given a contextual location problem (1-D or 2-D), find the optimal solution by calculating the median weight and determining which location's cumulative sum of weights exceeds the median weight or by using Excel Solver
- Use the geometric circumcenter of equally weighted, non-collinear points to find the optimal location
- Use binary programming with equally weighted, non-collinear points to find the optimal location
- Analyze and interpret the optimal location; make decisions based on results

Chapter 8 Study Guide

1.

References

Alexander, D. (2003). *Elementary geometry for college students* (3rd ed.). Boston: Houghton Mifflin.

Francis, R. L., McGinnis, F., & White, J. A. (1991). *Facility layout and location: An analytical approach* (2nd ed.). Englewood Cliffs, New Jersey: Prentice Hall.

Kay, M. (2008). *ISE 453 & ISE 754 Course Notes*. North Carolina State University, Raleigh, NC.

NOAA. (2010, January 12). Monthly and annual U.S. tornado summaries. *Storm prediction center monthly and annual summaries*. Retrieved December 1, 2010, from http://www.spc.ncep.noaa.gov/climo/online/monthly/newm.html

Tornado Climatology. (n.d.). *Questions and answers about tornadoes: Climatology*. Retrieved November 29, 2010, from http://www.nssl.noaa.gov/primer/tornado/tor_climatology.html

Winston, W. (2004). *Operations research: Applications and algorithms* (4th ed.). London: Thomson Learning.

Section 9.0: Introduction to Graph Theory

This chapter provides an introduction into **graph theory—**the study of graphs. Graph theory is a large area of research for mathematicians and computer scientists. However, only two mathematical techniques involving graphs are presented in this chapter.

First, a technique called Kruskal's algorithm is used to find paths between houses after a tornado destroyed a town's road system. The roads need to be rebuilt so that emergency workers and volunteers have a way to travel from one house to another. However, the lengths of these rebuilt roads need to be minimized due to time and money constraints. Kruskal's algorithm is used to find the minimum total lengths of the rebuilt roads.

Second, a mathematical technique called Dijkstra's algorithm is used to determine the best route for a company that transports urgently needed medical supplies. With the high cost of gasoline and sometimes short notice to transport the supplies, the route used to transport these medical supplies is an important consideration. Dijkstra's algorithm is used to find the shortest routes to transport medical supplies from a supplies company to medical offices. Next, this same medical supplies company wants to build a new warehouse next to one of its customers. The possible locations for the new warehouse are found by performing Dijkstra's algorithm repeatedly.

Third, Dijkstra's algorithm is used again, this time to explore how long it would take for a rumor to spread. Then, the algorithm is used repeatedly to find which person should start the rumor in order for it to spread most quickly. Although the medical supplies context and the rumor context are very different, both problems can be solved by utilizing Dijkstra's algorithm to find the shortest path.

Before Kruskal's algorithm and Dijkstra's algorithm are presented, graph theory is introduced through the context of social networks.

9.0.1 Six Degrees of Separation

The phrase **six degrees of separation** is common in our country. The idea is that, through social networks, any person in the United States can be connected to any other person by an average of six people.

A Social Experiment

The theory of *six degrees of separation* has been around for several generations, but in 1967, Stanley Milgram performed an experiment to study this theory. In this experiment, Milgram sent out several hundred letters to people in Nebraska and Kansas. These letters contained instructions for the recipients to follow. The eventual goal was for the letter to reach a specific person in Boston, Massachusetts. Therefore, the letter instructed the recipients to either (1) send this letter to the person in Boston if they were familiar with them (i.e. if they were on a first-name basis with them), or (2) send it to someone who they were on a first-name basis with and who may have a better chance of knowing the person in Boston. If the recipient did not know the person in Boston, they may choose to send it to someone who lives closer to Boston than they do, or they may decide to send the letter to someone who has a large group of friends and acquaintances so that there was a better chance that the letter eventually arrived at the person in Boston. For the people who chose to participate, the average number of people it took to get to the person in Boston was 5.5. Hence, this experimented supported the idea of *six degrees of separation*.

Although this theory is sociological, it has leaked into the world of pop culture. For example, in the early 1990s, many people began to play the game **six degrees of Kevin Bacon**. In this game, people choose a random actor/actress and try to find a connection through movies to the popular actor Kevin Bacon. Kevin Bacon is known for his roles in Mystic River, Apollo 13, A Few Good Men, and Footloose.

To better understand the six degrees of Kevin Bacon game, consider the following example. If Angelina Jolie is chosen as the beginning actor/actress, she was in *Mr. & Mrs. Smith* (2005) with Brad Pitt, and Brad Pitt was in *Sleepers* (1996) with Kevin Bacon. Thus, Angelina Jolie is connected to Kevin Bacon through one person, so her **Bacon number** is 2. The Bacon number refers to the minimum number of degrees of separation (e.g., Kevin Bacon's Bacon number is 0 and Brad Pitt's Bacon number is 1).

This is not the only connection from Angelina Jolie to Kevin Bacon. Figure 9.0.1 shows a graph with a few of the connections between Angelina Jolie and Kevin Bacon. Some paths are obviously longer than others. A **path**, in this context, refers to a connection between two people through movies. The goal, when playing this game, is to find the path with the minimum number of connections. There could be several connections from Angelina Jolie to Kevin Bacon, but the shortest connections are of length 2.

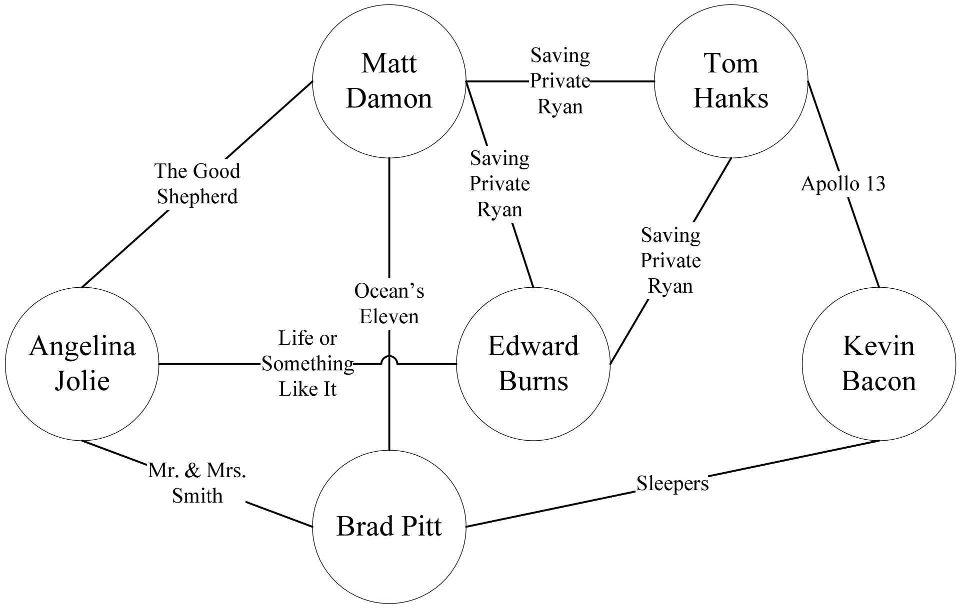


Figure 9.0.1: An example of a graph for the Kevin Bacon game

Q1. Choose an actor/actress and connect him/her to Kevin Bacon through movies.
- a. Create a graph similar to the one in Figure 9.0.1 showing more than one way to connect this actor/actress to Kevin Bacon.
- b. Determine the Bacon number (i.e., the minimum number of people it takes to get from this actor/actress to Kevin Bacon).

For the rock fans out there, there is also *six degrees of Dave Grohl*, where many rock music forums discuss the connections between various musicians and Dave Grohl, the drummer from Nirvana and the lead vocalist from Foo Fighters.

Also, social networking sites such as Friendster, MySpace, and Facebook use the idea of degrees of separation. For example, a graph similar to the one in Figure 9.0.1 can be created for Facebook friends. A fictitious example is given in Figure 9.0.2, where the connections between people represent Facebook friendships. In this example, Joey is not friends with Jonathan. However, they are connected through friends. Possible paths from Joey to Jonathan include:

- Joey-Jordan-Jonathan
- Joey-Donnie-Jonathan
- Joey-Danny-Donnie-Jonathan

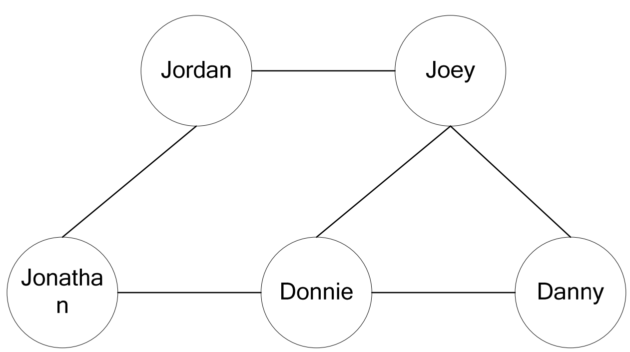


Figure 9.0.2: An example of a graph connecting Facebook friends

Figures 9.0.1 and 9.0.2 are examples of graphs. However, the kinds of graphs seen in this chapter are not the same as what you are used to seeing in Algebra (e.g., number lines or Cartesian planes). A **graph** is a broad term meaning a set of points, called **nodes**, connected by segments, called **arcs** (note: nodes are sometimes called *vertices* and arcs are sometimes called *edges*, but in this chapter, only the terms *nodes* and *arcs* are used). Figure 9.0.3 shows an example of a graph with 6 nodes and 10 arcs. Note that arcs can cross, like arc *AD* and arc *BC* in the figure. Crossing arcs do not form another node.

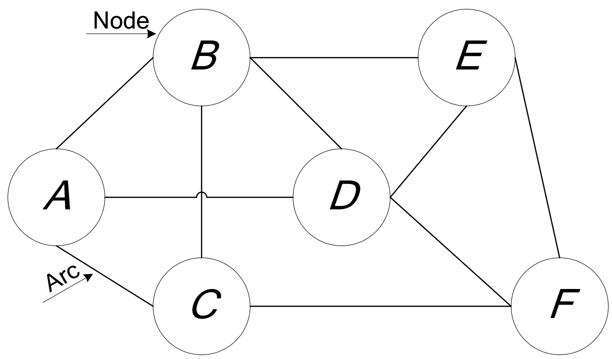


Figure 9.0.3: Example of a graph with 6 nodes and 10 arcs

Graphs, such as the one shown in Figure 9.0.3, are explored throughout this chapter. In the next section, a graph is used to determine which roads should be reconstructed after a tornado came through a town.

Section 9.1: Road Reconstruction

A tornado passed through a small Midwestern town called Nashatuck, destroying the entire road system in the town. Some roads need to be reconstructed so that it is possible to get from each house to every other house. The path from one house to another may be indirect, but there needs to be some sort of path between each house for emergency workers and volunteers to travel.

In the graph shown in Figure 9.1.1, the nodes represent houses and the arcs represent the roads between the houses before the tornado. The numbers on the arcs correspond to the distance, in miles, between each house. For example, house *A* is two miles from house *B*.

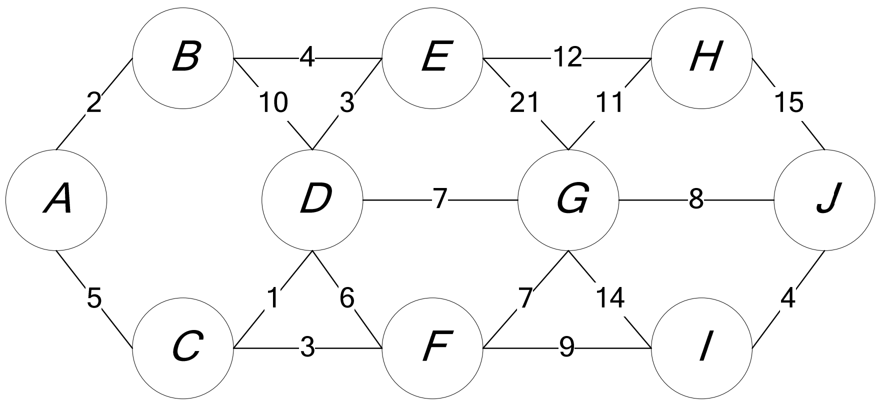


Figure 9.1.1: A graph representing the road system in Nashatuck

Note that the graph is not scaled to represent the distances as lengths. For example, the distance between house *C* and house *D* is 1 mile, and the distance between house *D* and house *F* is 6 miles. However, in Figure 9.1.1, arc *CD* and arc *DF* have the same length. This is common practice in graph problems. However, the distances and not the lengths are used to work through the problem.

To reduce the cost of reconstruction, the Nashatuck government officials want to minimize the lengths of the roads they rebuild. They need to determine which roads should be reconstructed. To do so, they create a minimum spanning tree.

9.1.1 Minimum Spanning Trees

Minimum spanning trees are used to determine which roads should be reconstructed. To understand minimum spanning trees, some definitions need to be given.

First, a **tree** is a graph in which there are no circuits. A **circuit** is a path that starts and ends in the same node. Figures 9.1.2 and 9.1.3 show examples and non-examples of trees, respectively. The graphs shown in Figure 9.1.3 are not trees because they contain circuits, shown with dotted lines.

Next, a **spanning tree** is a tree (so there are no circuits) that connects all of the nodes. Thus, a path exists from each node to every other node, but it may not be a direct path. Examples of spanning trees are given in Figure 9.1.4.

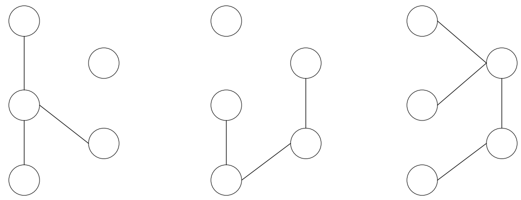


Figure 9.1.2: Examples of trees

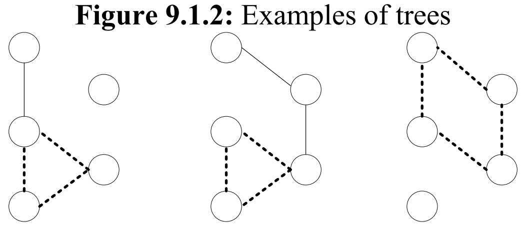


Figure 9.1.3: Non-examples of trees

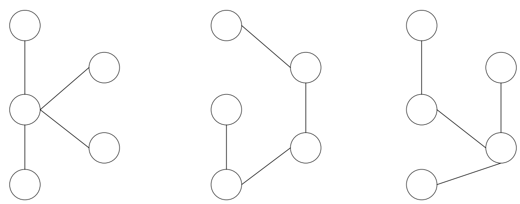


Figure 9.1.4: Examples of spanning trees

If a tree is **weighted** (i.e., the arcs are given a "weight" such as distance or cost), then a minimum spanning tree can be found. A **minimum spanning tree** is a spanning tree of the least total weight. Therefore, in a minimum spanning tree, every node is connected directly or indirectly to every other node in the graph, and the total weight as small as possible.

Minimum spanning trees can be used to determine which roads need to be rebuilt after being destroyed by the tornado. The weights in this problem refer to the number of miles between houses (shown on the arcs in Figure 9.1.1). Some, but not all, roads in the town need to be reconstructed to allow emergency workers and volunteers to have access to all of the houses. They need to determine which roads to rebuild so that the cost of reconstruction is minimized. To minimize this cost, the total number of miles of the rebuilt roads should be as small as possible. Therefore, it makes sense that minimum spanning trees should be used to solve this problem.

Q1. In the definition of a minimum spanning tree, circuits are not allowed. Why would circuits not be needed when rebuilding roads after a tornado?

Q2. Using Figure 9.1.1, find a spanning tree that connects all of the houses. Find the total weight of your spanning tree (i.e., add up the number of miles on the arcs in your spanning tree).

Q3. Compare your spanning tree and the total weight of your spanning tree with a neighbor.

Q4. What is the minimum total weight of the spanning trees for all students in the class? Is this the minimum possible total weight? How do you know?

To solve this problem, we need to implement an algorithm. An **algorithm** is a set of steps followed to solve a problem. In many cases, these steps are repeated until there is a reason to end. For example, if someone was asked how to walk to a wall, he/she might say, "Take a step and repeat until you reach the wall." That is a very simple algorithm.

For this problem, Kruskal's algorithm is used. Kruskal's algorithm is an example of a **greedy algorithm** because at each step of the problem, the best choice is made for that particular step. When this is done, the result is a minimum spanning tree that connects each of the houses in the network.

9.1.2 Kruskal's Algorithm

Step 1: Find the arc of the least weight and mark it.
Note: the arc of the least weight does not necessarily correspond with a particular node. For example, the algorithm does not necessarily begin with node *A*.

In this example, the arc of the least weight is *CD*, which is of weight 1, meaning that house *C* is 1 mile from house *D*. Since this road is relatively short, it makes sense that the government officials would make sure to rebuild this road. The graph in Figure 9.1.5 shows arc *CD* marked.

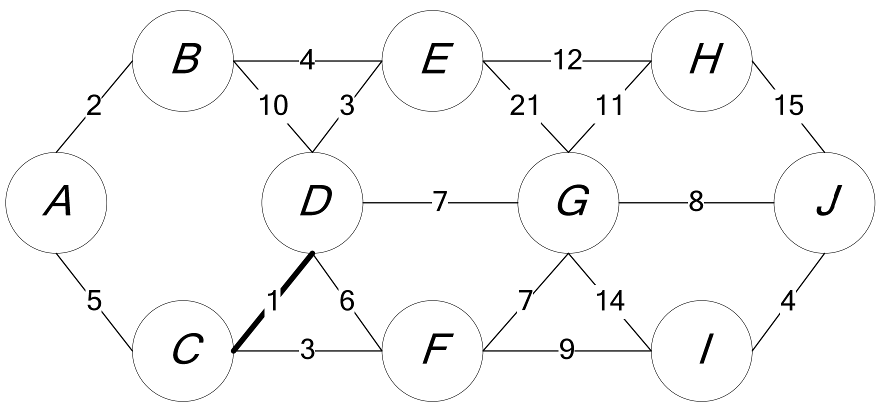


Figure 9.1.5: Step 1 of Kruskal's algorithm, where the arc of the least weight is arc *CD*

Step 2: Find the arc of the next least weight and mark it.
In this example, the arc of the next least weight is arc *AB*. Since house *A* is only 2 miles from house *B*, this road should also be reconstructed. Arc *AB* is marked in Figure 9.1.6.

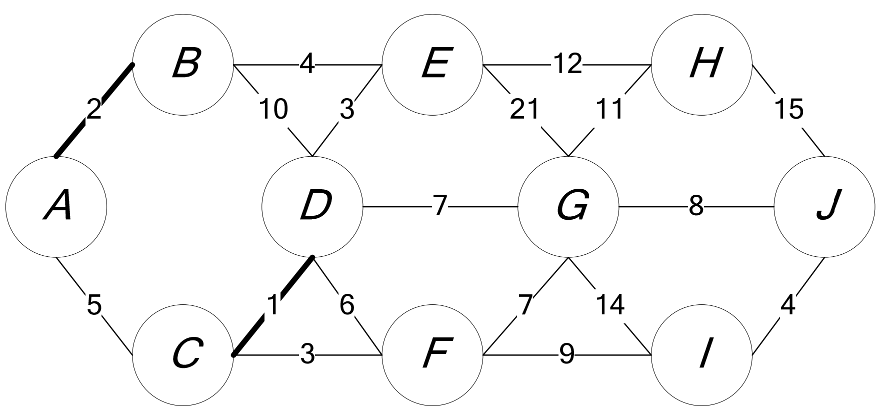


Figure 9.1.6: Step 2 of Kruskal's algorithm, where the arc of next least weight is arc *AB*

Step 3: Repeat Step 2 until each node is connected. If the arc with the next least weight creates a circuit, skip it and go to the arc with the next least weight after that.

Remember that a circuit is a path that starts and ends at the same node. If a circuit is created, then the graph is no longer a tree.

For the road reconstruction example, there are two arcs of with the next least weight; *DE* and *CF* are both of length 3. Both of these arcs can be marked, as shown in Figure 9.1.7.

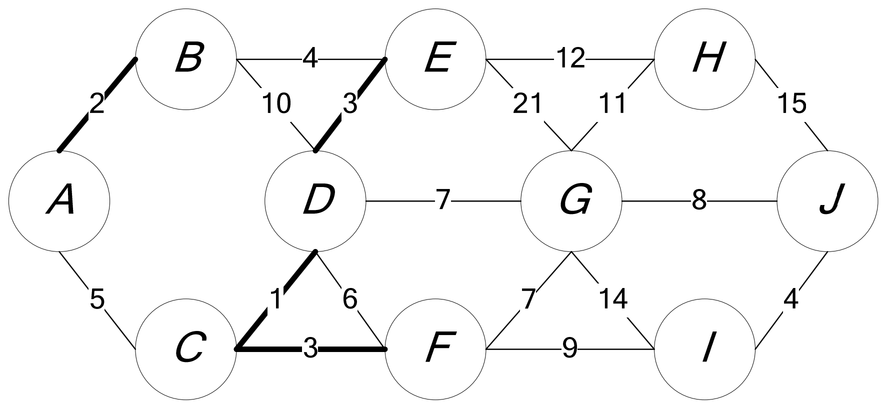


Figure 9.1.7: Step 3 of Kruskal's algorithm, where arcs *DE* and *CF* are both marked

Again, two arcs share the next least weight: *BE* and *IJ* are both of length 4. That is, the length of the road between houses *B* and *E* is four miles, and the length of the road between houses *I* and *J* is also four miles. Both of these arcs are marked in Figure 9.1.8.

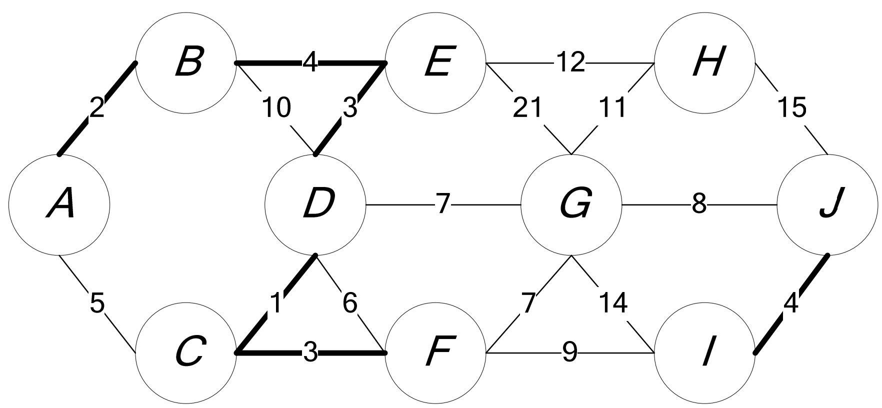


Figure 9.1.8: Continuing Step 3 of Kruskal's algorithm, where arcs *BE* and *IJ* are both marked

At this point, houses *A*, *B*, *C*, *D*, *E*, and *F* are connected to one another as are houses *I* and *J*. However, the emergency workers are still unable to access all of the houses in the town. For example, if an emergency worker is at house *A*, they would have no way to get to houses *G*, *H*, *I*, or *J*. Therefore, the algorithm is continued until all houses can be accessed.

The arc of next least weight is arc *AC*, with a weight of 5. However, if *AC* is marked, there would be a circuit, as shown in Figure 9.1.9. Since houses *A* and *C* are already connected to one another via the path *A*-*B*-*E*-*D*-*C*, it would be unnecessary to rebuild the road connecting house *A* to house *C*.

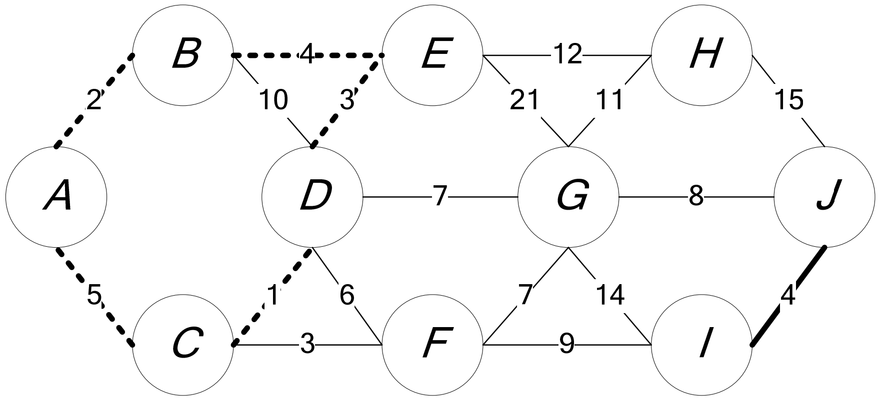


Figure 9.1.9: Choosing arc *AC* creates a circuit

Therefore, the government officials skip this arc and go on to the arc of next least weight, which is *DF*, of weight 6. Notice again that a circuit is created if this arc is marked, as shown in Figure 9.1.10.

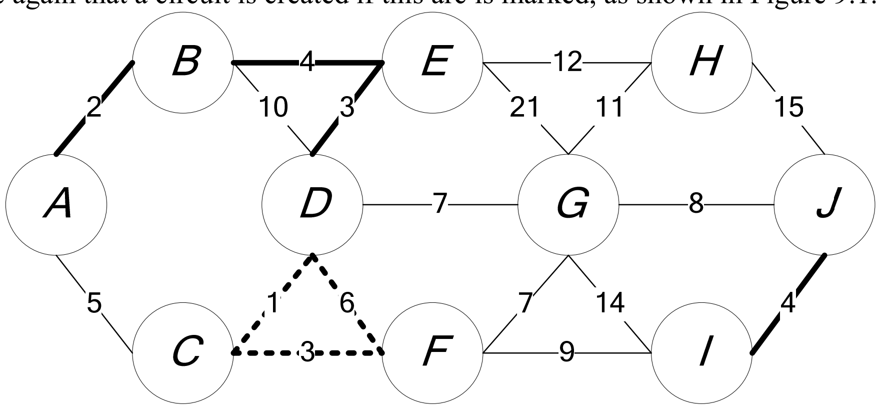


Figure 9.1.10: Choosing arc *DF* creates a circuit

Thus, the government officials skip arc *DF* and move on to the next least weight. Both arcs *DG* and *FG* have a weight of 7. However, if *both* arcs are marked (as was done earlier), a circuit is created, as shown in Figure 9.1.11.

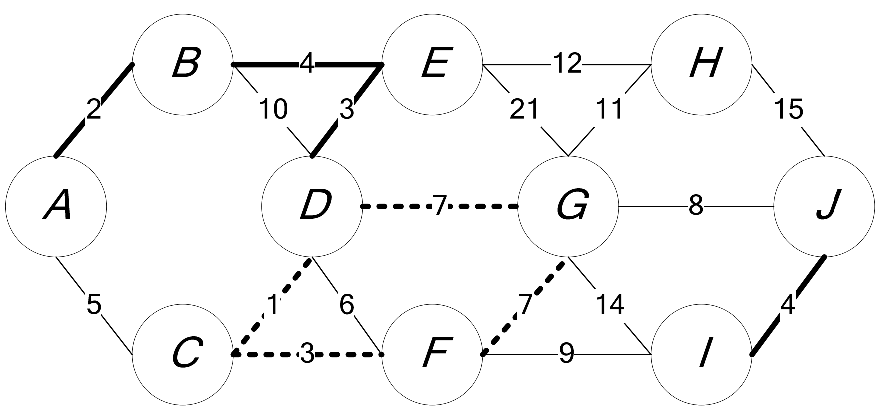


Figure 9.1.11: Choosing both arcs *DG* and *FG* creates a circuit

Therefore, either arc *DG* or arc *FG* must be marked, but not both. It is not clear which one to choose because they are of the same weight. Therefore, the government officials choose *FG* for now. Later, they go back, choose *DG*, and see which arc was the better choice. Figure 9.1.12 shows the graph with arc *FG* marked.

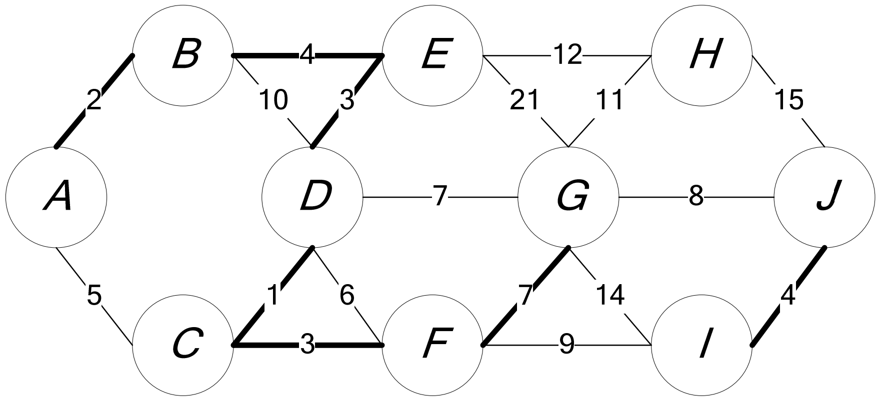


Figure 9.1.12: Continuing with Kruskal's algorithm after choosing arc *FG*

Looking at the graph in Figure 9.1.12, only two more roads need to be reconstructed so that each house can be accessed. Continuing with Kruskal's algorithm, the last two arcs chosen are *GJ* and *GH*, respectively. Figure 9.1.13 shows the complete minimum spanning tree.

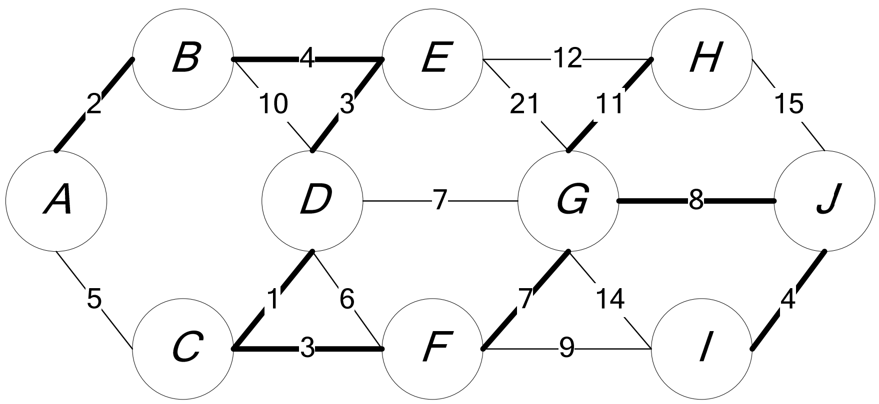


Figure 9.1.13: Completed minimum spanning tree

Q5. How do you know when to stop the algorithm?

Q6. Find the total weight of this minimum spanning tree. What does this mean in terms of the problem?

The steps of Kruskal's' algorithm are given below.

Kruskal's Algorithm

Step 1: Find the arc of the least weight and mark it.

Step 2: Find the arc of the next least weight and mark it.

Step 3: Repeat Step 2 until each node is connected. If the arc with the next least weight creates a circuit, skip it and go to the arc with the next least weight after that.

Recall that the government officials also wanted to find the minimum spanning tree using arc *DG* rather than arc *FG*, as shown in Figure 9.1.14.

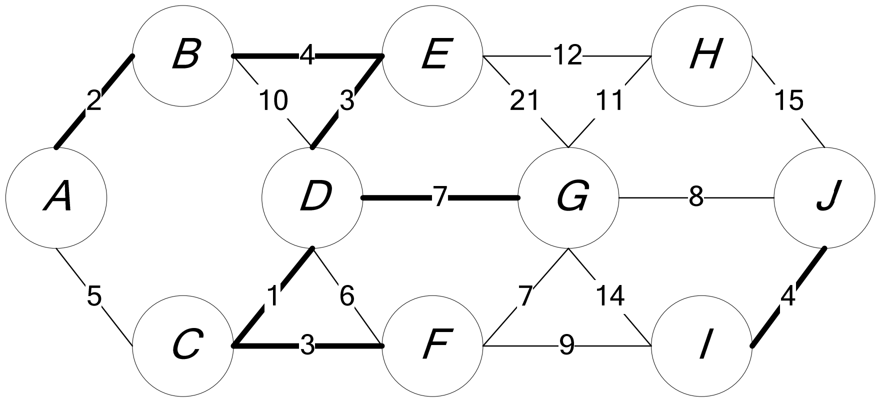


Figure 9.1.14: Continuing with Kruskal's algorithm after choosing arc *DG*, instead of arc *FG*

Q7. Find the minimum spanning tree using arc *DG* instead of arc *FG*.

Q8. Find the total weight of this minimum spanning tree.

Hopefully, you found that the total weight did not change depending on whether arc *FG* or arc *DG* was chosen. This will always be the case. In other words, if you have to choose between two arcs of the same length when creating a minimum spanning tree, which arc you choose will not make a difference in the total weight.

Q9. Using the minimum spanning tree you found in Q7, determine the path an emergency worker would follow if he started at house *A* and needed to get to house *G*.

Q10. Using the minimum spanning tree you found in Q7, determine the path an emergency worker would follow if he started at house *E* and needed to get to house *F*.

Q11. Using the minimum spanning tree you found in Q7, determine the path an emergency worker would follow if he started at house *F* and needed to get to house *I*.

Q12. Suppose the state police headquarters are located along arc *GI*. Therefore, the government officials need to ensure that this arc is part of the minimum spanning tree. What new solution would you provide? Explain how you found this new spanning tree.

Q13. Suppose arc *BE* contains a bridge and is therefore too difficult to rebuild quickly (also assume that the spanning tree needs to include arc *GI* for the state police headquarters). What new solution would you provide? Explain how you found this new spanning tree.

Q14. Using the spanning tree you created in the previous question, determine the path an emergency worker would follow if he started at house *A* and needed to get to house *H*. How long is this path?

Q15. Suppose all of the roads in the town were rebuilt. Determine the shortest path from house *A* to house *H*. How is this question different from the previous question?

In Q15, there were many different paths from house *A* to house *H*. The shortest path may not always be obvious. In the next section, a new algorithm is introduced that makes it easier to find the shortest path between two nodes.

Section 9.2: Medical Supplies

The Triangle Medical Supplies Company, in Raleigh, North Carolina, is a supplier to many doctors' offices located in and around Raleigh, North Carolina. Sometimes emergencies come up, and they need to know the shortest distance from the company to each of the doctors' offices.

The situation in this problem is different from the situation presented in the last problem (Section 9.1). Here, the goal is to find the shortest path from the supplier's warehouse to each doctor's office; previously, the goal was to find the minimum spanning tree connecting houses in a town.

Figure 9.2.1 shows the travel alternatives to nine of the major clients served by the Triangle Medical Supplies Company and the distance between them (in miles). Note that, like in the previous section, the arc lengths are not scaled to represent the value of the weights.

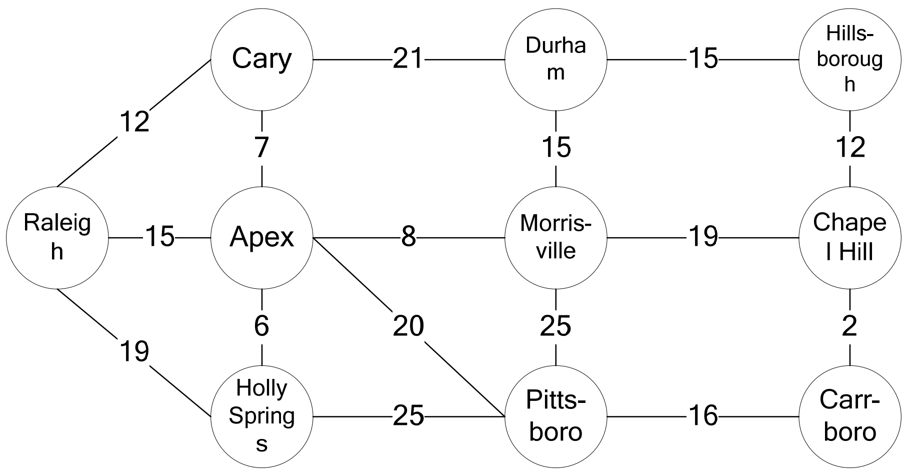


Figure 9.2.1: The medical supplies warehouse, its major clients, and the distance between them

To make this graph easier to discuss, the nodes can be represented with letters rather than the names of the cities, as shown in Figure 9.2.2.

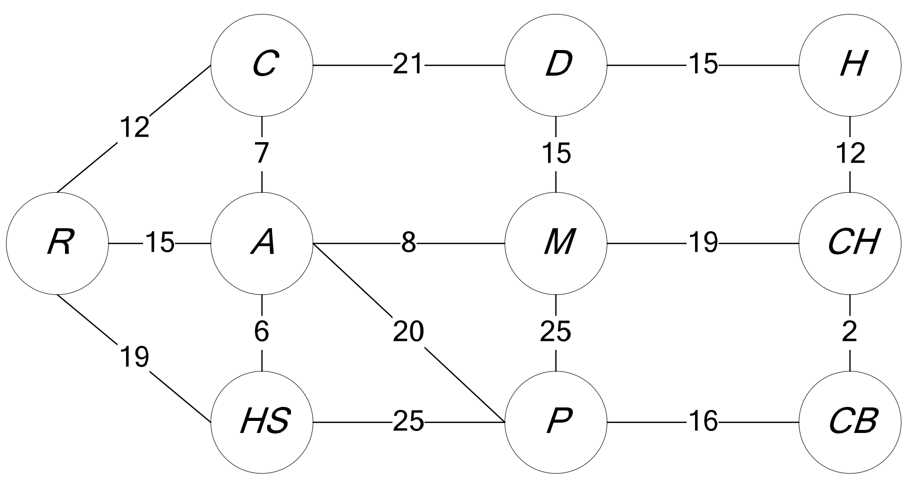


Figure 9.2.2: The supplies company graph, with nodes represented by letters

In this graph, *R* is the supplies warehouse and the shipping supply point for the Triangle Medical Supply Company. The other nodes correspond to the locations of the medical offices. The lines that connect the nodes correspond to the routes between the locations. The numbers on each line represent the travel distances in miles. The manager, Ms. Clark, wants to minimize the travel distance from the supplies warehouse to each of the customer's medical offices.

To get a feel for this, consider the travel distance from Raleigh to Carrboro (*R* to *CB*). There are several different paths, and each path has a different total travel distance. For example,

- The path *R-A-HS-P-CB* has a total distance of 15 + 6 + 25 + 16 = 62 miles
- The path *R-C-D-M-P-CB* has a total distance of 12 + 21 + 15 + 25 + 2 = 75 miles
- The path *R-HS-P-CB* has a total distance of 19 + 25 + 16 = 60 miles

Q1. Choose two other paths from Raleigh to Carrboro and calculate the travel distance.
  a. Is either of the paths you selected shorter than the ones listed above?
  b. Is either of them the shortest path? How can you be sure?

Q2. Using Figure 9.2.1, find the shortest path and the total distance from the supplies company in Raleigh (*R*) to:
  a. Morrisville (*M*)
  b. Pittsboro (*P*)

The above questions show that brute force is not necessarily an efficient way to find the shortest path between two nodes. Instead, a technique called Dijkstra's algorithm can be used to find the shortest path from the supplies warehouse to each of the medical offices.

9.2.1 Dijkstra's Algorithm

Recall that an algorithm is a step-by-step procedure. Like Kruskal's algorithm, Dijkstra's algorithm is a **greedy algorithm** because at each stage of the problem, the algorithm works by taking the best action for that stage of the problem. When this is done, the result is the shortest path from the starting node to each ending node.

Before beginning the algorithm, there are a few important things to remember:

- The word *permanent* is used to indicate when a node is "finished." A permanent node will be colored in.
- The word *temporary* is used when the node is in the "working stage."
- Throughout the algorithm, each node will be assigned a label with the following notation: [*distance value*, *preceding node*], such as [15, *C*].

Step 1: Determine the starting node. Assign this node the permanent label [0, *S*] and color the node. Because this is the starting node, there is no distance value and no preceding node. Therefore, the proper notation for the starting node's label is always [0, *S*], where *S* indicates the starting node.

In the medical supplies example, the starting node is the warehouse in Raleigh. In Figure 9.2.3, this node is labeled and colored.

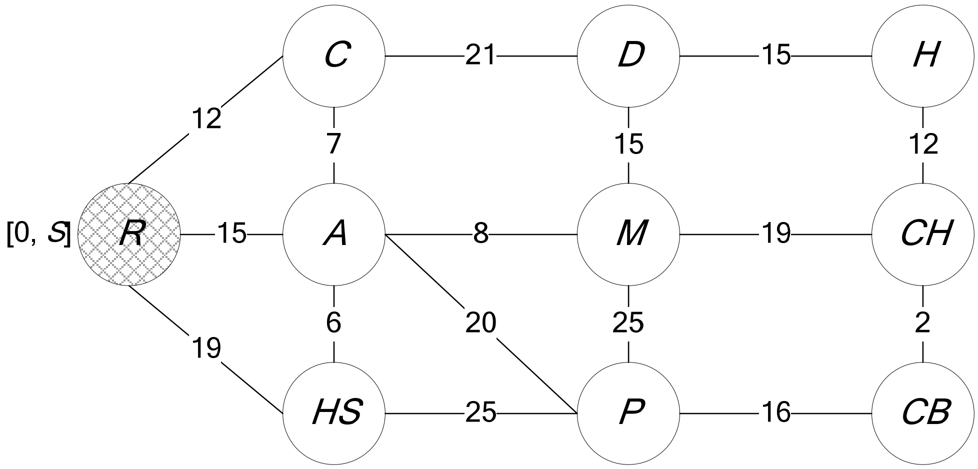


Figure 9.2.3: Identifying and labeling the starting node

Step 2: Assign temporary labels for the nodes that can be reached directly from the starting node. Each of these temporary labels will be in the form [*distance value*, *S*], where the *distance value* refers to the distance from the starting node to this node.

In Figure 9.2.3, there are three nodes that are directly connected to the starting node. In other words, there are three medical offices (Cary, Apex, and Holly Springs) that can be directly reached from the warehouse in Raleigh. In Figure 9.2.4, these three nodes have been assigned temporary labels.

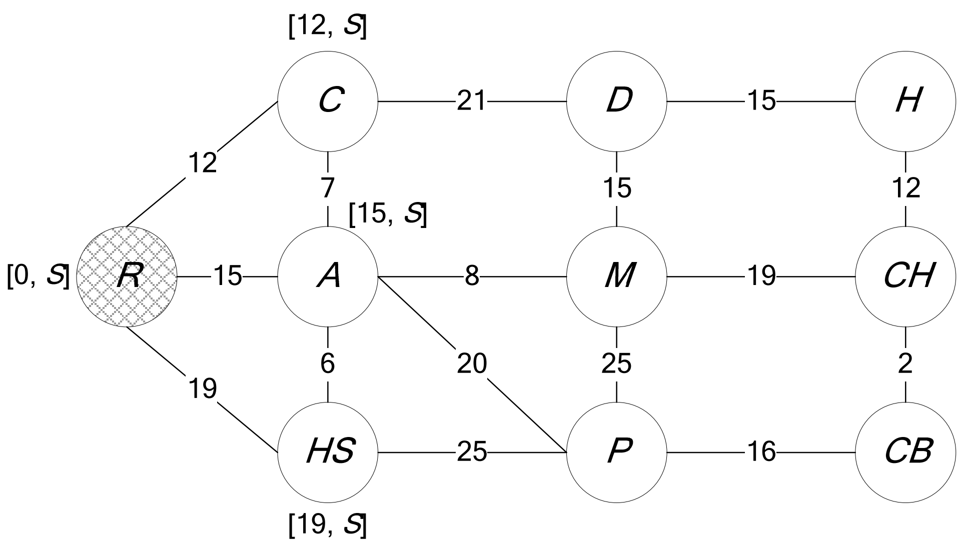


Figure 9.2.4: Computing temporary labels

Step 3: Identify the temporarily labeled node with the smallest distance value, and declare that node permanently labeled by coloring it in.
Note: If there is more than one with the same distance, you can choose either one.

In the medical supplies example, the node with the smallest distance value is *C*, which has a distance value of 12 miles. That is, Cary is the closest office to the supplies warehouse. Therefore, this node should be permanently labeled, as shown in Figure 9.2.5.

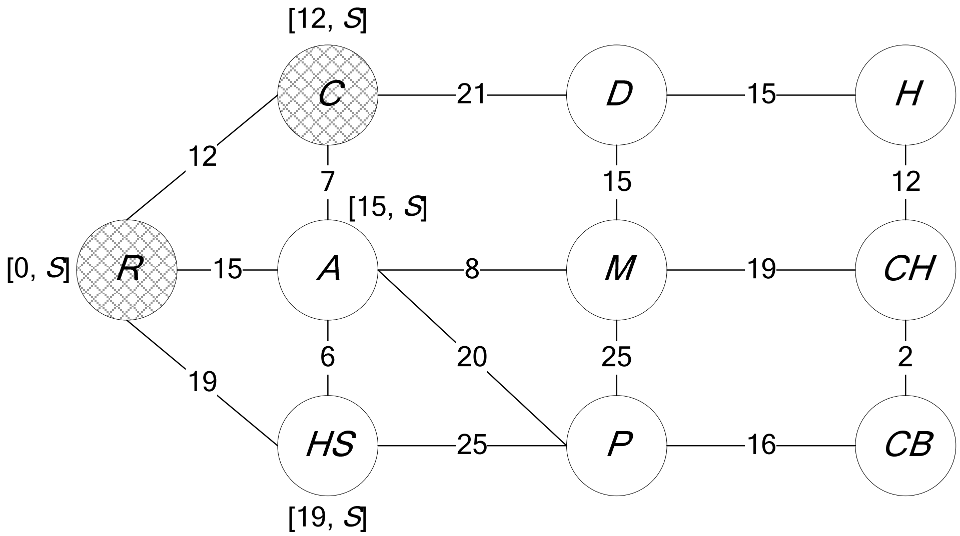


Figure 9.2.5: Changing temporarily labeled nodes to permanently labeled nodes

Step 4: Consider all nodes that are not permanently labeled and can be reached directly from the new permanently labeled node (identified in Step 3). Assign temporary labels as appropriate.
Assign labels as follows:

A. If the node does not yet have a temporary label, assign a temporary label with the following sum to as the distance value:
[distance value at the new permanently labeled node]
+ [direct distance from the new permanently labeled node to the node in question]

In the medical supplies example, the new permanently labeled node from Step 3 is node *C* (Cary). The nodes that can be directly reached from node *C* are node *D* (Durham) and node *A* (Apex). Node *D* does not have yet have a temporary label.

Since Ms. Clark is interested in the distance starting from the warehouse in Raleigh, she needs to determine the total distance from Raleigh to Durham. Therefore, she adds the distance value from Cary to Durham to the distance value from Raleigh to Cary: 21 + 12 = 33 miles. Therefore, node *D* should be temporarily labeled [33, *C*] because it takes 33 miles to get to Durham via Cary, as shown in Figure 9.2.6.

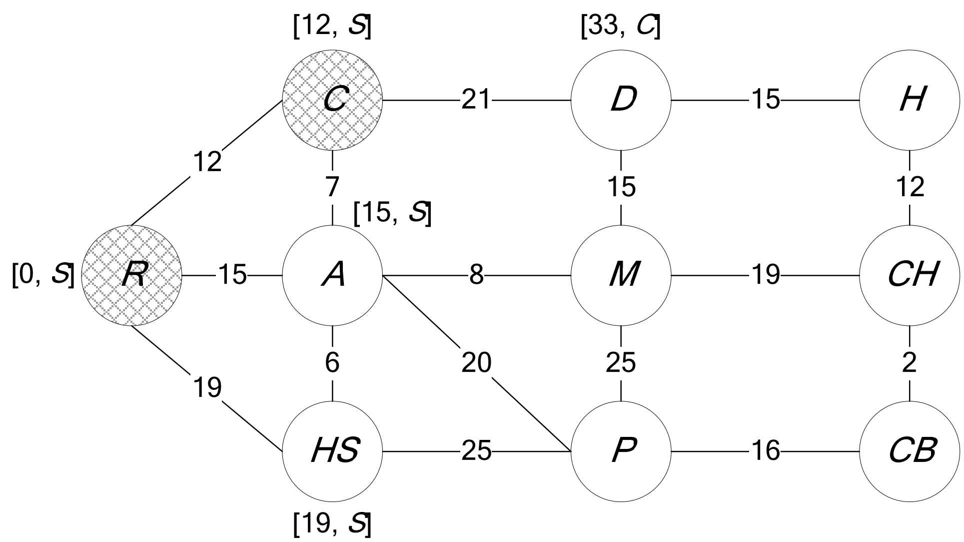


Figure 9.2.6: Computing temporary labels

B. If the node already has a temporary label, compute the same sum as in part A ([the distance value at the new permanently labeled node] + [the direct distance from the new permanently labeled node to the node in question]) and consider the following:
   1. If the sum just computed is equal to or greater than the distance value of the node in question, do nothing.
   2. If the sum just computed is less than the distance value already listed for the node in question, do the following:
      a. Change the distance value for the node in question to make it equal to the sum just computed.
      b. Change the "preceding node value" for the node in question to the letter of the new permanently labeled node.

Node *A* already has a temporary label. Therefore, Ms. Clark adds the distance from Raleigh to Cary to the distance from Cary to Apex: 12 + 7 = 19 miles. Since 19 > 15, she will do nothing. In other words, if the driver at Triangle Medical Supplies travels directly to Apex from Raleigh, it will take 15 miles. However, if the driver went through Cary first, it would take 19 miles. Since 19 is greater than 15, there is no reason for the driver to go through Cary. Thus, the temporary label for node *A* remains [15, *S*], as shown in Figure 9.2.6.

Repeat Step 3 and Step 4 until all nodes are permanently labeled.
If all nodes are permanently labeled, go to step 5.

Repeat Step 3
In the medical supplies example, the temporarily labeled node with the smallest distance value is *A*, which has a distance value of 15 miles. In other words, Apex is the next closest office to the supplies warehouse in Raleigh. Figure 9.2.7 shows this node permanently labeled.

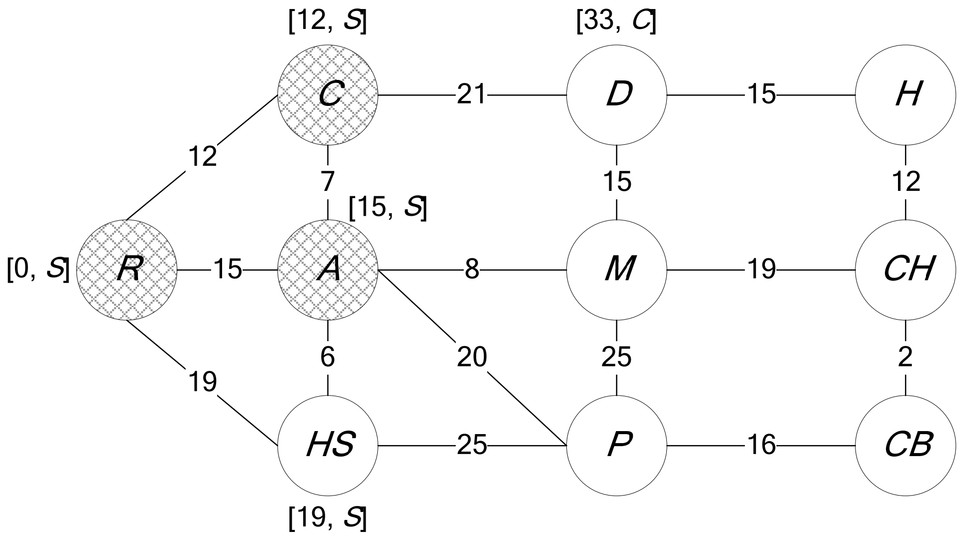


Figure 9.2.7: Changing temporarily labeled nodes into permanently labeled nodes

Repeat Step 4
The new permanently labeled node is node *A* (Apex). The temporarily labeled nodes that can be directly reached from node *A* are nodes *HS*, *M*, and *P*. Neither node *M* nor node *P* has temporary labels yet, so they are calculated by adding the distance value at *A* to the direct distance from *A*, as shown in Figure 9.2.8.

Node *HS* already has a temporary label. Computing the sum from Apex to Holly Springs, Ms. Clark finds this distance value to be 15 + 6 = 21 miles. Since the current temporary label has a distance value of 19 miles, she will not replace this label with a new one. In other words, the distance directly from Raleigh to Holly Springs is shorter (19 miles) than the distance from Raleigh to Apex to Holly Springs (21 miles). Therefore, the temporary label on node *HS* will remain [19, *S*], as shown in Figure 9.2.8.

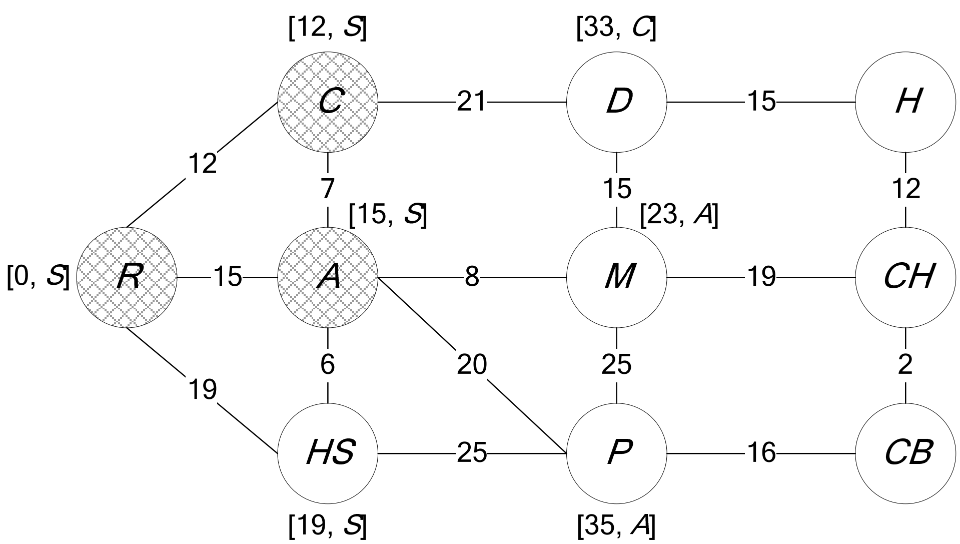


Figure 9.2.8: Computing temporary labels

Repeat Step 3
The temporarily labeled node with the next smallest distance value is node *HS* (Holly Springs). This node is permanently labeled in Figure 9.2.9.

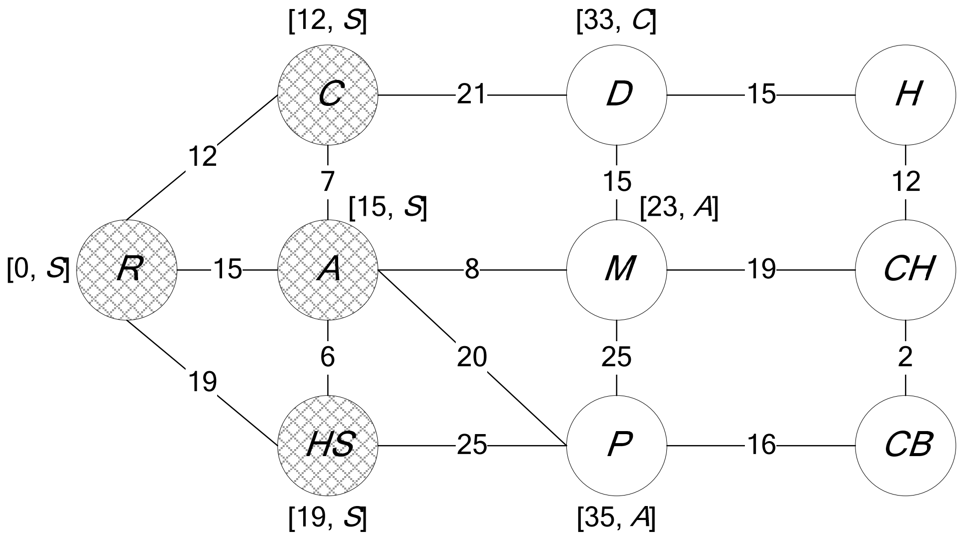


Figure 9.2.9: Changing temporarily labeled nodes into permanently labeled nodes

Repeat Step 4
The only temporarily labeled node directly connected to node *HS* is node *P*. The distance from Raleigh to Pittsboro via Holly Springs is 19 + 25 = 44 miles. Since 44 is greater than 35 miles, this temporary label should not be changed.

Repeat Step 3
The temporarily labeled node with the next smallest distance value is node *M* (Morrisville). This node is permanently labeled in Figure 9.2.10.

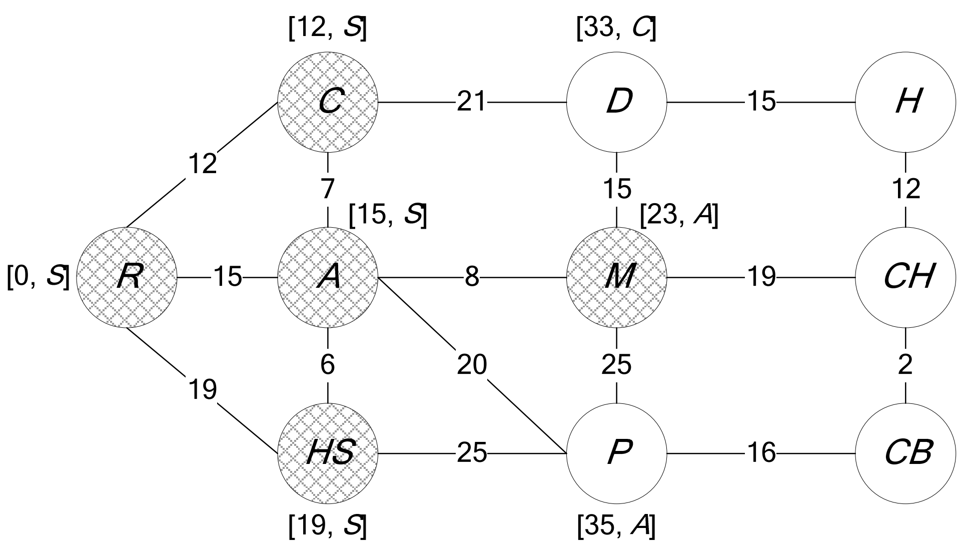


Figure 9.2.10: Changing temporarily labeled nodes into permanently labeled nodes

Repeat Step 4
Three nodes can be reached from Morrisville: Durham, Chapel Hill, and Pittsboro. Chapel Hill is the only node without a temporary label. Ms. Clark finds the distance value for Chapel Hill by adding 23 miles (the distance to Morrisville) to 19 miles (the distance from Morrisville to Chapel Hill) and obtains 42 miles. The temporary label for node *CH* is shown in Figure 9.2.11.

Nodes *D* and *P* already have temporary labels. Ms. Clark computes the total distance to Durham via Morrisville (23 + 15 = 38 miles) and the total distance to Pittsboro via Morrisville (23 + 25 = 48 miles). Neither of these distance values are less than the current distance values, so the temporary labels will be left unchanged, as shown in Figure 9.2.11.

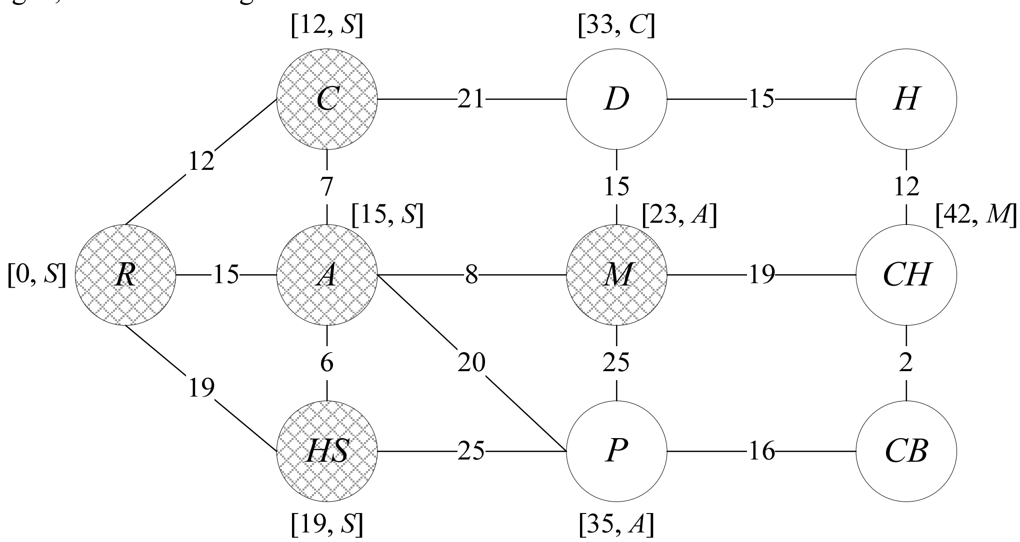


Figure 9.2.11: Computing temporary labels

Repeat Step 3
Next, node *D* (Durham) should be permanently labeled because it has the smallest distance value. This is shown in Figure 9.2.12.

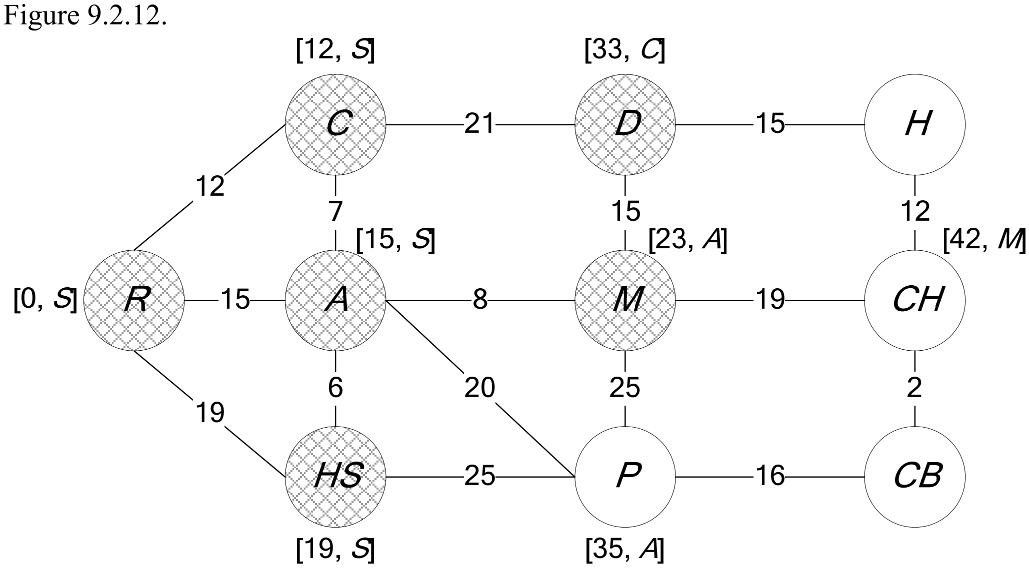


Figure 9.2.12: Changing temporarily labeled nodes into permanently labeled nodes

Repeat Step 4
The only node that can be directly reached from node *D* is node *H* (Hillsborough). Node *D* does not yet have a temporary label. Therefore, node *H* is given a temporary label with the distance value of 33 + 15 = 48 miles, as shown in Figure 9.2.13.

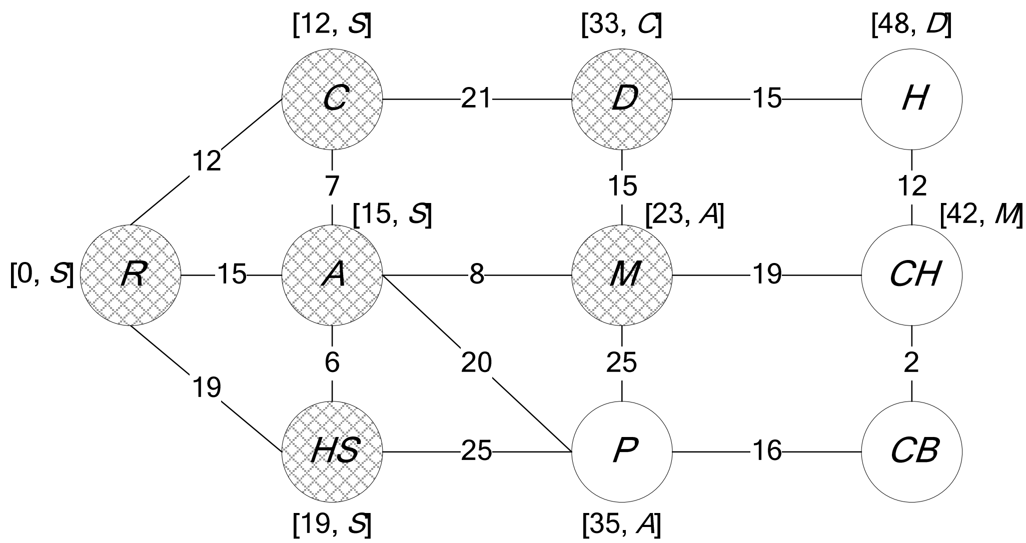


Figure 9.2.13: Computing temporary labels

Repeat Step 3
The next smallest distance value belongs to node *P* (Pittsboro). This node is permanently labeled in Figure 9.1.14.

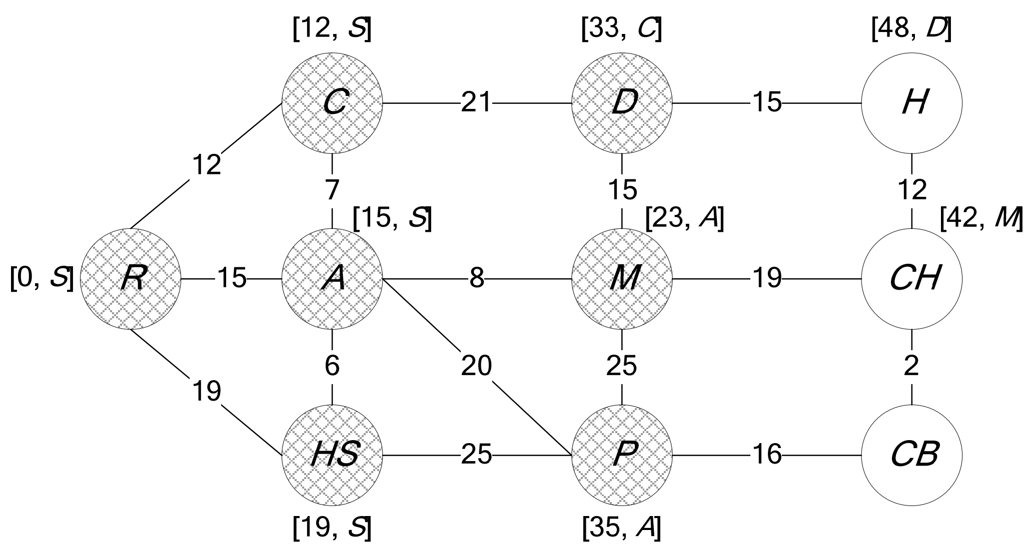


Figure 9.2.14: Changing temporarily labeled nodes into permanently labeled nodes

Repeat Step 4
The only node directly connected to node *P* is node *CB* (Carrboro). This node does not yet have a temporary label. Ms. Clark computes the distance value for this node by adding the distance value of Pittsboro (35 miles) to the distance from Pittsboro to Carrboro (16 miles). The distance value for Carrboro is therefore 51 miles, as shown in Figure 9.2.15.

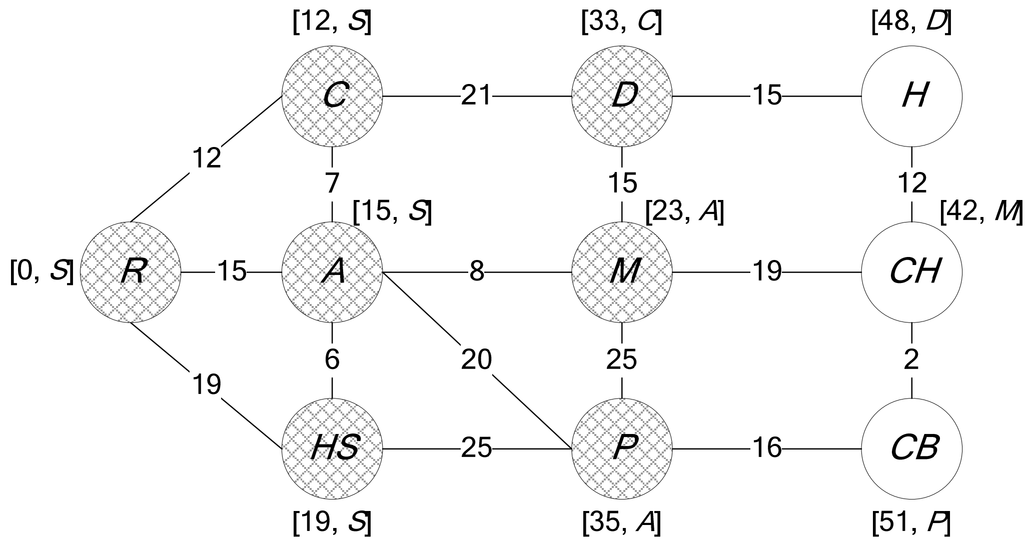


Figure 9.2.15: Computing temporary labels

Repeat Step 3
The next smallest distance value is 42 miles, which belongs to node *CH* (Chapel Hill). Therefore, this node is permanently labeled, as shown in Figure 9.2.16.

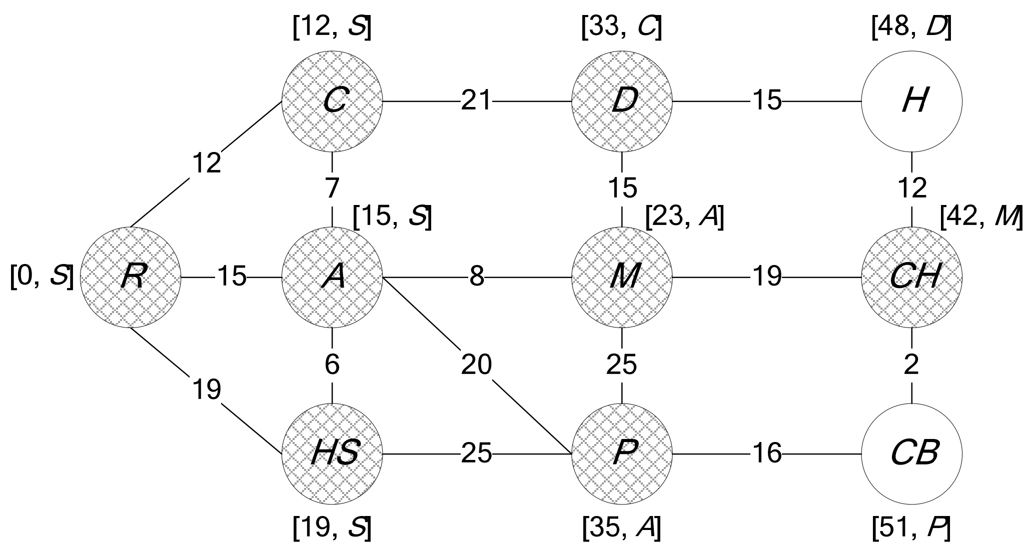


Figure 9.2.16: Changing temporarily labeled nodes into permanently labeled nodes

Repeat Step 4
From Chapel Hill, both Hillsborough and Carrboro can be directly reached. Currently, the distance value for Hillsborough is 48 miles. If the driver came from Chapel Hill, the distance value would instead be 42 + 12 = 54 miles. Since this is greater than 48, this temporary node is not changed.

However, the current distance value for Carrboro is 51 miles. If the driver drove through Chapel Hill, rather than through Pittsboro, to get to Carrboro, the distance value would instead be 42 + 2 = 44 miles. Recall from Step 4B that if this sum is less than the distance value already listed, then the distance value should be changed to this sum (44 miles), and the preceding node value should be changed to the letter of the new permanently labeled node (*CH*). These changes are shown in Figure 9.2.17.

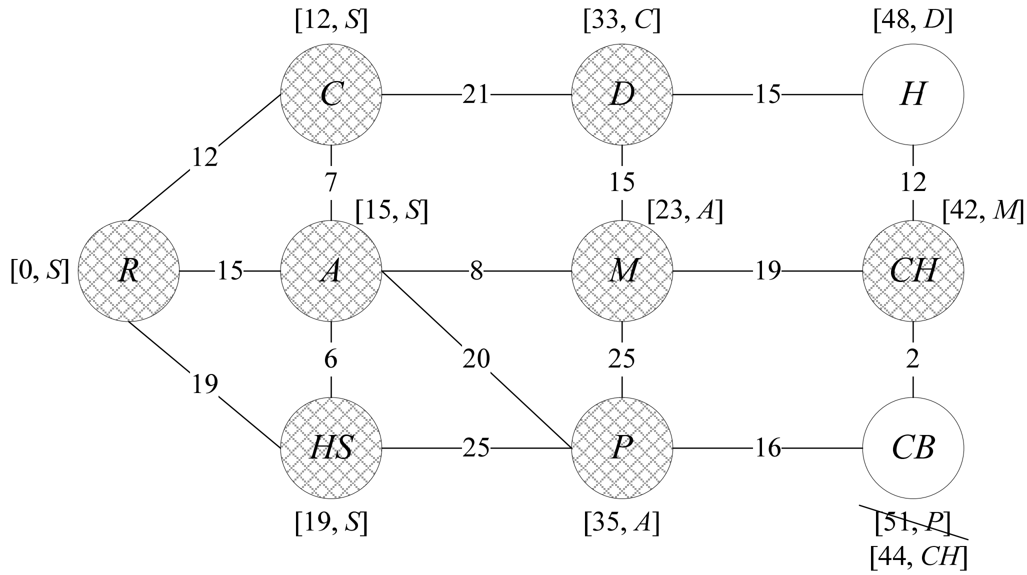


Figure 9.2.17: Computing temporary labels

Repeat Step 3
The node with the next smallest distance value is *CB* (Carrboro). Therefore, this node is permanently labeled, as shown in Figure 9.2.18.

Since no temporarily labeled nodes can be directly reached from node *CB*, Step 4 is skipped and Step 3 is repeated again. This time, node *H* (Hillsborough) is permanently labeled. Figure 9.2.18 shows all nodes are now permanently labeled.

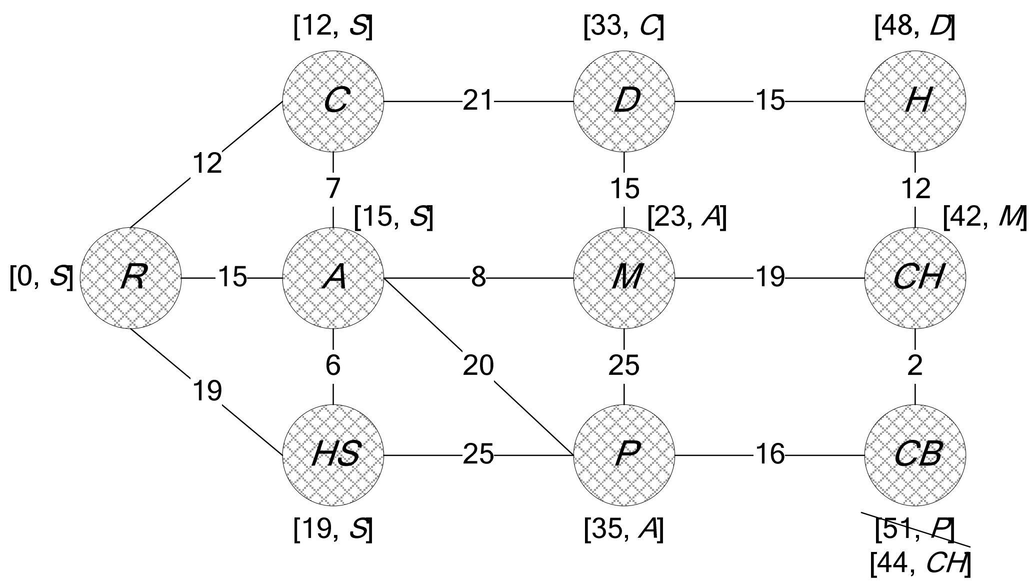


Figure 9.2.18: Graph with permanent labels for all nodes

Step 5: Identify the shortest path from the starting node to each node.
The permanent labels show the shortest distance from the starting node to each node and the preceding node on the shortest route. The shortest path to a given node can be found by starting at the given node and moving to its preceding node. Continuing this backward movement through the graph provides the shortest route from the starting node to the node in question.

For example, in Figure 9.2.18, the shortest distance from Raleigh to Pittsboro is 35 miles. Before arriving at Pittsboro, the supplies driver must go through Apex. Therefore, the shortest path from Raleigh to Pittsboro is *R-A-P* (Raleigh to Apex to Pittsboro).

Q3. Using Figure 9.2.18, find the shortest path and the total distance from the supplies company in Raleigh (*R*) to:

a. Chapel Hill (*CH*).
b. Carrboro (*CB*).
c. Hillsborough (*H*).

It seems like using Dijkstra's algorithm is a long process, but using it assures that the minimum distance from the starting node to each of the other nodes is found. Dijkstra's algorithm works for any other situation like this, and, if necessary, it can be done on a computer.

The steps of Dijkstra's algorithm are summarized below.

Dijkstra's Algorithm

Step 1: Determine the starting node. Assign this node the permanent label [0, *S*] and color the node. Because this is the starting node, there is no distance value and no preceding node. Therefore, the proper notation for the starting node's label is always [0, *S*], where *S* indicates the starting node.

Step 2: Assign temporary labels for the nodes that can be reached directly from the starting node. Each of these temporary labels will be in the form [*distance value*, *S*], where the *distance value* refers to the distance from the starting node to this node.

Step 3: Identify the temporarily labeled node with the smallest distance value, and declare that node permanently labeled by coloring it in. If all nodes are permanently labeled, go to step 5. Note: If there is more than one with the same distance, you can choose either one.

Step 4: Consider all nodes that are not permanently labeled and can be reached directly from the new permanently labeled node (identified in Step 3). Then, assign labels as follows:

- **A.** If the node does not yet have a temporary label, compute the following sum to determine the distance value:
  [distance value at the new permanently labeled node]
  + [direct distance from the new permanently labeled node to the node in question]
- **B.** If the node already has a temporary label, compute the same sum as in part A ([the distance value at the new permanently labeled node] + [the direct distance from the new permanently labeled node to the node in question]) and consider the following:
  - **1.** If the sum just computed is equal to or greater than the distance value of the node in question, do nothing.
  - **2.** If the sum just computed is less than the distance value already listed for the node in question, do the following:
    - **a.** Change the distance value for the node in question to make it equal to the sum just computed.
    - **b.** Change the "preceding node value" for the node in question to the letter of the new permanently labeled node.

Repeat Step 3 and Step 4 until all nodes are permanently labeled. Then, proceed to Step 5.

Step 5: Identify the shortest path from the starting node to each node.

9.2.2 New Warehouse Location

The president of the Triangle Medical Supplies Company, Mr. Kudzik, wants to build a new warehouse next to one of their existing customers. He asks Ms. Clark, the manager, to determine the best location for the new warehouse. Mr. Kudzik wants the new warehouse to be as close as possible to all of their customers in the area so that the distances between the warehouse and the doctors' offices are minimized.

In order to solve this problem, Ms. Clark needs to find the shortest path from each doctor's office to every other doctor's office. The driver will not need to travel from the new warehouse to the existing warehouse (in Raleigh); thus, the graph only needs to include nodes for the doctors' offices, as shown in Figure 9.2.19.

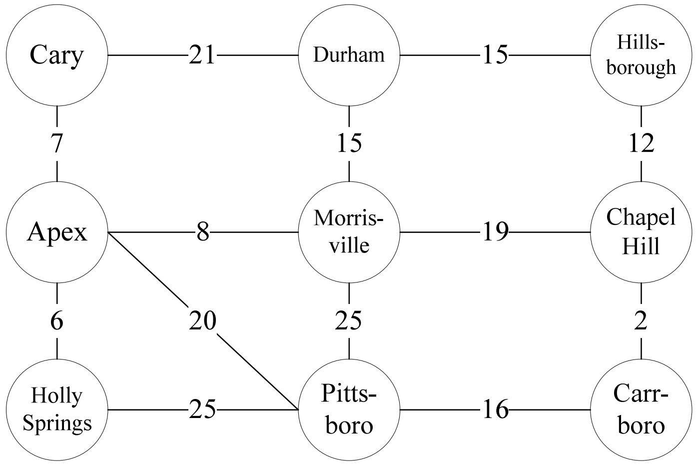


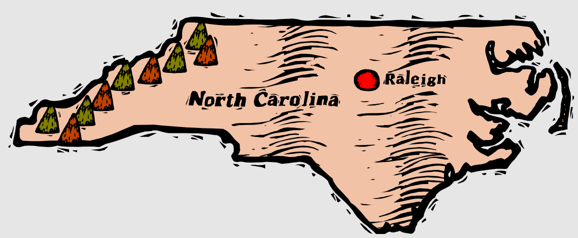
Figure 9.2.19: The major clients of the medical supplies company and the distances between them

There are two ways for Ms. Clark to think about this problem. First, she could locate the new warehouse at the doctor's office that minimizes the maximum distance to all the other offices. That is, she could find the office that, if the new warehouse was located there, the driving distance to all other offices is as small as possible. In terms of the graph, the node that *minimizes the maximum distance* to all the other nodes is called the **center location**.

Second, Ms. Clark could locate the new warehouse at the doctor's office that minimizes the average distance to all the other offices. In other words, she could find the office such that the average driving distance to each other is as small as possible. In terms of the graph, the node that is *on average closest to all of the other nodes* is called the **median location**.

Median Location

In the previous chapter, *median location* referred to the first location where the cumulative weight up to that point is at least half of the total weight of all other locations. In this chapter, *median location* has a similar but different definition; it refers to the location that is, on average, closest to the other locations.



These two locations, center and median, may or may not be the same, but they provide two different ideas of the "middle" of the graph.

To find the center and median location, Dijkstra's algorithm must be employed repeatedly. Ms. Clark uses Dijkstra's algorithm to find all the shortest distances between every node and every other node. The completed graphs are given in Figures 9.2.20, 9.2.21, and 9.2.22, where the starting nodes are Cary, Apex, and Holly Springs, respectively.

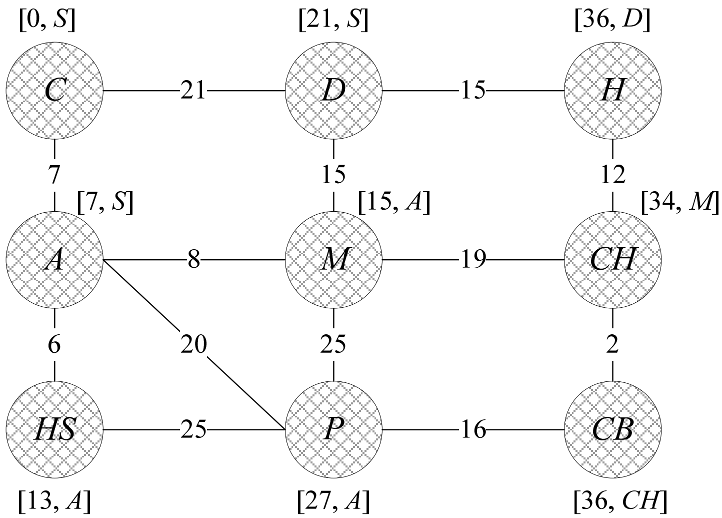


Figure 9.2.20: Final version of the graph, with Cary as the starting node

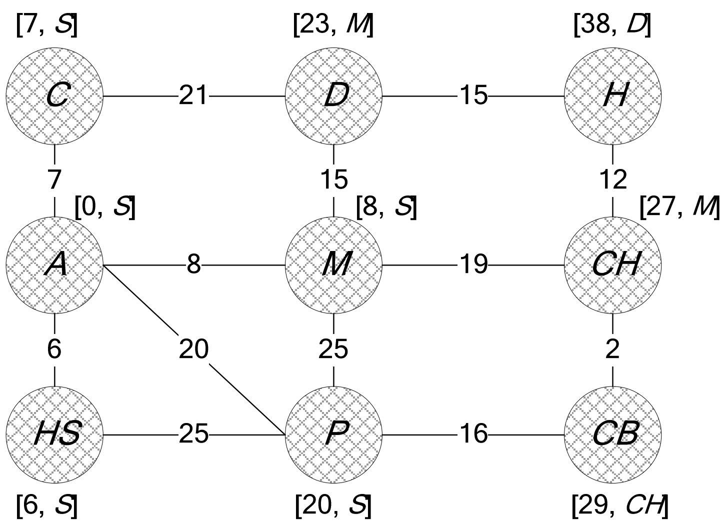


Figure 9.2.21: Final version of the graph, with Apex as the starting node

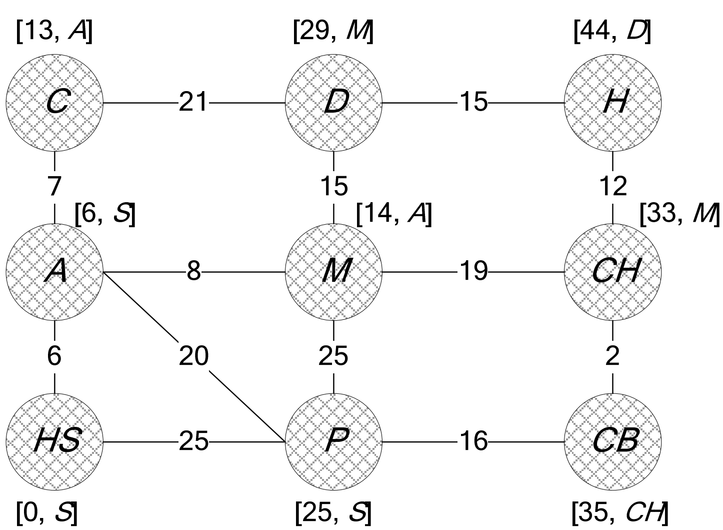


Figure 9.2.22: Final version of the graph, with Holly Springs as the starting node

The first few distances are shown in Table 9.2.1, where the leftmost column represents the starting node and the top row represents the ending node. The maximum distance and the average distance for each row are also given.

For example, if the new warehouse is located in Cary, the maximum distance from Cary to each of the offices is 36 miles (the distance from Cary to Hillsborough and also from Cary to Carrboro). The average distance from Cary to each of the offices is

$$\frac{0+7+13+21+15+27+36+34+36}{9} = 21 \text{ miles.}$$

	C	**A**	**HS**	**D**	**M**	**P**	**H**	**CH**	**CB**		**Max**	**Average**
C	0	7	13	21	15	27	36	34	36		36	21
A	7	0	6	23	8	20	38	27	29		38	17.56
HS	13	6	0	29	14	25	44	33	35		44	22.11
D												
M												
P												
H												
CH												
CB												

Table 9.2.1: The shortest paths from each office to every other office

Q4. Use Dijkstra's algorithm to complete Table 9.2.1 for the remaining offices.

Q5. What are some patterns you notice in the table?

Q6. Which office represents the center location? How do you know?

Q7. Which office represents the median location? How do you know?

Q8. Based on this information, where should Ms. Clark decide to locate the new warehouse? Why?

Q9. Suppose the doctor's office in Hillsborough has moved. Now, the distance from Durham to Hillsborough is 20 miles, and the distance from Chapel Hill is Hillsborough is 17 miles, as shown in Figure 9.2.23. Do you think this change would impact the optimal location of the new warehouse? Why or why not?

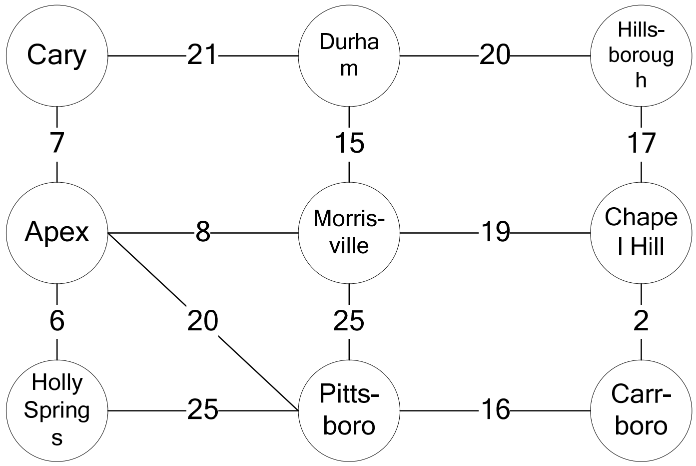


Figure 9.2.23: The major clients and the distances between them, with a new Hillsborough office

In this section, Ms. Clark first needed to determine the shortest path from the medical supplies warehouse in Raleigh to each of the doctor's offices in the area. She used Dijkstra's algorithm to determine these paths.

Then, Ms. Clark needed to find the best location for a new warehouse, next to one of the existing clients. She utilized Dijkstra's algorithm repeatedly to determine the locations that (1) minimized the maximum distance traveled and (2) minimized the average distance traveled.

In the next section, Dijkstra's algorithm is used again to find the shortest path, the center location, and the median location. However, rather than finding the distances between locations, the algorithm is used to determine how quickly a rumor would spread in a group of people.

Section 9.3: How Quickly do Rumors Spread?

Suppose there are 10 people in a class, and someone starts a rumor. Rumors tend to spread quickly, but how quickly they spread depends on how often people communicate with one another. In this section, Dijkstra's algorithm is used to determine how long it takes for a rumor to spread.

In the graph below (Figure 9.3.1), the nodes represent people and the arcs represent how often (in minutes) they communicate with each other. For example, the maximum amount of time before person *A* speaks with, sends a text message to, calls, or emails person *B* is 5 minutes, so the edge connecting *A* to *B* has a weight of 5. The goal is to examine how quickly the rumor will spread throughout the class.

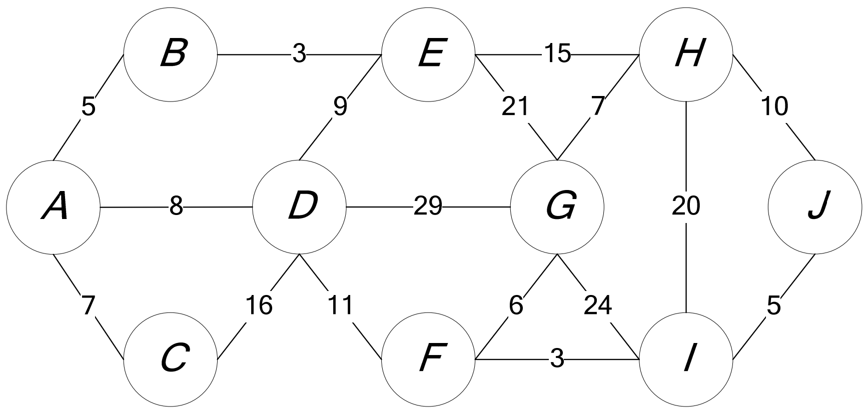


Figure 9.3.1: A graph representing a social network, where the nodes represent people and the arcs represent how often the people communicate with each other (in minutes)

Suppose person *A* begins a rumor. To determine how long it will take for the rumor to reach the most distant member in the class from *A*, the shortest path from person *A* to each person in the class must be found. Using Dijkstra's algorithm, the shortest paths from *A* to each node is given in Figure 9.3.2.

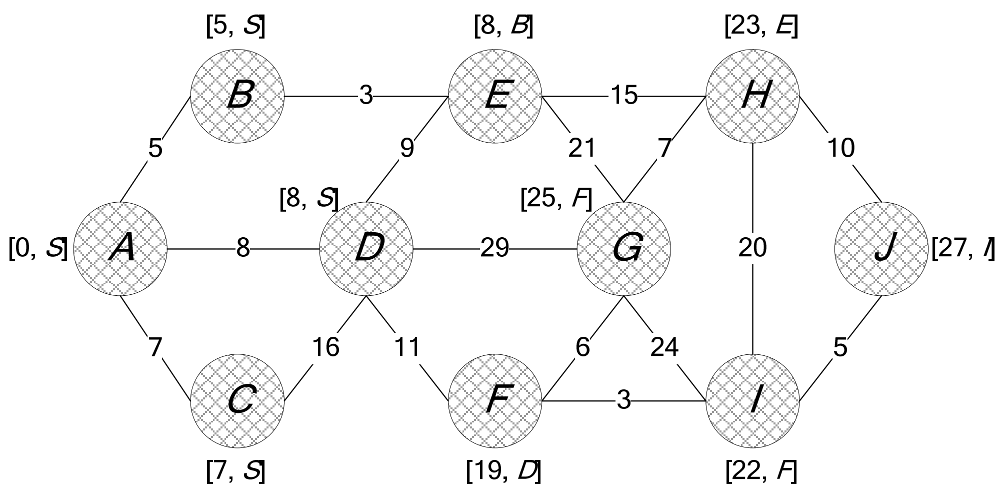


Figure 9.3.2: Final version of the graph where person *A* started the rumor

Q1. How long will it take for everyone in the class to hear the rumor? How do you know?

Q2. What is the average amount of time it takes for each member to first hear the rumor if person *A* starts the rumor?

Q3. How long would it take the rumor to spread if person *J* started it instead of person *A*? That is, find the shortest path from *J* to each node and determine which shortest path has the greatest value.

Q4. What is the average amount of time it takes for each member to first hear the rumor if person *J* starts the rumor?

Center and median locations can be used to determine which individual would need to start the rumor in order for it to spread as quickly as possible. Just like in the medical supplies problem, the center location refers to the node that minimizes the maximum distance to all the other nodes; the median location refers to the node that minimizes the average distance to all the other nodes.

Q5. Explain center and median location in terms of the rumor example.

Dijkstra's algorithm can be used to find all the shortest distances between every node and every other node These distances are given in Table 9.3.1.

	A	B	C	D	E	F	G	H	I	J		Max	Average
A	0	5	7	8	8	19	25	23	22	27		27	14.4
B	5	0	12	12	3	23	24	18	26	28		28	15.1
C	7	12	0	15	15	26	32	30	29	34		34	20
D	8	12	15	0	9	11	17	24	14	19		24	12.9
E	8	3	15	9	0	20	21	15	23	25		25	13.9
F	19	23	26	11	20	0	6	13	3	8		26	12.9
G	25	24	32	17	21	6	0	7	9	14		32	15.5
H	23	18	30	24	15	13	7	0	15	10		30	15.5
I	22	26	29	14	23	3	9	15	0	5		29	14.6
J	27	28	34	19	25	8	14	10	5	0		34	17

Table 9.3.1: The shortest paths from each person to every other person

Q6. Determine which individual's node is the center. What does this mean in terms of the problem?

Q7. Determine which individual's node is the median. What does this mean in terms of the problem?

Q8. If you were in this class and you wanted a rumor to spread quickly, who would you want to start the rumor? Why?

Q9. Suppose person *D* is absent. Determine the center and median locations in this new scenario (see Figure 9.3.3).

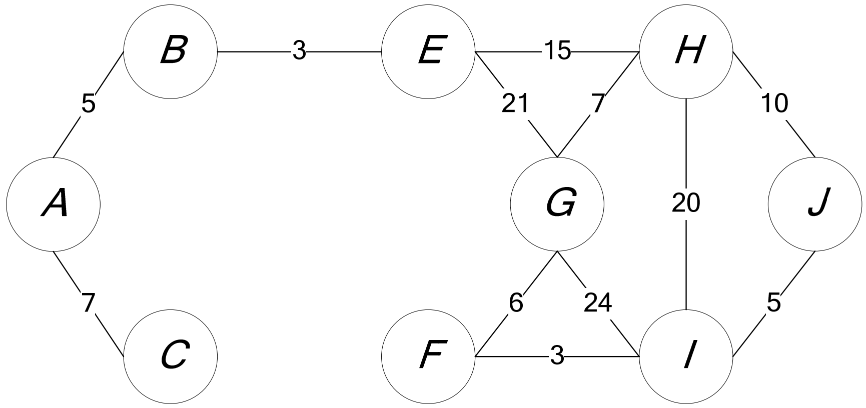


Figure 9.3.3: A graph representing a social network, with person *D* absent

Although this rumor example was quite different from the medical supplies example, both could be solved using Dijkstra's algorithm. Furthermore, both used the ideas of center and median to get an idea about which node is the "middle" of the graph.

The concepts discussed in this chapter (i.e., minimum spanning tree, shortest path, center, and median) give an introduction into the large field known as graph theory.

Section 9.4: Chapter 9 (Graph Theory) Homework Questions

1. Find the Bacon number, using imdb.com, of the following actors/actresses. Create a graph with at least **three** paths from the actor/actress to Kevin bacon; write the movie and actor/actress for each connection.
    a. Jodie Foster
    b. Natalie Portman
    c. Harrison Ford
    d. Nicole Kidman

2. A company has a business with several offices in the New York City area. The owner needs to set up phone lines connecting the offices. The phone company charges the company based on distances of the phone line connections. The following mileage chart shows the distances between each of the offices. Using this chart, create a tree representing the offices and the distance between them. Then use Kruskal's algorithm to find the minimum spanning tree connecting the offices.

	Manhattan	Bronx	Brooklyn	Queens	Newark, NJ
Manhattan	**	8.1	8.0	10.2	13.4
Bronx	8.1	**	15.6	14.0	21.8
Brooklyn	8.0	15.6	**	14.1	13.9
Queens	10.2	14.0	14.1	**	19.8
Newark, NJ	13.4	21.8	13.9	19.8	**

Table 9.4.1: Distances between cities in miles

3. The figure below represents a plan for a network of streets. Each node represents a place that needs water, i.e. a residence, a fire hydrant, a commercial property, etc., and the distance between nodes is measured in miles. To minimize the cost of laying down pipes, use Kruskal's algorithm to find the minimum spanning tree connecting the nodes.

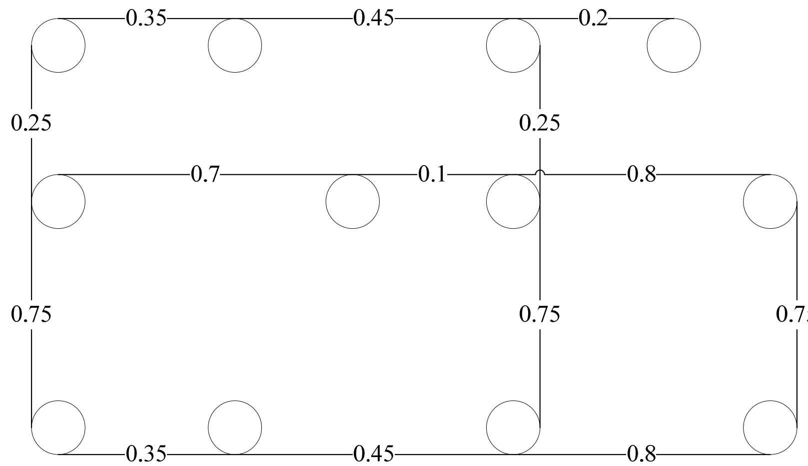


Figure 9.4.1: Network of streets

4. Here is another medical supply graph. The *O* is the warehouse, and the other letters are locations needing emergency supplies. The numbers on the arcs represent miles. Using Dijkstra's algorithm, we can create the final version below it. Identify the shortest path as identified for each of the locations, *A*, *B*, *C*, *D*, *E*, and *T*.

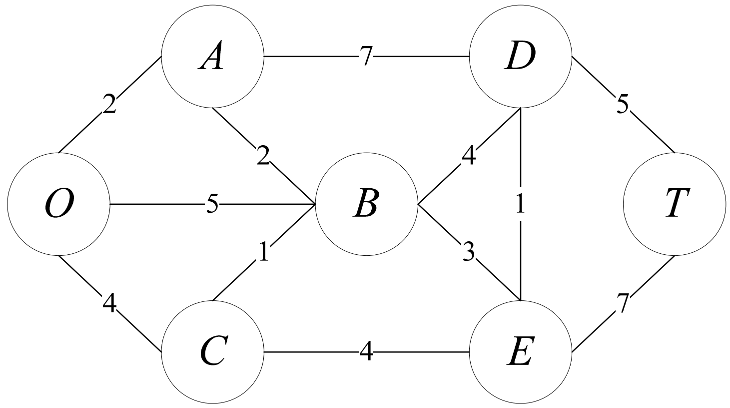


Figure 9.4.2: Graph for example medical supply problem

a. Find the Shortest Paths and the distances from *O* to each of the medical supply houses.

Where to	Path	Shortest Distance in miles
To *A*		
To *B*		
To *C*		
To *D*		
To *E*		
To *T*		

b. Would the answer for the shortest path to *T* change if the distance from *E* to *T* were 3 instead of 7? How would it change?

c. Would the answer for the shortest path to *E* change in this case? How would it change?

5. Steve would like to visit some of his friends at college. Due to money and time constraints, he can only travel 150 miles on any given trip. Use Dijkstra's algorithm on the graph below to find the shortest distance between Steve's home and each of his friends' colleges, where the nodes are labeled with various colleges and the arcs are labeled with the distance between the colleges, in miles. Based on this graph, which colleges could he visit? Which of these trips include more than one stop?

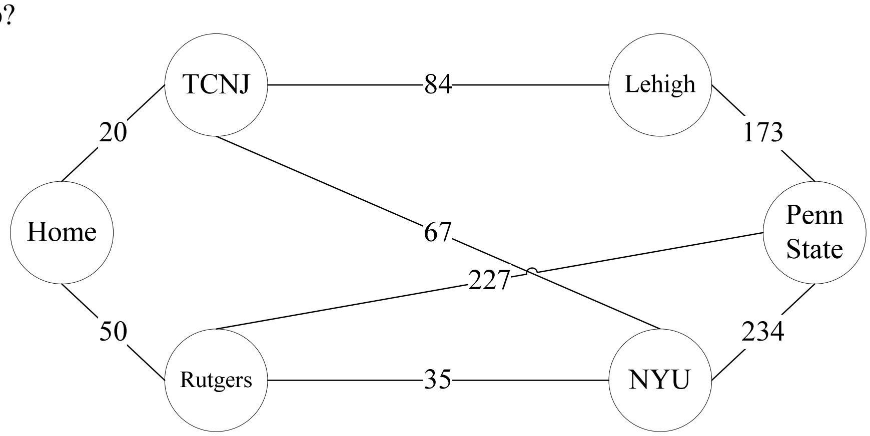


Figure 9.4.3: Distances between colleges

6. Below is a graph of a neighborhood. The nodes represent houses and the arcs represent roads. The weights on the arcs represent the length of time it takes to walk from one house to the next. Rochelle needs to take her little sister trick-or-treating in the neighborhood. The node labeled T represents the house the gives the best and the most candy. Use Dijkstra's algorithm to find the shortest path from home to node T. How many houses will they visit along the way?

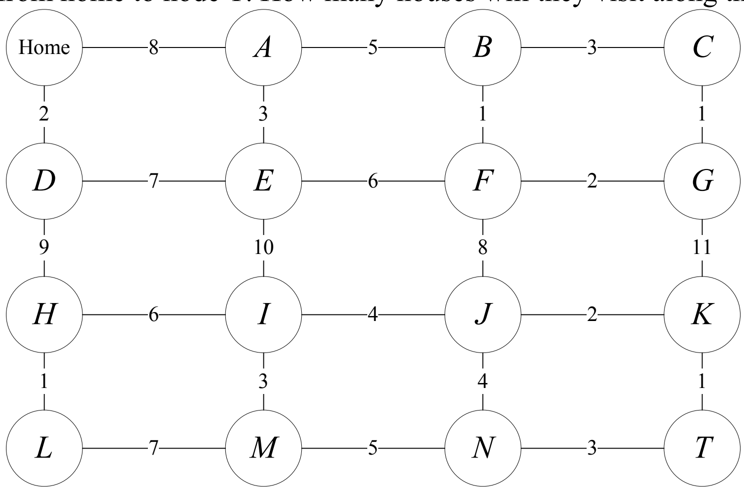


Figure 9.4.4: Graph of a neighborhood

7. Use Dijkstra's algorithm to find the shortest path from each node to every other node in the graphs below. Then determine the median and the center for each graph.
   a.

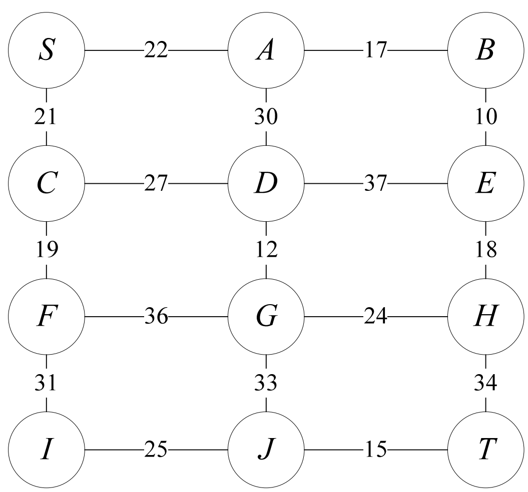


Figure 9.4.5: Dijkstra's algorithm example

   b.

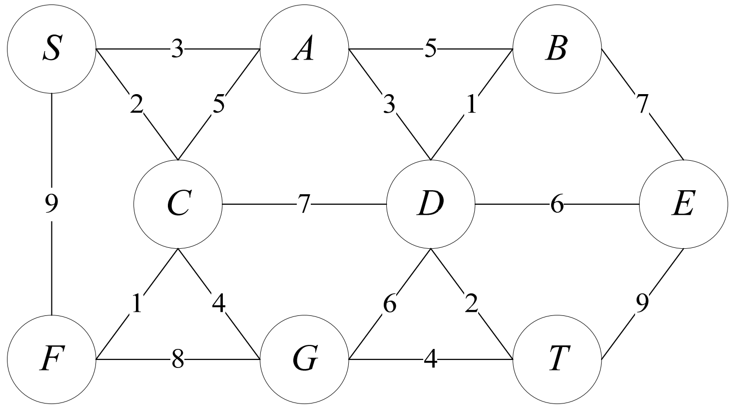


Figure 9.4.6: Dijkstra's algorithm example

Chapter 9 Summary

What have we learned?

Terms

Algorithm	A step-by-step process for solving a problem
Arc	A line segment on a graph
Bacon Number	The minimum number of degrees of separation between two individuals when playing the six degrees of Kevin Bacon game
Center Location	The node in a graph that minimizes that maximum distance to all other nodes in the graph
Circuit	A path in a graph that begins and ends at the same node
Dijkstra's Algorithm	An algorithm created by Edsger Dijkstra that finds the shortest path between two points in a graph
Graph	A set of nodes and arcs
Graph Theory	The study of graphs
Greedy Algorithm	A method of finding an optimal solution, where at each step, the best action is taken for that step of the problem
Kruskal's Algorithm	An algorithm created by Joseph Kruskal that finds the minimum spanning tree of a given graph
Median Location	The node in a graph that is on average closest to all of the other nodes in the graph
Minimum Spanning Tree	The spanning tree of a graph with the least amount of total weight
Node	Points on a graph
Path	A connection between two nodes along arcs
Shortest Path Problems	Problems where the goal is to find the shortest path between two nodes in a graph
Six Degrees of Kevin Bacon	A game in which actors are connected, through movies, to the actor Kevin Bacon with as few connections as possible; this game is an application of the "six degrees of separation" idea
Six Degrees of Separation	The idea that, through social networks, every person is connected to every other person by an average of six people
Spanning Tree	A tree that connects all of the nodes

Tree	A graph with no circuits
Weights	A number assigned to an arc that represents the cost, distance, or time related to that arc

Chapter 9 (Graph Theory) Objectives

You should be able to:

- Investigate and define graphs, arcs, nodes, networks, spanning trees, and weights and be able to identify them in a contextual problem.

- Given a contextual problem, find the minimum spanning tree using Kruskal's algorithm and analyze and make decisions based on the results.

- Given a contextual problem, find the shortest path using Dijkstra's algorithm and analyze and make decisions based on the results.

Chapter 9 Study Guide

1.

References

Anderson, D. R., Sweeney, D. J., & Williams, T. A. (1991). *An introduction to management science: Quantitative approaches to decision making* (6th ed.). St. Paul, MN: West Publishing Company.

Films And TV. (2010). Retrieved January 12, 2011, from http://www.filmsandtv.com/

The Internet Movie Database (IMDb). (2011). Retrieved January 12, 2011, from http://www.imdb.com/

Reynolds, P. (2010). The Oracle of Bacon. Retrieved January 12, 2011, from http://oracleofbacon.org/

SixDegrees.org | It's a small world. You can make a difference. (n.d.). Retrieved January 12, 2011, from http://www.sixdegrees.org/

Section 10.0 Introduction

In the previous chapter, graph theory was used to solve problems such as determining which roads to rebuild after a disaster, finding the best route for a medical supplies company, and exploring how a rumor spreads. In this chapter, graph theory is used again. This time, graphs are used to plan large projects.

First, project planning is introduced through a morning routine. A person's morning routine consists of several activities, some of which are dependent on others. For example, one would not put on their shoes and then put on their socks. This example introduces project planning and activity dependence.

Second, a group of friends is preparing a dinner. In this example, the *Critical Path Method (CPM) algorithm* is introduced. Through this algorithm, the group of friends determine how long it will take to prepare the dinner and how much time flexibility they have. Then, a *Gantt chart* is shown, which is a visual representation that will help with them plan the dinner.

The third section presents a large project-planning problem: the manufacturing of a part of a weather satellite known as a flight propulsion system. A group of engineers lists the activities, the activity dependencies, and the times to complete this project. The CPM algorithm and Gantt charts are utilized once again to determine how long the project will take and how much time flexibility they have.

In this chapter, the activity time lengths have been determined by people who completed the project previously. Another method, called Program Evaluation and Review Technique (PERT), considers projects that use probability to estimate the time activities will take. This method is presented in a later chapter.

Section 10.1: Getting Ready for School

A project that may need to be planned is one's morning routine. The activities that make up individuals' morning routines vary. However, the goal of a morning routine is typically to sleep as long as possible and still get to work or school on time. Tardiness is the consequence for not reaching this goal.

Q1. Under the "Activity Description" column in Table 10.1.1, list all the activities you need to accomplish in the morning in order to get to school on time. For example, brushing your teeth might be one of the things you do.

Q2. How would you figure out how much time you need to complete all these activities?

Q3. Estimate how much time each individual activity takes. Write these estimates under the "Completion time (minutes)" column in Table 10.1.1.

Activity Code	**Activity Description**	**Activity Dependence**	**Completion Time (minutes)**
A			
B			
C			
D			
E			
F			
G			
H			
I			
		Total Time	

Table 10.1.1: Time estimates and activity dependence for activities accomplished before school

One thing that needs to be considered when planning a morning routine is activity dependence. **Activity dependence** addresses the idea that some activities are dependent on others. In other words, one activity cannot begin before a previous one is completed. For example, the activity of getting dressed is *dependent* on showering. A person certainly would not get dressed and then shower.

Q4. Looking at the activities you listed in Table 10.1.1, determine which activities are dependent on other activities. Write the letter of the activity that must happen before the activity in the "Activity Dependence" column next to the activity. For example, if getting dressed is activity *C* and showering is activity *A*, then "*A*" is written in the row for activity *C*. Remember, for some activities, it does not matter when you fit them in your morning routine. Hence, some activities will not have any letters in the "Activity Dependence" column.

Q5. Determine how much time you need to get ready by adding all the completion times.

This particular set of activities, getting ready for school, is not too difficult because each activity occurs after the previous one. Therefore, the total time is the sum of the activity times.

This plan assumes that you have no one to help (or hinder) you. However, most projects are not that simple. For example, a student may have to compete with her older sister for time in the only bathroom;

another student may have a mother who makes breakfast for her; and another student may have a little brother make his bed. In these cases, simply adding up these times will not give you the absolute time you need to get ready in the morning.

Q6. Suppose someone helped or hindered your morning routine. Why would adding up the times not give you the actual amount of time it would take to complete your morning routine?

10.1.1: Order-Requirement Digraphs

Graphs can be used to determine how to plan a project. Recall from Chapter 9 that a **graph** is set of nodes connected by arcs. An example of a graph is shown in Figure 10.1.1.

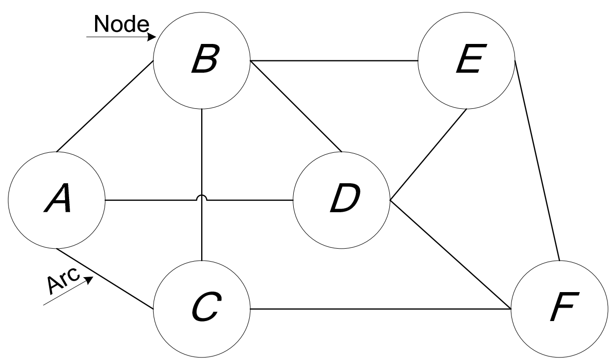


Figure 10.1.1: Example of a graph with 6 nodes and 10 arcs

Usually, when using graphs like this, movement from one node to the next can occur in either direction. This was the case in Chapter 9, when finding shortest paths and minimum spanning trees. However, in this chapter, movement along arcs occurs in only *one direction*. Here, graphs are used to model the situation of several activities in a project; the nodes represent an activity and the arcs represent the directional paths between activities.

Therefore, the order of the movement from node to node matters, and arrows are placed on the arcs to indicate the direction of the path. When direction is added to a graph, it is called a **digraph**, or a directed graph. The direction could refer to traffic on a one-way street or to an activity that needs to be completed before another activity begins. For instance, in the getting ready for school example, showering should occur before getting dressed. The activities could not work in the reverse direction.

When considering the specific order of the digraph (i.e., the dependencies between the activities), the digraph is called an order-requirement digraph. An **order-requirement digraph** is a digraph that does not loop back onto itself at any time. Figure 10.1.2 shows an example of a looping digraph; Figure 10.1.3 shows an example of an order-requirement digraph.

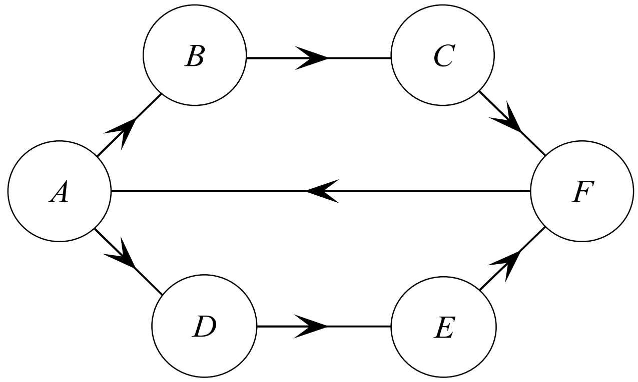


Figure 10.1.2: Example of a looping digraph

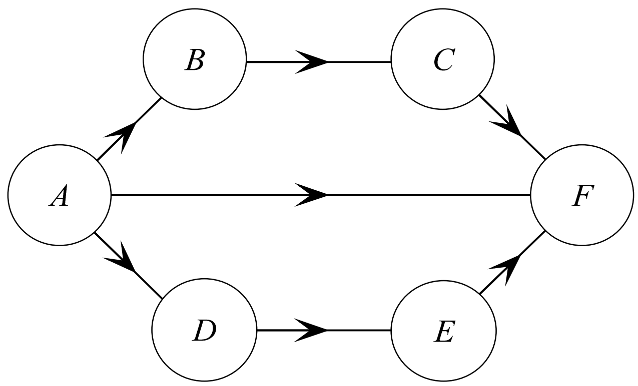


Figure 10.1.3: Example of an order-requirement digraph

Figure 10.1.4 shows an order-requirement digraph with the example of "taking a shower" occurring before "getting dressed."

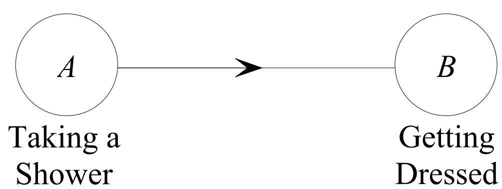


Figure 10.1.4: An example of a morning routine order-requirement digraph

Q7. Create an order-requirement digraph for your morning routine, where the nodes represent the activities from Table 10.1.1 and the arcs represent the directional paths between activities.

Next, weights are added to the nodes in the graph. Recall from previous chapters, **weights** are values that represent different units such as time or cost. In this chapter, all of the weights represent time in minutes, hours, days, weeks, and so on. The structure of the nodes includes six sections, where the activity and the weight are placed in the middle two sections, as shown in Figure 10.1.5.

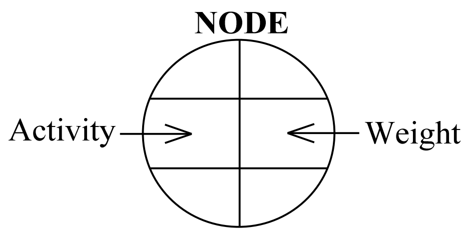


Figure 10.1.5: The structure of a node for an order-requirement digraph

Figure 10.1.6 shows an order-requirement digraph with the appropriate node structure. In this case, taking a shower takes 10 minutes and getting dressed takes 8 minutes.

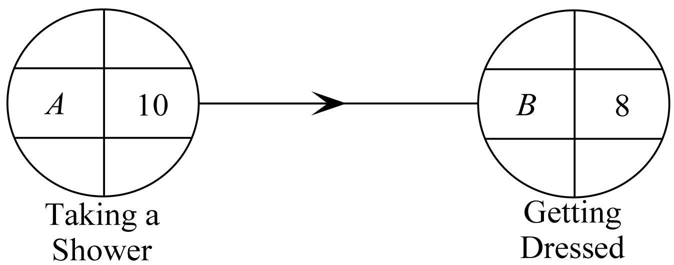


Figure 10.1.6: An example of an order-requirement digraph with the appropriate node structure

Q8. Revise your order-requirement digraph from the previous question to include the appropriate node structure, where the weights represent the completion times from Table 10.1.1.

Order-requirement digraphs, such as the ones seen in this section, can be used to solve planning problems. Remember that order-requirement digraphs, rather than looping digraphs, are appropriate to solve such problems because once an activity is finished that activity is not repeated. Order-requirement digraphs are a mathematical tool used to solve a **critical path method (CPM)** problem. The next section is an example of a CPM problem.

Section 10.2: Preparing a Taco Dinner

A simple project to plan is making a dinner. Maya and three of her friends are preparing a meal. Chefs who have cooked this dinner before have determined the order of the activities and the activity dependence. Remember that activity dependence explains which activity/activities must happen before the one in the row. For example, before "Prepare the pudding", one must "Find recipe for dessert in Mom's recipe box". The completion times were calculated by averaging the time it takes for different chefs to prepare this meal.

Maya and her friends use the information from the chefs to develop the activity chart shown in Table 10.2.1.

Activity Code	**Activity Description**	**Activity Dependence**	**Completion Time (minutes)**
A	Find recipe for taco salad on Internet	—	5
B	Find recipe for dessert in Mom's recipe box	—	6
C	Make graham cracker crust	*B*	4
D	Clean lettuce	*A*	3
E	Find cheese in refrigerator	*A*	1
F	Shred cheese	*E*	4
G	Brown hamburger and assemble tacos (including cheese)	*D, F*	14
H	Prepare the pudding	*B*	8
I	Assemble the pudding pie	*C, H*	4
J	Put out all the food for dinner	*G, I*	2

Table 10.2.1: Activity chart for making a taco dinner

Q1. Based on the information in Table 10.2.1, predict how long it will take to complete the dinner.

In general, an activity chart—like the one in Table 10.2.1—must be developed before creating an order-requirement digraph that reflects the information in the chart. The activity chart shows which activities are dependent on others, and this information is necessary when determining the direction of the arcs in the order-requirement digraph.

When drawing an order-requirement digraph, a box is used to indicate the node at the beginning of the project (start) and circles to represent other activities. Knowing this, Maya develops the order-requirement digraph for preparing a taco dinner shown in Figure 10.2.1.

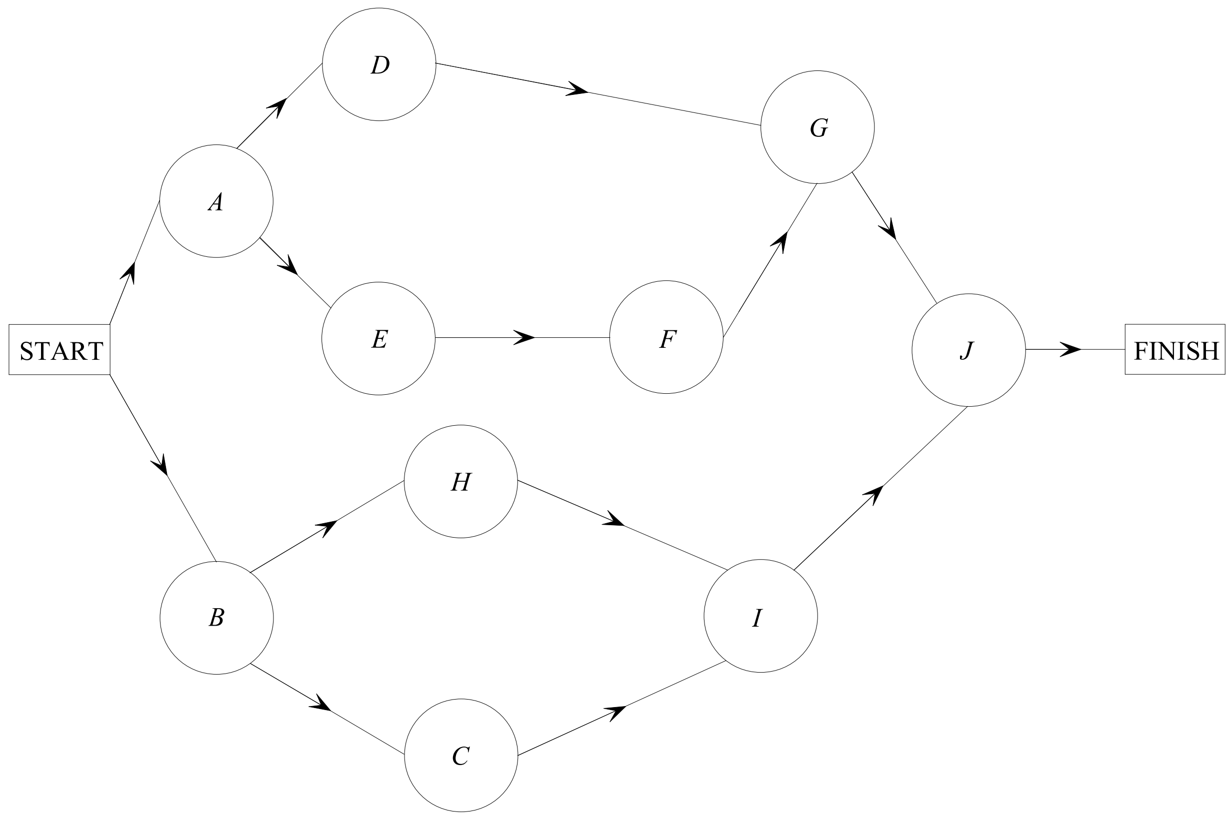


Figure 10.2.1: Order-requirement digraph representing the project of preparing a taco dinner

Maya knows the arrows force the direction of the path. That way when looking at the order-requirement digraph, she and her friends know they must complete activity *A* before reaching activity *D*. In other words, they need to find a recipe for taco salad on Internet (activity *A*) before they can clean the lettuce (activity *D*).

Recall from the previous section that the nodes should include weights. In this context, the weight of the nodes in the order-requirement digraph should be completion times, in minutes, because the completion times will determine when to move from one activity to the next. The structure of the node is shown in Figure 10.2.2.

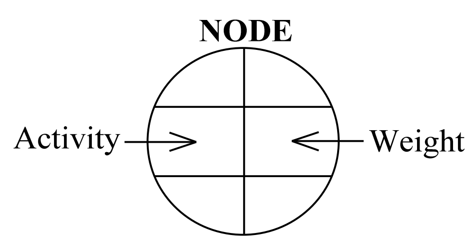


Figure 10.2.2: The structure of a node for an order-requirement digraph

The middle left hand side always represents the activity; the middle right hand side always represents the weight, which is completion time in this problem. The top and bottom regions will be explained later.

Maya applies this structure to her order-requirement digraph, with completion times as the weights, and produces Figure 10.2.3 below.

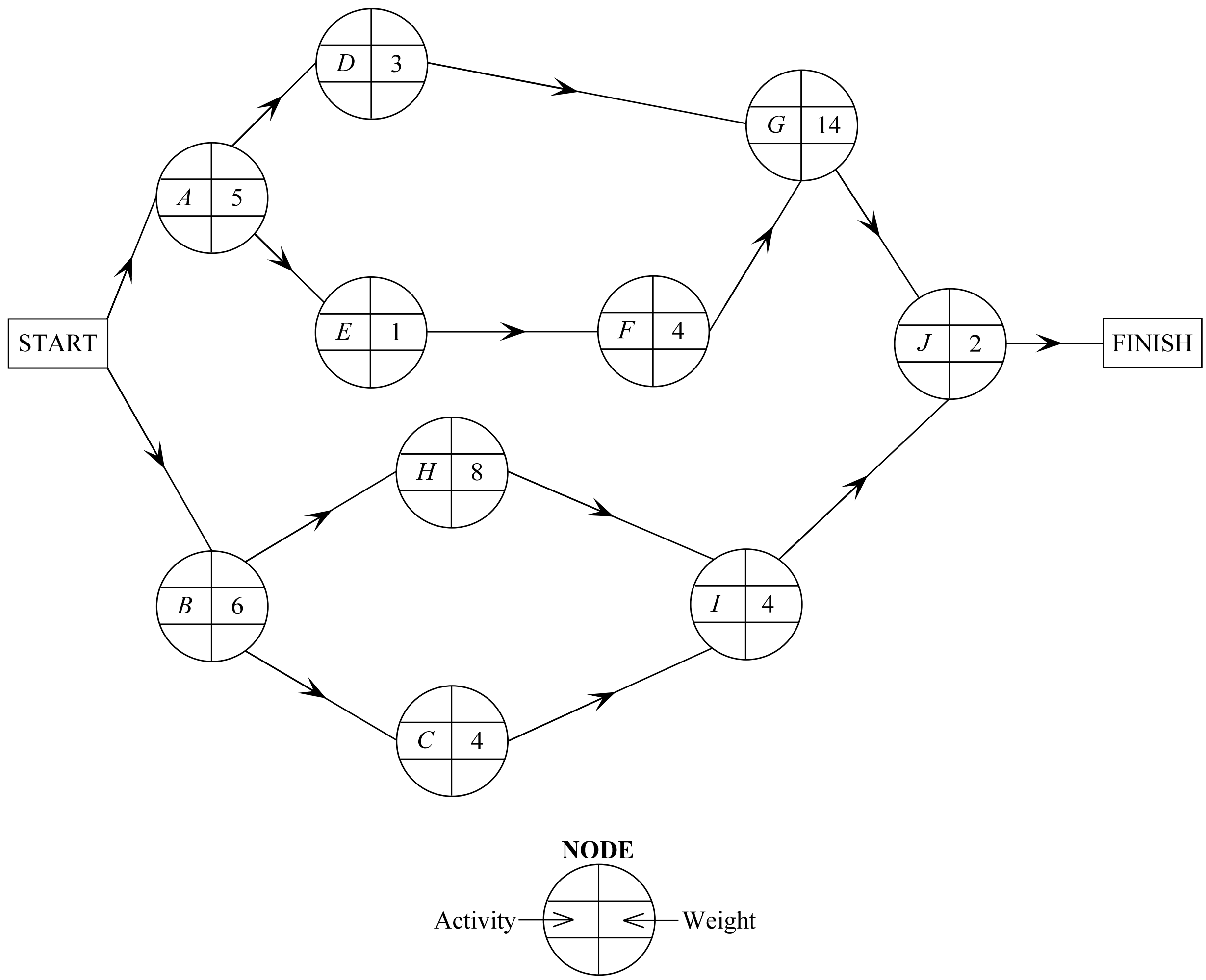


Figure 10.2.3: Order-requirement digraph representing preparing a taco dinner

Now Maya has developed the activity chart and the order-requirement digraph. The next step is to perform calculations to determine the least possible total time required to make the dinner. Since some activities can occur simultaneously, the total time to prepare dinner will be smaller than the sum of all the activity times.

A **path** is a sequence of connected activities that start at the first node. When it comes time for Maya and her three friends to make dinner, they will follow along the paths in the order-requirement digraph. Since all paths must be traveled in order to complete all the activities of the project, Maya studies the amount of time the different paths require.

Q2. Moving from left to right in Figure 10.2.3, follow all the paths that exist in the order-requirement digraph. Describe the possible paths from start to finish using letters and activities (e.g., path *A-D-G-J*: Find recipe for taco salad on Internet; Clean lettuce; Brown hamburger and assemble tacos (including cheese); Put out all the food for dinner).

Q3. Add up the total time to go through each path.

Q4. Suppose that Maya and her three friends each follow a different path.
   a. Who will be finished first? When will this person finish?

b. Who will be finished last? When will this person finish?
c. How long will it take for the entire meal to be finished? Explain your reasoning.

The longest path through the order-requirement digraph is the one of most interest. Given that all of the other paths are shorter in duration, the longest path determines the total time required to complete the project. If the activities on the longest path are delayed, the entire project will be delayed. Therefore, the activities on the longest path are the **critical path activities** of the project, and the longest path is called the **critical path**.

It may seem confusing that the longest path, rather than the shortest, is the critical path. However, the activities along the longest path are the critical ones because their path takes the longest time to complete, and all of the activities must be completed in a particular order.

Q5. Do you think it is possible for Maya to prepare this meal in the minimum completion time by herself? Why or why not?

Q6. Complete Table 10.2.2 to determine what activities Maya and her friends need to do to be sure they all finish in the shortest amount of time. For example, either Maya or one of her friends could find a recipe for taco salad on the Internet.

Activity Code	**Activity Description**	**Person**
A	Find recipe for taco salad on Internet	Maya / Friend 1
B	Find recipe for dessert in Mom's recipe box	
C	Make graham cracker crust	
D	Clean lettuce	
E	Find cheese in refrigerator	
F	Shred cheese	
G	Brown hamburger and assemble tacos (including cheese)	
H	Prepare the pudding	
I	Assemble the pudding pie	
J	Put out all the food for dinner	

Table 10.2.2: Activity chart for making a taco dinner

Q7. Complete Table 10.2.3 to determine what time each activity should start if dinner is to be served promptly at 6:00pm. For example, since it takes 2 minutes to put out all the food for dinner, activity *J* should begin at 5:58pm.

Activity Code	**Activity Description**	**Person**	**Time**
A	Find recipe for taco salad on Internet	Maya / Friend 1	
B	Find recipe for dessert in Mom's recipe box		
C	Make graham cracker crust		
D	Clean lettuce		
E	Find cheese in refrigerator		
F	Shred cheese		
G	Brown hamburger and assemble tacos (including cheese)		
H	Prepare the pudding		
I	Assemble the pudding pie		
J	Put out all the food for dinner		5:58pm

Table 10.2.3: Activity chart for making a taco dinner

Q8. Suppose browning the hamburger meat was done in a microwave, which cut down the browning and assembling time from 14 minutes to 6 minutes. What is the critical path now? (Remember that the critical path is the *longest* path determined by summing the weights along the path from start to finish.) Describe the path both using letters and the activities themselves.

As the number of activities and the dependence of one activity on others become more and more complicated, a procedure is needed to find the critical path that can be applied when there are more than paths and nodes.

10.2.1 Critical Path Method (CPM) Algorithm

Maya already determined the critical path in the taco dinner project by calculating the total time of each path in the order-requirement digraph. However, she could have instead used the critical path method (CPM) algorithm. Recall that an algorithm is a step-by-step procedure. There are four main steps in the CPM algorithm.

Step 1: Compute the earliest start time (EST) and the earliest finish time (EFT) for each activity in the order-requirement digraph.

Recall that there is a certain structure to the nodes in an order-requirement digraph (see Figure 10.2.2). The earliest start time and earliest finish time are placed at the top of the node, as shown in Figure 10.2.4.

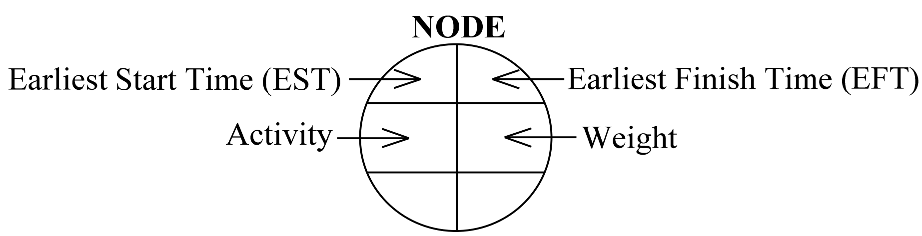


Figure 10.2.4: The structure of a node for an order-requirement digraph including EST, EFT

An activity cannot start until all preceding activities have been completed. Therefore, when using this algorithm, the **earliest start time** for an activity is the sum of the completion time for all the activities before it in its path. For example, Maya and her friends cannot begin cleaning the lettuce (activity *D*) until they find a recipe for the taco salad on the Internet (activity *A*). Therefore, the earliest start time (EST) of activity *D* is minute 5 because activity *A* finishes at minute 5.

The **earliest finish time** is the earliest time an activity can be completed, meaning

EFT = EST + [completion time of activity].

For example, activity *D*'s earliest finish time is minute 8 since Maya and her friends will start cleaning the lettuce at minute 5, and it takes three minutes to complete this activity (8 minutes = 5 minutes + 3 minutes).

Maya determines the ESTs and EFTs for all activities for the event of preparing dinner, as shown in Figure 10.2.5. However, she finds that some calculations are easier than others are.

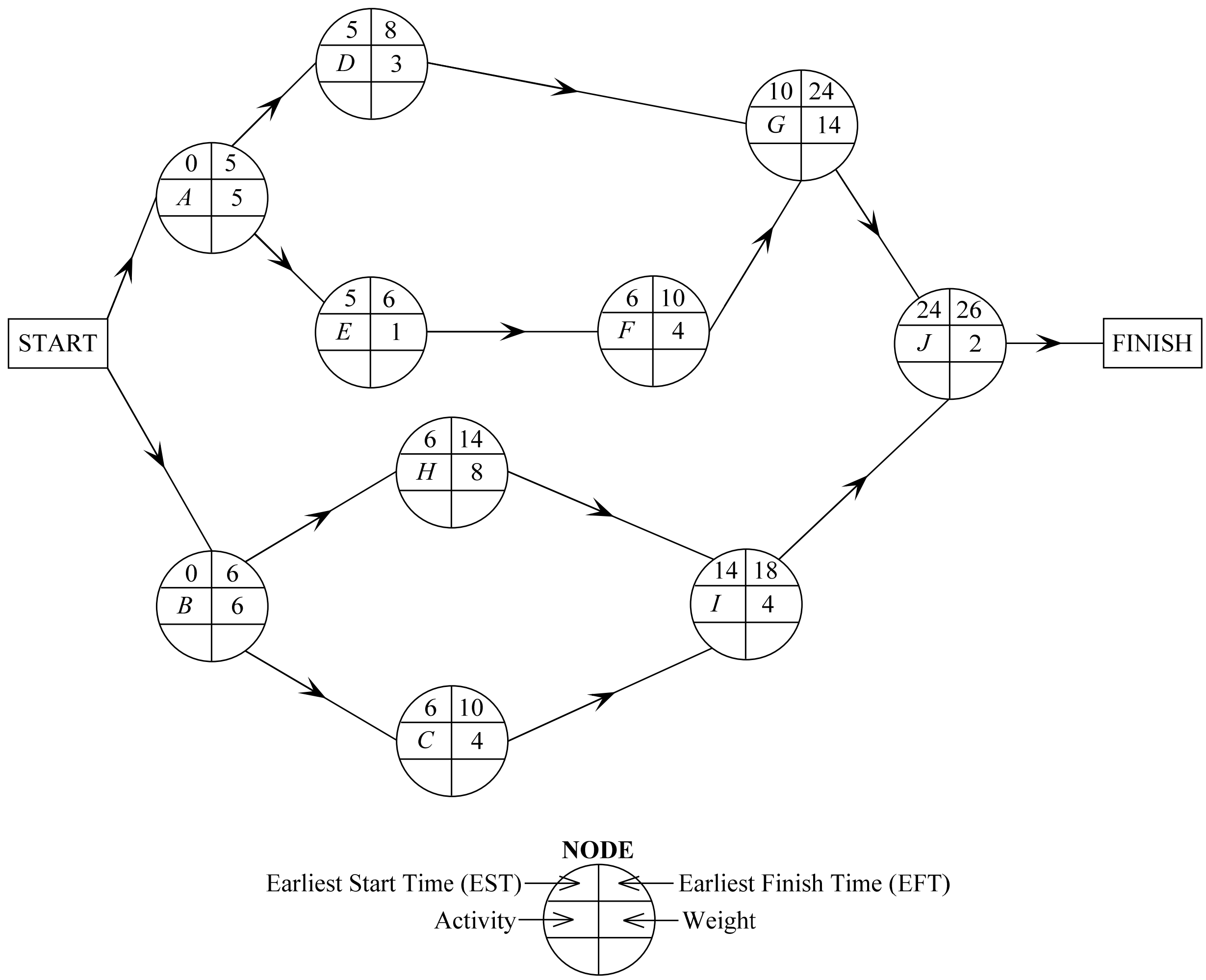


Figure 10.2.5: Order-requirement digraph with EFT, EST

For example, Maya has a difficult time determining the earliest time she and her friends can start browning the hamburgers and assembling the tacos (activity *G*). Maya and her friends cannot brown the hamburger and assemble the tacos until they finish cleaning the lettuce (activity *D*) and shredding the cheese (activity *F*). That is, activity *G* is dependent on both activities *D* and activity *F* meaning that she cannot start activity *G* until both activities are completed (see Figure 10.2.6). Activity *G* has two options for start time: minute 8 (when activity *D* finishes), and minute 10 (when activity *F* finishes). Since Maya and her friends have to wait until they finish *both* cleaning the lettuce (minute 8) and shredding the cheese (minute 10), they cannot begin browning the hamburgers and assembling the tacos until minute 10. Therefore, the earliest start time for activity *G* is 10 minutes.

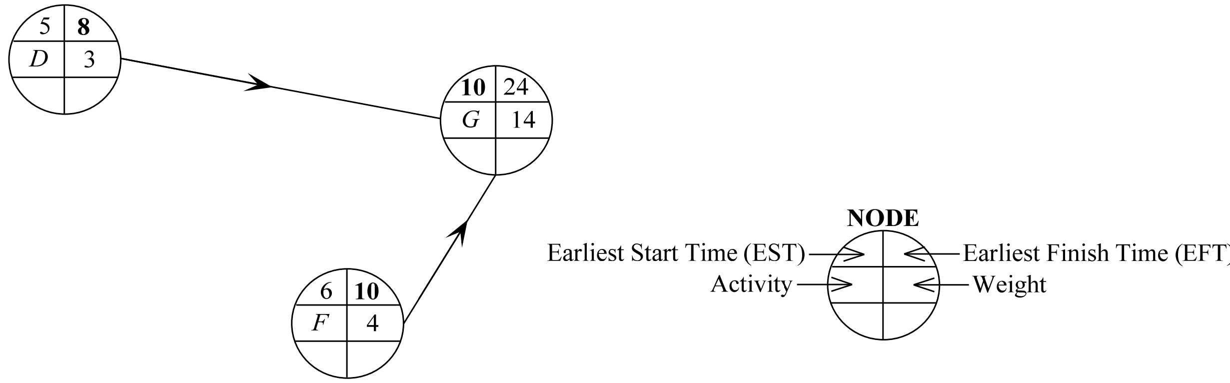


Figure 10.2.6: A closer look at the order-requirement digraph with EST, EFT

Maya looks at the last node in Figure 10.2.5 (activity *J*) and sees that the critical path (i.e., the longest path through the order-requirement digraph) is 26 minutes long. By computing the EST and EFT values, Maya automatically calculates the length of the critical path as she works left to right through the order-requirement digraph.

Now Maya follows the activities *backwards* through the order-requirement digraph (right to left) and looks for the path of activities that will take the most time. This process will allow her to identify the activities that make up the critical path.

She starts with activity *J*, the final activity, and then decides which activity is activity *J* waiting on to complete before it can start. In this case, it is activity *G*. That is, Maya and her friends want to put out all the food for dinner, but they cannot do so until they finish browning the hamburger and assembling the tacos.

Next, Maya determines which activity is activity *G* waiting on before it can start, and so on. She notices, when there are two or more choices as she works backwards, she chooses the node whose end time matches the start time of the successive node.

By continuing this pattern, she discovers that the longest path is *A-E-F-G-J*. Therefore, the critical path is *A-E-F-G-J*. Since this path is 26 minutes long, it will take Maya and her friends—working simultaneously on different paths—26 minutes to make all parts of dinner.

Q9. Is that the answer you got before? If so, what are some benefits of using the CPM algorithm rather than calculating the length of each possible path?

Maya knows that if she was only interested in finding the critical path (i.e., the shortest time it takes to prepare dinner), then she is done. However, Maya also wants to know how much flexibility she has in the schedule. For example, she wonders if she and her friends can take a break at any point during the dinner preparations. Therefore, she decides to continue the algorithm.

Step 2: Compute the latest start time (LST) and the latest finish time (LFT) for each activity in the order-requirement digraph.

The latest start time and latest finish time are placed at the bottom of the node, as shown in Figure 10.2.7.

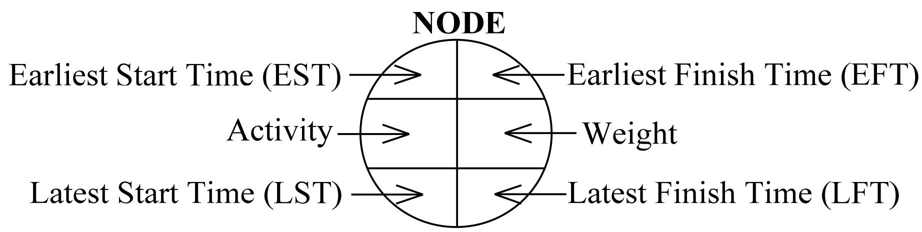


Figure 10.2.7: The structure of a node for an order-requirement digraph including LST, LFT

Maya continues onto the second part of the critical path method (CPM) algorithm by making a series of calculations while taking a *backward pass* through the order-requirement digraph. This is different from just working backwards as she did above because not only is she moving backwards, right to left, through the order-requirement digraph, but she is also calculating the finish time before the start time.

Starting at *the last node* (activity *J*), and using the minimum finish time of 26 minutes, Maya traces backwards through the order-requirement digraph, computing a latest start time (LST) and latest finish time (LFT) for each activity.

The **latest finish time** for an activity refers to the latest time a certain activity can finish. For example, the latest finish time for activity *J* is minute 26 since 26 minutes is the amount of time to complete the longest path. That is, the latest time Maya and her friends can finish putting out the food for dinner is at minute 26.

The **latest start time** is the latest time an activity can be begin, meaning

LST = LFT – [completion time of activity].

For example, the latest time Maya and her friends can start to put out the food for dinner is at minute 24 because it takes 2 minutes to complete this activity and this activity ends at minute 26 (24 minutes = 26 minutes – 2 minutes). Thus, the latest start time (LST) for activity *J* is minute 24.

Next, Maya and her friends want to know when they should finish browning the hamburger and assembling the tacos (activity *G*). Moving backwards from activity *J* to activity *G*, Maya realizes the latest activity *G* can finish occurs at the latest time activity *J* starts. That is, they finish browning the hamburger and assembling the tacos at the same time that they start putting the food out for dinner. Therefore, activity *G*'s latest finish time is minute 24.

Then, to finish at minute 24, Maya and her friends need to start browning the hamburger and assembling the tacos at minute 10 because it takes fourteen minutes to complete this activity (activity *G*).

Figure 10.2.8 shows the LST and LFT values for activities *G* and *J*.

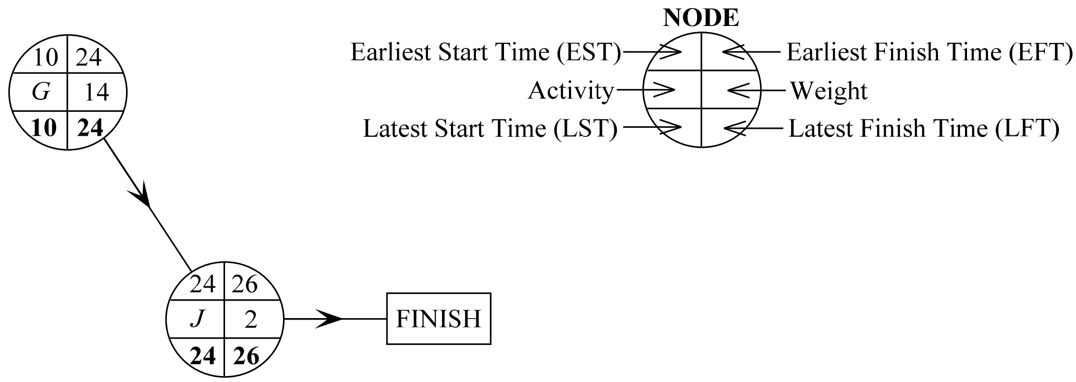


Figure 10.2.8: A closer look at the order-requirement digraph with LST, LFT

Next, Maya finds the LSTs and LFTs for all activities for the event of preparing dinner, as shown in Figure 10.2.9.

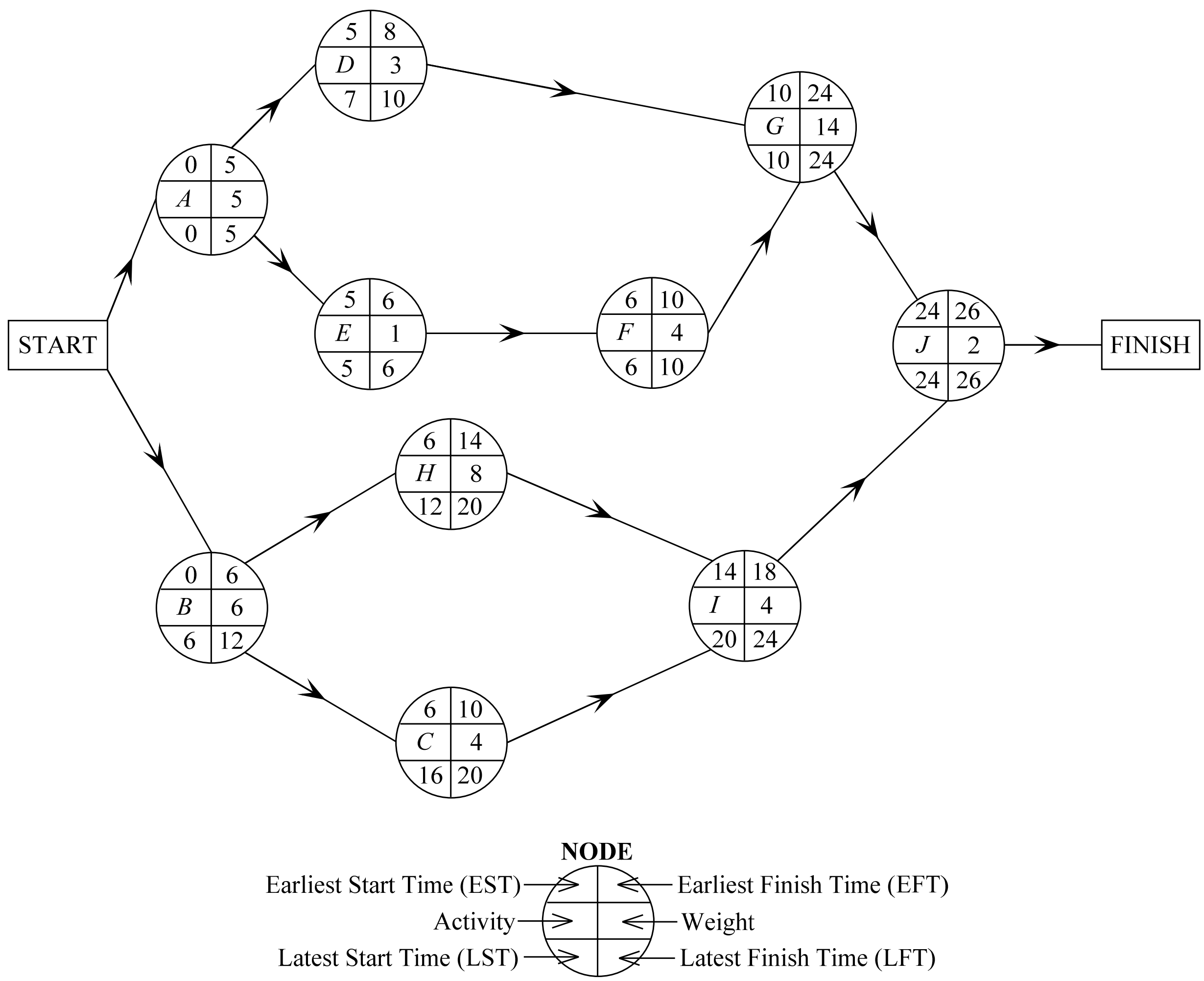


Figure 10.2.9: Order-requirement digraph with LFT, LST

Again, Maya finds that some values are easier to compute than others are. For example, she notices that activity *A* has two options for latest finish time (see Figure 10.2.10). Both activity *D* and activity *E* are dependent on activity *A* meaning that Maya and her friends cannot start cleaning the lettuce (activity *D*) or finding the cheese in the refrigerator (activity *E*) until they find a recipe for taco salad on the Internet (activity *A*).

Maya focuses on how activity *D* and activity *E* influence activity *A*. The latest activity *D* starts is at minute 7 and the latest activity *E* starts is at minute 5. Maya notices that if activity *A* were to finish at minute 7, then activity *E* would not be able to start on time, which would delay the project. That is, if Maya and her friends finish finding the recipe for taco salad at minute 7, they could not start finding the cheese in the refrigerator at minute 5.

On the other hand, if activity *A* finished at minute 5, then both activity *E* and activity *D* could start on time. In other words, if Maya and her friends finish finding the recipe for taco salad at minute 5, they could start finding the cheese in the refrigerator (activity *E*) at minute 5 *and* they could start cleaning the lettuce (activity *D*) at minute 7. No activity would be delayed.

Therefore, the latest time activity *A* can finish is minute 5.

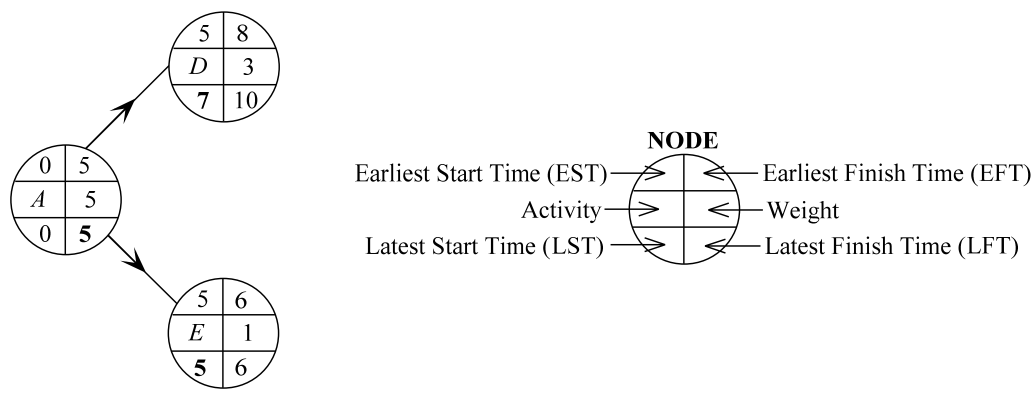


Figure 10.2.10: A closer look at order-requirement digraph with LST, LFT

Now that Maya has determined the EST, EFT, LST, and LFT values, she can determine how much time flexibility she and her friends have when making dinner.

Step 3: Compute the slack for each activity in the order-requirement digraph.

After obtaining the start and finish activity times and labeling the nodes as such, Maya continues on with the critical path method (CPM) algorithm and finds the amount of slack associated with each of the activities. **Slack** is defined as the length of time a particular activity can be delayed without affecting the total time required to complete the project. There are two possible equations to compute the amount of slack for each activity:

$$\text{Slack} = \text{LST} - \text{EST}$$
or
$$\text{Slack} = \text{LFT} - \text{EFT}$$

It does not matter which equation Maya chooses to compute slack, both equations will result in the same quantity when she subtracts the corresponding values in the nodes.

For example, the slack time for finding a recipe for taco salad (activity *A*) can be found by calculating LST – EST = 0 – 0 = 0 minutes or by calculating LFT – EFT = 5 – 5 = 0 minutes. In either case, activity *A* has no slack. On the other hand, the slack time for finding a recipe for dessert (activity *B*) is 6 – 0 = 6 minutes (or 12 – 6 = 6 minutes). Therefore, Maya and her friends could goof off for up to six minutes when finding a dessert recipe and still finish dinner on time.

Maya creates a table of the activities with the earliest start time (EST), earliest finish time (EFT), latest start time (LST), latest finish time (LFT), and slack time columns added (see Table 10.2.5). Maya prefers the table—rather than the order-requirement digraph—when calculating slack times because the order-requirement digraph sometimes becomes complicated when the layout is cluttered with the EST, EFT, LST, and LFT values filled in.

Activity	**EST**	**LST**	**EFT**	**LFT**	**Slack (minutes)**		**Critical Activity**
					LST – EST	**LFT – EFT**	
A	0	0	5	5	0 – 0 = **0**	5 – 5 = **0**	Yes
B	0	6	6	12	6 – 0 = **6**	12 – 6 = **6**	
C	6	16	10	20	16 – 6 = **10**	20 – 10 = **10**	
D	5	7	8	10	7 – 5 = **2**	10 – 8 = **2**	
E	5	5	6	6	5 – 5 = **0**	6 – 6 = **0**	Yes
F	6	6	10	10	6 – 6 = **0**	10 – 10 = **0**	Yes
G	10	10	24	24	10 – 10 = **0**	24 – 24 = **0**	Yes
H	6	12	14	20	12 – 6 = **6**	20 – 14 = **6**	
I	14	20	18	24	20 – 14 = **6**	24 – 18 = **6**	
J	24	24	26	26	24 – 24 = **0**	26 – 26 = **0**	Yes

Table 10.2.5: Application of CPM algorithm to determine the critical path

Step 4: Identify the critical path and find the slack along each path.

Maya notices that the slack times along the critical path are zero. This makes sense to her in two ways. First, if she follows the rule, slack is LFT – EFT or LST – EST and in each case, the difference is zero. If an activity is on a critical path, then there is no extra time to get it done. Another way of stating this is that activities in the critical path are essential to the timing of dinner. If any activity on the critical path is delayed, the entire dinner is delayed.

Second, the numbers on the other activities are the extra minutes Maya and her friends can use without affecting the finish time. For example, Maya or a friend can take up to 10 extra minutes to make a graham cracker crust (activity *C*), and it will not delay dinner. However, slack is the accumulation of extra time along a path. As Maya and her friends move along a path, the slack time represents the *total* amount of time they have to "slack off" along that path, not the amount of time they have to slack off for an individual activity. For example, the slack time for activity *I* is 6 minutes, so they can take a total of 6 extra minutes for activities *B*, *C*, *H*, and *I* combined. This idea becomes clearer in the next section, when Maya creates a Gantt chart.

The steps of the CPM algorithm are given below.

CPM Algorithm

Step 1: Compute the earliest start time (EST) and the earliest finish time (EFT) for each activity in the order-requirement digraph.

Step 2: Compute the latest start time (LST) and the latest finish time (LFT) for each activity in the order-requirement digraph.

Step 3: Compute the slack for each activity in the order-requirement digraph.

Step 4: Identify the critical path and find the slack along each path.

The critical path method (CPM) is often more complicated than the above example. For example, Maya did not worry about issues like who can do which activity better or if they have the resources needed. In later real-life CPM situations, more complexity will come into play.

10.2.2 Using a Gantt Chart for the Critical Path Method (CPM) Algorithm

Once the activity dependence and completion times for each activity in the project has been determined, another approach to finding the critical path, earliest start time (EST), earliest finish time (EFT), latest start time (LST), and latest finish time (LFT) is to use a Gantt chart instead of an order-requirement digraph.

The History of Gantt Charts

Henry Laurence Gantt (1861-1919) published the Gantt chart in 1910 as a visual tool to display a project's schedule and progress of the project. Gantt charts have been used on many large construction projects, including the Hoover Dam in 1931 and the interstate highway network, which started in 1956.

The **Gantt chart**, in essence, is a graph where the horizontal axis represents the time for each project, and the vertical axis represents the activities of the project.

Maya decides to make a Gantt chart for the taco dinner project. She begins by drawing a horizontal axis and marking off increments in the appropriate time unit. For preparing a taco dinner example, Maya marks off increments of one for an axis that measures minutes, as shown in Figure 10.2.11. At this point, she does not know how long the horizontal axis will be, so she leaves herself some space.

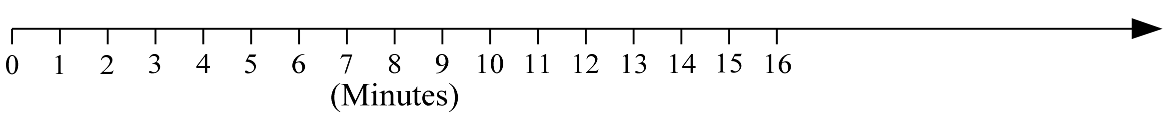


Figure 10.2.11: The horizontal axis of a Gantt chart

Next, Maya draws the vertical axis and marks off equal increments for activities required to prepare a taco dinner, as shown in Figure 10.2.12. Again, she does not know how long the vertical axis will be, so she leaves herself space.

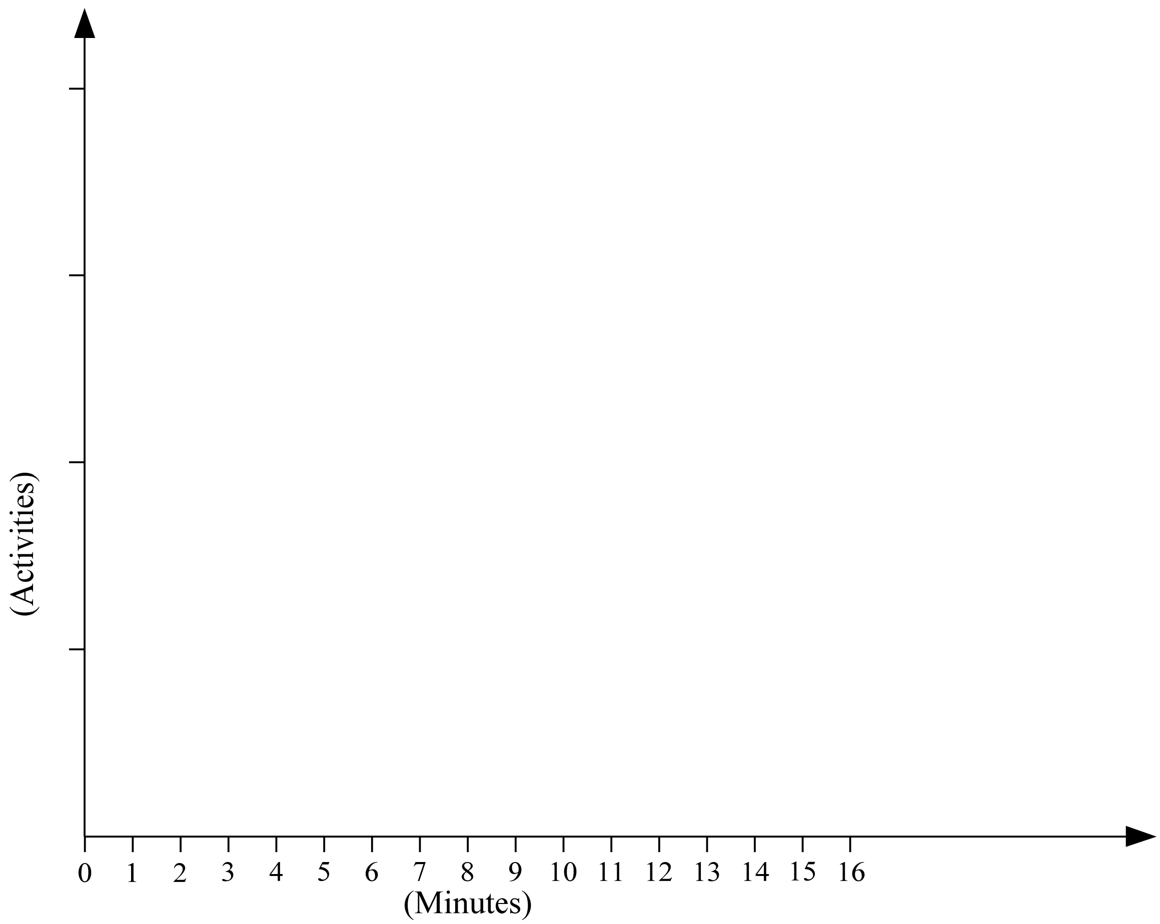


Figure 10.2.12: The horizontal and vertical axes of a Gantt chart

For the next part, Maya uses a pencil because she may need to erase. She knows it will take her a couple of tries for the Gantt chart to be correctly drawn.

Maya reminds herself of the activities that make up the taco dinner project (Table 10.2.6)

Activity Code	**Activity Description**	**Activity Dependence**	**Completion Time (minutes)**
A	Find recipe for taco salad on Internet	—	5
B	Find recipe for dessert in Mom's recipe box	—	6
C	Make graham cracker crust	*B*	4
D	Clean lettuce	*A*	3
E	Find cheese in refrigerator	*A*	1
F	Shred cheese	*E*	4
G	Brown hamburger and assemble tacos (including cheese)	*D, F*	14
H	Prepare the pudding	*B*	8

I	Assemble the pudding pie	*C, H*	4
J	Put out all the food for dinner	*G, I*	2

Table 10.2.6: Activity chart for making a taco dinner

She notices that both activity *A* and activity *B* are not dependent on any other activity; Maya and her friends do not need to wait before they start finding the taco salad recipe and the dessert recipe. Thus, she starts both of these activities at minute 0 in her Gantt chart, as shown in Figure 10.2.13.

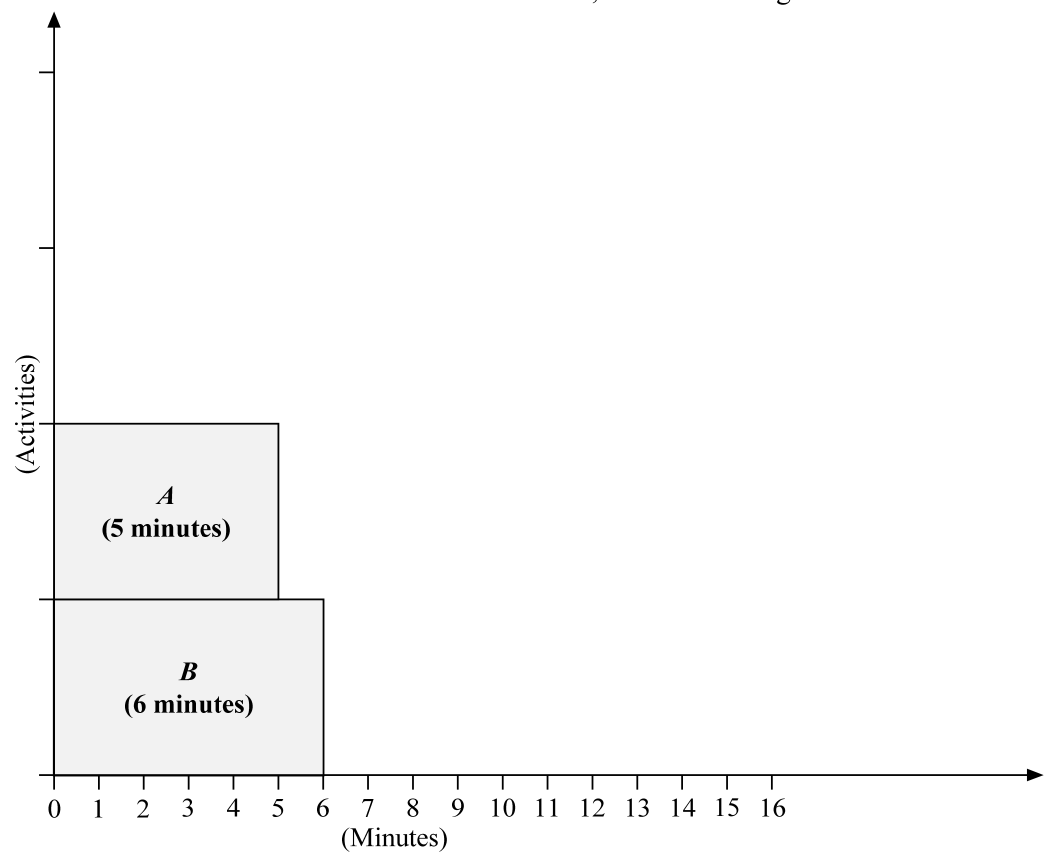


Figure 10.2.13: Activity *A* and activity *B* positioned on a Gantt chart

As shown in Figure 10.2.13, Maya draws the height of each rectangle the same (1 unit on the vertical axis), and she draws the width of the rectangle as the amount of time it takes to complete each activity. Since it takes 5 minutes to find a recipe for the taco salad, the rectangle for activity *A* is 5 units long; since it takes 6 minutes to find a recipe for dessert, the rectangle for activity *B* is 6 units long.

Next, Maya adds on the activities that are dependent on activity *A* and activity *B*. In this example, activity *D* and activity *E* are dependent on activity *A* because Maya and her friends must find a recipe for taco salad before they can clean the lettuce and find the cheese in the refrigerator. Activity *H* and activity *C* are dependent on activity *B* because they must find a recipe for dessert before they can make the graham cracker crust and prepare the pudding.

Because two activities are dependent on one activity, Maya has to expand the height of the rectangles representing activities *A* and *B* in order to accommodate for this dependence. She then attaches activities *D*, *E*, *H*, and *C* to the appropriate box, as shown in Figure 10.2.14.

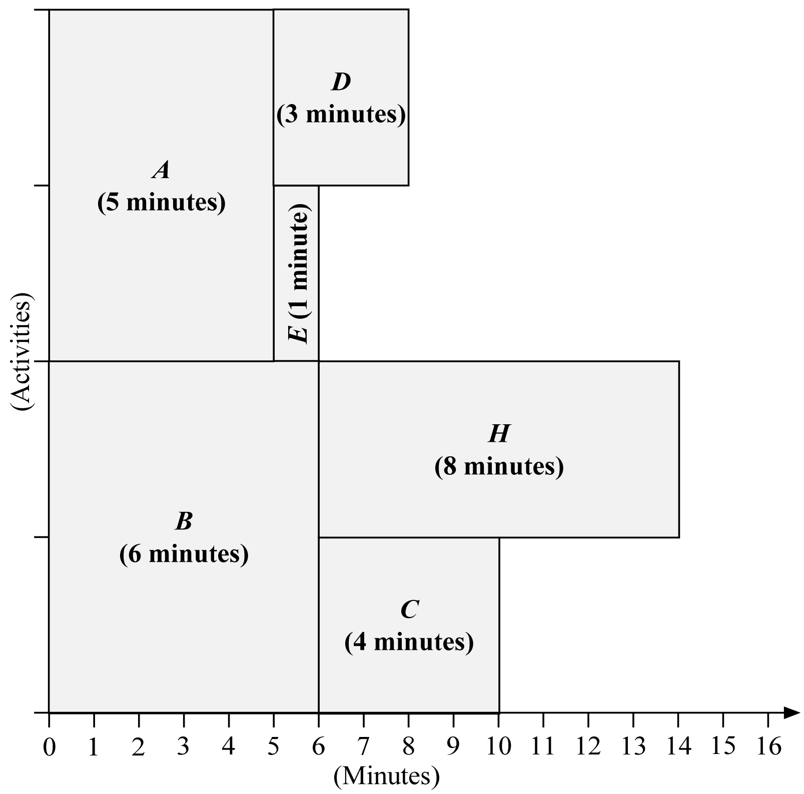


Figure 10.2.14: Activities *A*, *B*, *D*, *E*, *H*, and *C* positioned on a Gantt chart

Looking at the Gantt chart above, Maya notices the start times for activities *D*, *E*, *H*, and *C*. Activity *D* and activity *E* start at minute 5 and activity *H* and activity *C* start at minute 6. These times refer to the earliest start times (ESTs) for each activity. For example, the earliest Maya and her friends can start cleaning lettuce (activity *D*) and finding cheese in the refrigerator (activity *E*) is 5 minutes after the project started. Then, the earliest Maya and her friends can start preparing the pudding (activity *H*) and making the graham cracker crust (activity *C*) is 6 minutes after the project started.

Moreover, Maya sees the earliest finish times (EFTs) by looking at where each activity ends along the horizontal axis: minute 8 for activity *D*, minute 6 for activity *E*, minute 14 for activity *H*, and minute 10 for activity *C*. Therefore, Maya and her friends will finish cleaning the lettuce (activity *D*) at minute 8, finding the cheese (activity *E*) at minute 6, preparing the pudding (activity *H*) at minute 14, and making the graham cracker crust (activity *C*) at minute 10.

The last thing Maya considers when drawing the Gantt chart is slack time. Slack time exists when there are gaps between boxes. For example, Maya and her friends cannot begin to assemble the pudding pie (activity *I*) until they make the graham cracker crust (activity *C*) and prepare the pudding (activity *H*); that is, activity *I* is dependent on both activity *H* and activity *C*. Thus, activity *I* is attached to both activities *H* and *C*. When activity *I* is attached to both activities *H* and *C*, a gap is created between activity *C* and activity *I*. This gap represents a slack time of 4 minutes. In other words, since making the graham cracker crust (activity *C*) only takes 4 minutes to complete, Maya and her friends will have some slack time while they wait for the pudding to be prepared, which takes 8 minutes. This leaves a slack time of 4 minutes. The gap illustrating the slack time is shown in Figure 10.2.15.

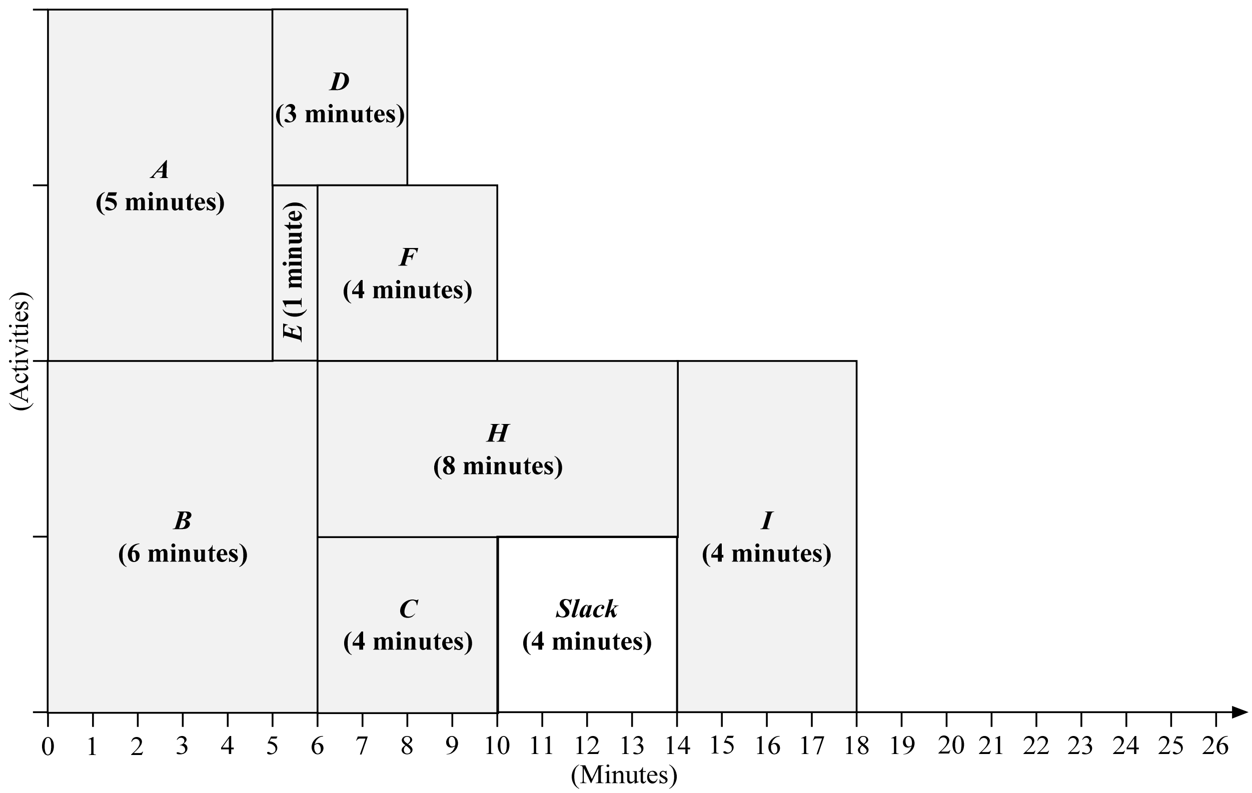


Figure 10.2.15: Slack time for the taco dinner Gantt chart

Maya continues the Gantt chart by adding on activities *G* and *J* and gets the completed Gantt chart below shown in Figure 10.2.16.

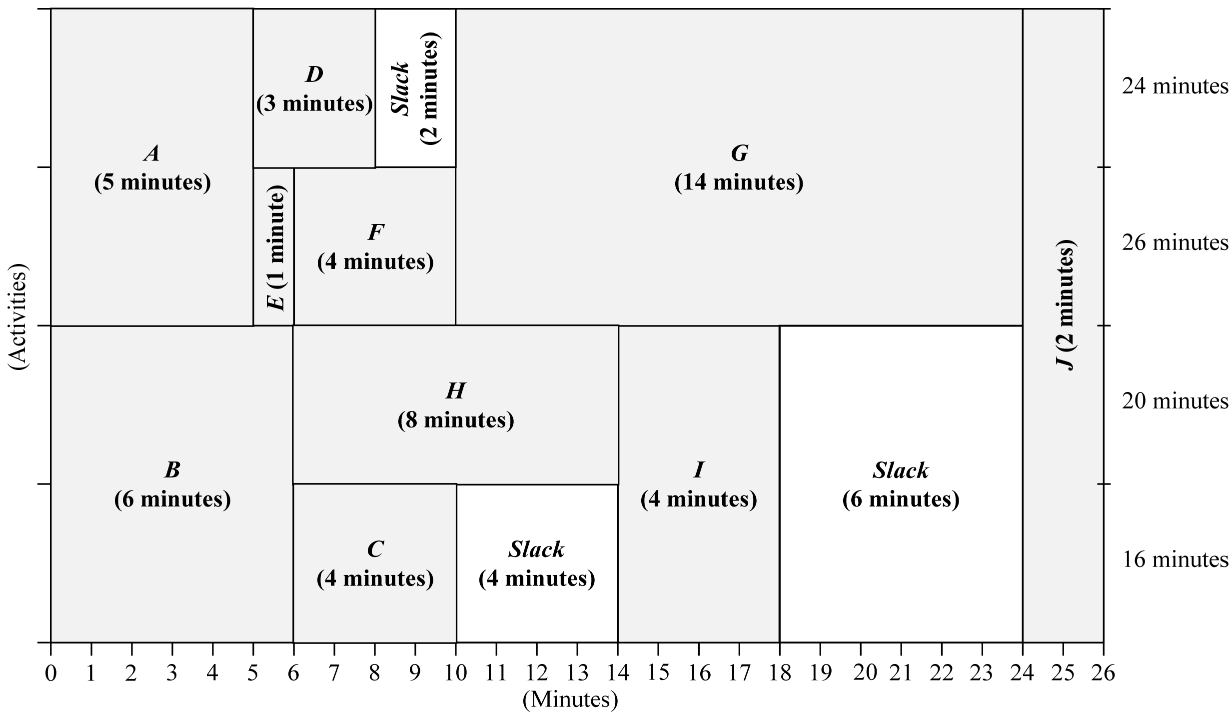
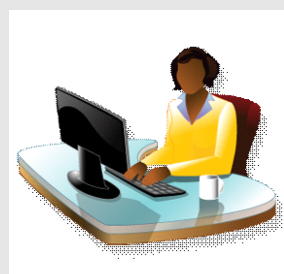


Figure 10.2.16: Completed Gantt chart for EST, EFT: Preparing a taco dinner

Gantt Charts

People often create Gantt charts using Excel or other software designed specifically for their creation. The reason they are so helpful is that the visual representation allows project managers to quickly see each path and the time allotted for a specific path.



The Gantt chart displays slack time so that Maya and her friends know exactly how much time they have to goof off on certain activities and still put dinner on the table in the minimum amount of time.

Moreover, the Gantt chart informs Maya how many paths there are and how long each path takes. She determines how many paths there are by focusing on the vertical axis and counting the number of increments used to determine box height. In the Gantt chart above, there are four increments along the vertical axis; therefore, Maya will need four people to prepare the taco dinner (Maya and her three friends).

Maya can determine how long each path takes by choosing an increment on the vertical axis and summing up the minutes of each activity box that intersects the corresponding horizontal band as she moves left to right.

For example, Maya looks at the first increment on the vertical axis. Activities *B*, *C*, *I*, and *J* lie along this first increment (see Figure 10.2.17). The sum of these activity times is 6 + 4 + 4 + 2 = 16 minutes. Maya leaves out the slack boxes in the sum because the boxes represent extra time. Therefore, the first path takes a total of 16 minutes to complete. This is shown in Figure 10.2.17.

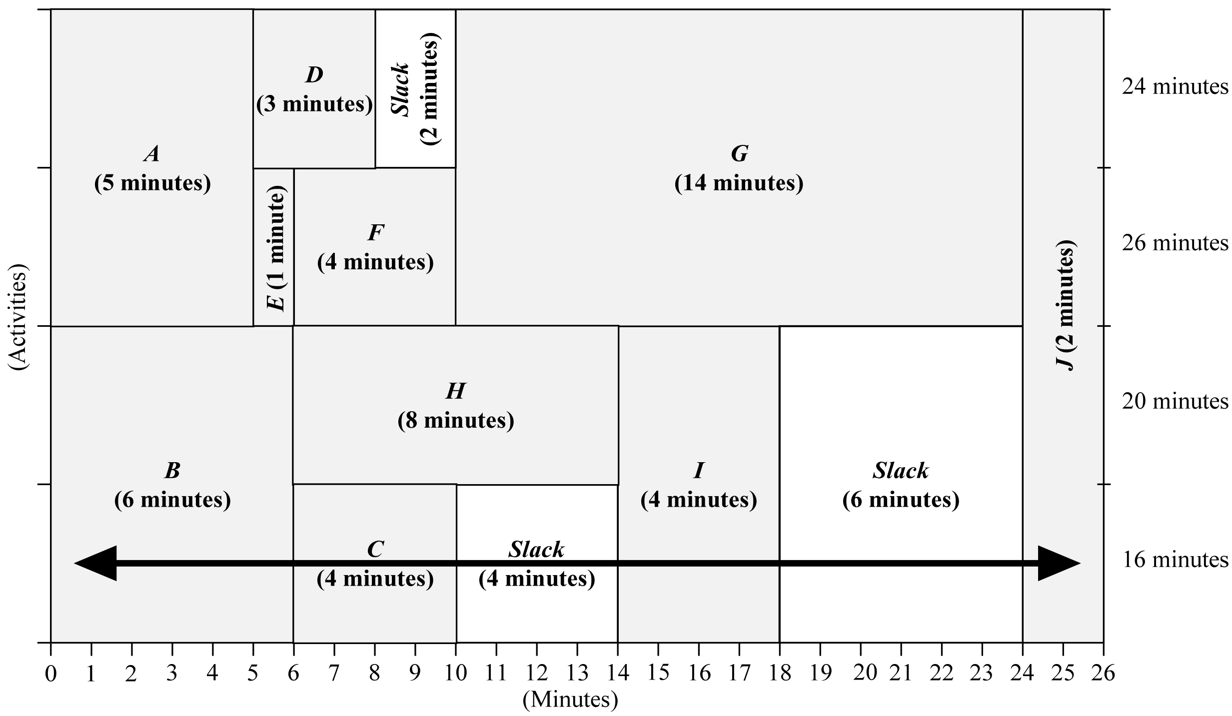


Figure 10.2.17: Length of first path on the taco dinner Gantt chart

Q10. Determine the lengths of the other paths on the taco dinner Gantt chart.

Q11. Which path is the critical path? How does this path differ from the others on the Gantt chart?

The Gantt chart in Figure 10.2.16 shows the earliest start times (ESTs) and earliest finish times (EFTs). Next, Maya creates the Gantt chart for latest start time and latest finish time.

Maya calculates the latest start time (LST) and latest finish time (LFT) for each activity by shifting the slack boxes to the left and the activity boxes to the right. For example, Maya slides activity *D* as far right as it will go. The start time for activity *D* (cleaning the lettuce) changes from minute 5 to minute 7; the finish time changes from minute 8 to minute 10. These modifications represent the difference between EST and LST and EFT and LFT. Figure 10.2.18 shows the Gantt chart for LST and LFT.

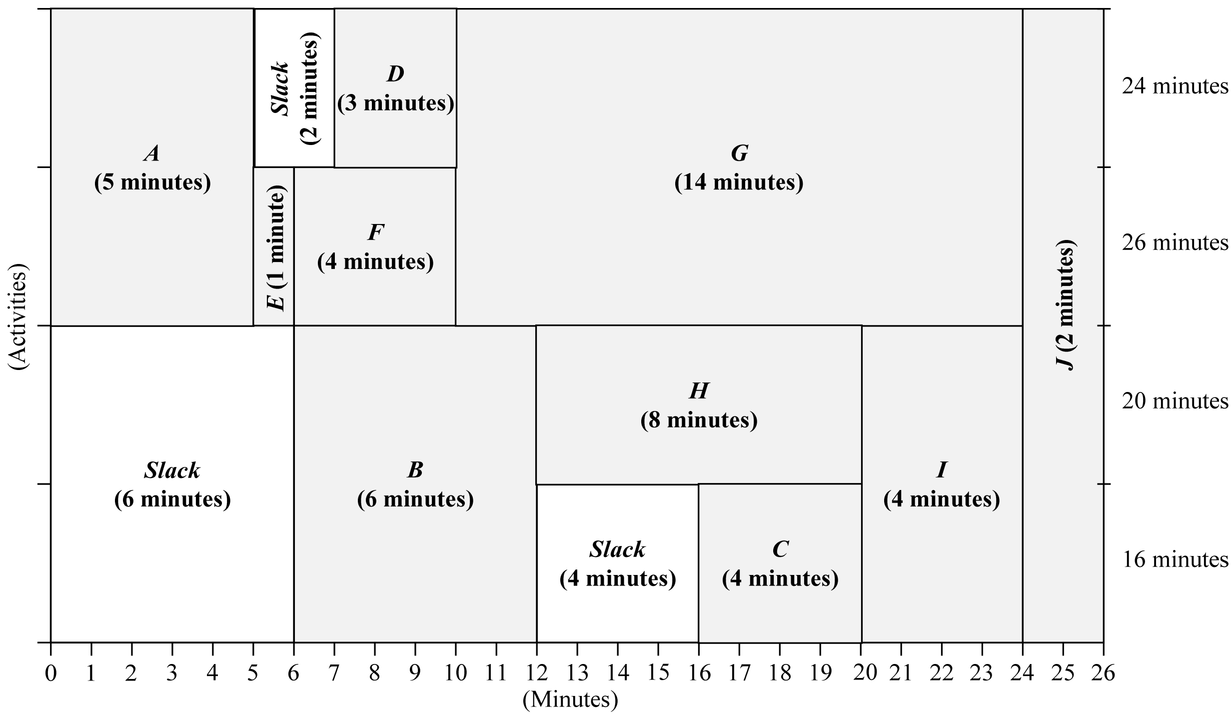


Figure 10.2.18: Gantt chart for LST, LFT: Preparing a taco dinner

Now that Maya has created the Gantt chart for EST and EFT (Figure 10.2.16) and the Gantt chart for LST and LFT (Figure 10.2.18), she wonders how much she and her friends can goof off and still finish the dinner in 26 minutes.

Maya finds the Gantt chart to be useful when thinking about slack time because she can shift the boxes back and forth. For example, Maya knows that activity *B* and activity *H* have slack times of 6 minutes (see Table 10.2.5). However, these activities do not *both* have this slack time. If activity *B* is shifted right 6 minutes, then there will be no slack left for activity *H*. Using the Gantt chart, Maya can visualize this scenario more easily than when she used a table.

Q12. Suppose Maya and her friends start finding the recipe for dessert (activity *B*) at minute 0 but then took an extra six minutes to complete this activity.
 a. Could they begin preparing the pudding (activity *H*) on time?
 b. In this scenario, how much extra time do they have to goof off when preparing the pudding?
 c. Could they begin making the graham cracker crust (activity *C*) on time?
 d. In this scenario, how much extra time do they have to goof of when making the graham cracker crust?

Q13. Suppose Maya and her friends start finding the recipe for dessert (activity *B*) at minute 0 and finish at minute 6; they start making the graham cracker crust (activity *C*) at minute 6 and finish at minute 14; they start preparing the pudding (activity *H*) at minute 6 and finish at minute 18.
 a. Calculate how long each of these activities took (activities *B*, *C*, and *H*).
 b. Calculate how much slack time they used up in each of these activities (i.e., how much longer did the activities take than was planned?).
 c. In this scenario, when will they be able to begin assembling the pudding pie (activity *I*)?
 d. When will they finish activity *I* assuming the time it takes for them to assemble a pie is:

1. 4 minutes?
2. 6 minutes?
3. 8 minutes?

e. Will the project be delayed in each of these cases? Explain.
1. 4 minutes
2. 6 minutes
3. 8 minutes

f. How much slack time do they have when assembling the pudding pie (activity *I*)?

The CPM algorithm and Gantt charts are very useful for small projects, like planning a dinner. More typically, however, these methods are used in very large projects. In the next section, the CPM algorithm and Gantt charts are used in a much larger project.

10.3 Scheduling with CPM for a Flight Propulsion System

Scott Williams has been requested to support a group of engineers working for a major aerospace company to help in the planning and scheduling of a critical subsystem for a weather satellite. A weather satellite sends important weather information to ground receiving stations. The flight propulsion system is one of the key subsystems in the satellite. The flight propulsion system, shown in Figure 10.3.1, consists of propellant tanks, rocket engines, and other components.

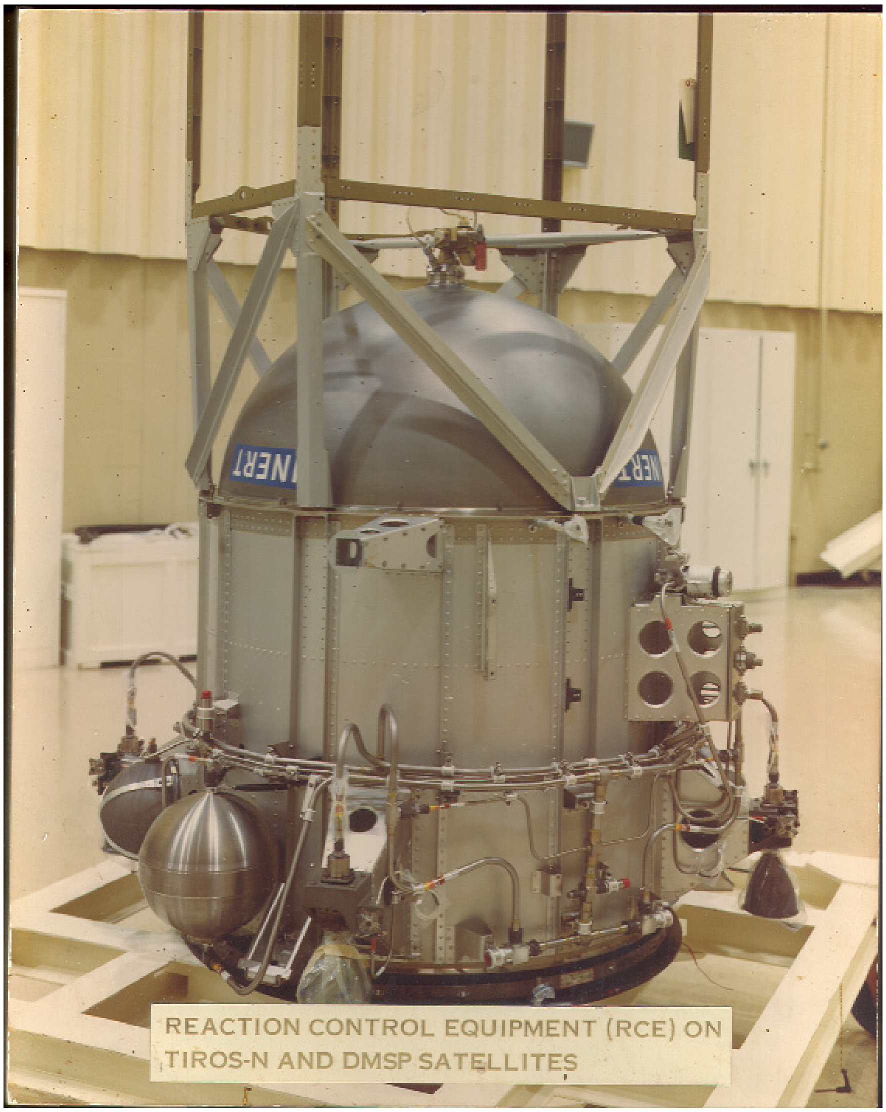


Figure 10.3.1: Flight propulsion system

The manufacturing of flight hardware for such a system is a complex task that requires frequent testing and inspections. This project is labor intensive and must meet strict schedule requirements. Scott Williams' job is to calculate which activities are critical and how various delays will affect the construction of the flight propulsion system. Scott Williams uses the critical path method (CPM) algorithm to help with the planning and scheduling of this project.

The engineers who are constructing the flight propulsion system must accomplish different activities sequentially. Thus, the overall program is constrained by time. Scott Williams' primary objective in this project is to ensure planned schedules are on time, and to determine how the entire project will be delayed if an activity is delayed.

First, Scott Williams lists the major activities that need to be completed. He creates this list based on the order in which the work is performed. This list of major activities is shown in Table 10.3.1.

Major Activities
Stainless steel tubing is procured and placed in stock
The tubing is cut, cleaned, and inspected
The tubing design is finalized
The tubing is formed into the proper shapes, cut, and cleaned, and the tube ends is prepared for brazing
The tubing is brazed into the manifolds, and the manifolds are x-rayed
The structure is prepared for mounting
Propellant tank supports are added to the structure
The propellant tanks are mounted
The components are installed and brazed
Rocket engines are accumulated
The engines are brazed into tubing manifolds
All brazes are x-rayed
Thermostats and blankets are installed
Tanks are brazed and x-rayed
Acceptance testing of the completed system are performed
A formal government buy-off of the system must occur

Table 10.3.1: Major activities for constructing a flight propulsion system

Brazing is a technique—similar to welding—for joining metal tubing, but instead of melting two pieces together, additional metal is melted and acts as glue between the two pieces. Manifolds are a type of brace that join many connections into one.

Second, Scott Williams ensures all activities are in the correct sequence and matches the requirements for the manufacturing and assembly procedures. He discusses the completion times and activity dependence with engineers working on the project and creates Table 10.3.2.

Activity code	**Activity description**	**Activity Dependence**	**Completion time (days)**
A	Accumulate parts to assemble rocket engines	—	1
B	Braze filters onto Rocket Engine Assemblies (REAs)	*A*	3
C	Braze REAs	*B*	5
D	X-ray and inspect	*C*	2
E	Tubing design and drawings	—	4
F	Bend tubing to drawings	*E*	8
G	Cut and clean tubing	*F*	6
H	Braze tubes into manifolds	*G*	12
I	X-ray and inspect manifolds	*D, H*	3
J	Prepare structure	—	2
K	Inspect structure	*J*	1
L	Add tank supports	*K*	3
M	Accumulate propellant tanks	—	1
N	Mount tanks to structure	*M*	2
O	Install all components	*N*	4
P	Install manifolds	*I, L, O*	4
Q	Braze tanks	*P*	3
R	Accumulate pressure tanks	—	1
S	Install pressure tanks	*R*	2

T	Install complete pressure system	*S*	3
U	Braze pressure system	*T*	5
V	X-ray and inspect	*U*	2
W	Final inspection	*Q, V*	2
X	Install thermal blankets	*W*	2
Y	Complete acceptance test	*X*	5
Z	Government inspection	*Y*	2

Table 10.3.2: Activity chart for manufacturing a flight propulsion system

Third, Scott Williams uses the activity chart of Table 10.3.2 to create the order-requirement digraph shown in Figure 10.3.2. This order-requirement digraph shows the interaction among the activities within the project.

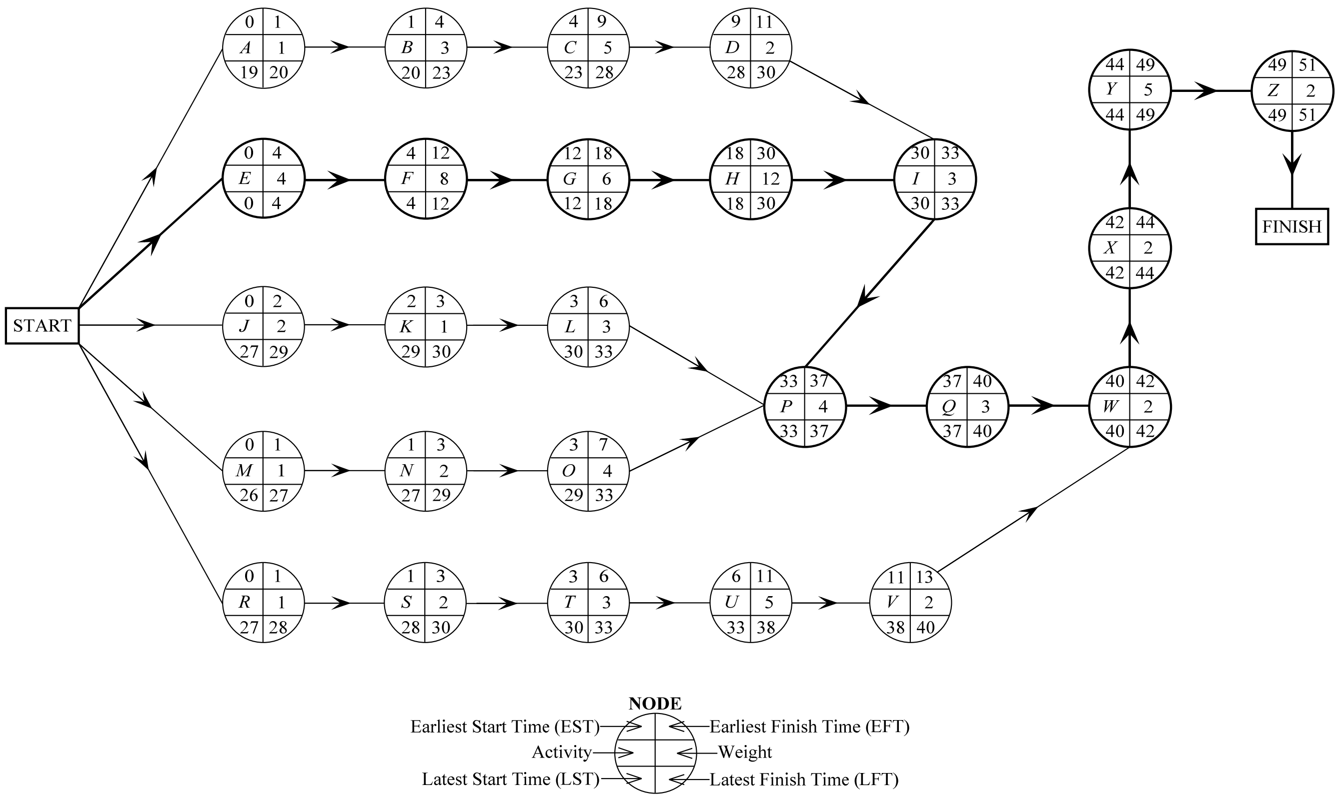


Figure 10.3.2: CPM order-requirement digraph for the manufacturing of a flight propulsion system

Fourth, Scott Williams needs to find the critical path through the order-requirement digraph for the manufacturing of the flight propulsion system. To do so, he follows the steps of the CPM algorithm. He calculates earliest start time, earliest finish time, latest start time, and latest finish time. He places these values into the appropriate sections of the nodes in Figure 10.3.2. Then, he determines the critical path for this order-requirement digraph by calculating the amount of slack for each activity.

Scott Williams creates Table 10.3.3, which shows the EST, EFT, LST, LFT, and slack time for each activity. By calculating the slack times, Scott Williams determines which activities make up the critical path for the manufacturing of the flight propulsion system.

Activity	EST	LST	EFT	LFT	Slack	Critical Activity
A	0	19	1	20	19	
B	1	20	4	23	19	
C	4	23	9	28	19	
D	9	28	11	30	19	
E	0	0	4	4	0	Yes
F	4	4	12	12	0	Yes
G	12	12	18	18	0	Yes
H	18	18	30	30	0	Yes
I	30	30	33	33	0	Yes
J	0	27	2	29	27	
K	2	29	3	30	27	
L	3	30	6	33	27	
M	0	26	1	27	26	
N	1	27	3	29	26	
O	3	29	7	33	26	
P	33	33	37	37	0	Yes
Q	37	37	40	40	0	Yes
R	0	27	1	28	27	
S	1	28	3	30	27	
T	3	30	6	33	27	
U	6	33	11	38	27	
V	11	38	13	40	27	
W	40	40	42	42	0	Yes
X	42	42	44	44	0	Yes
Y	44	44	49	49	0	Yes
Z	49	49	51	51	0	Yes

Table 10.3.3: Application of CPM algorithm to determine the critical path

Q1. What is the critical path for this project? Describe the path using letters and activities.

Q2. What is the length of the critical path for this project?

Q3. By looking at Figure 10.3.2 and Table 10.3.3, determine the second critical path (i.e., the second longest path) for the manufacturing of the flight propulsion system.
   a. How long is the second critical path?
   b. What is the significance of the second critical path in terms of the problem context?

Q4. What is the meaning of 19 days of slack in the activity path *A*-*B*-*C*-*D*?

Q5. What is the effect on the schedule for the manufacturing of the flight propulsion system if delays occur on the critical path?

Q6. Recall that the flight propulsion system is one part of the overall satellite project. What is the impact on the satellite project if the flight propulsion system required 60 days to complete?

Next, Scott Williams creates the Gantt charts for this project. These are shown in Figures 10.3.3 and 10.3.4.

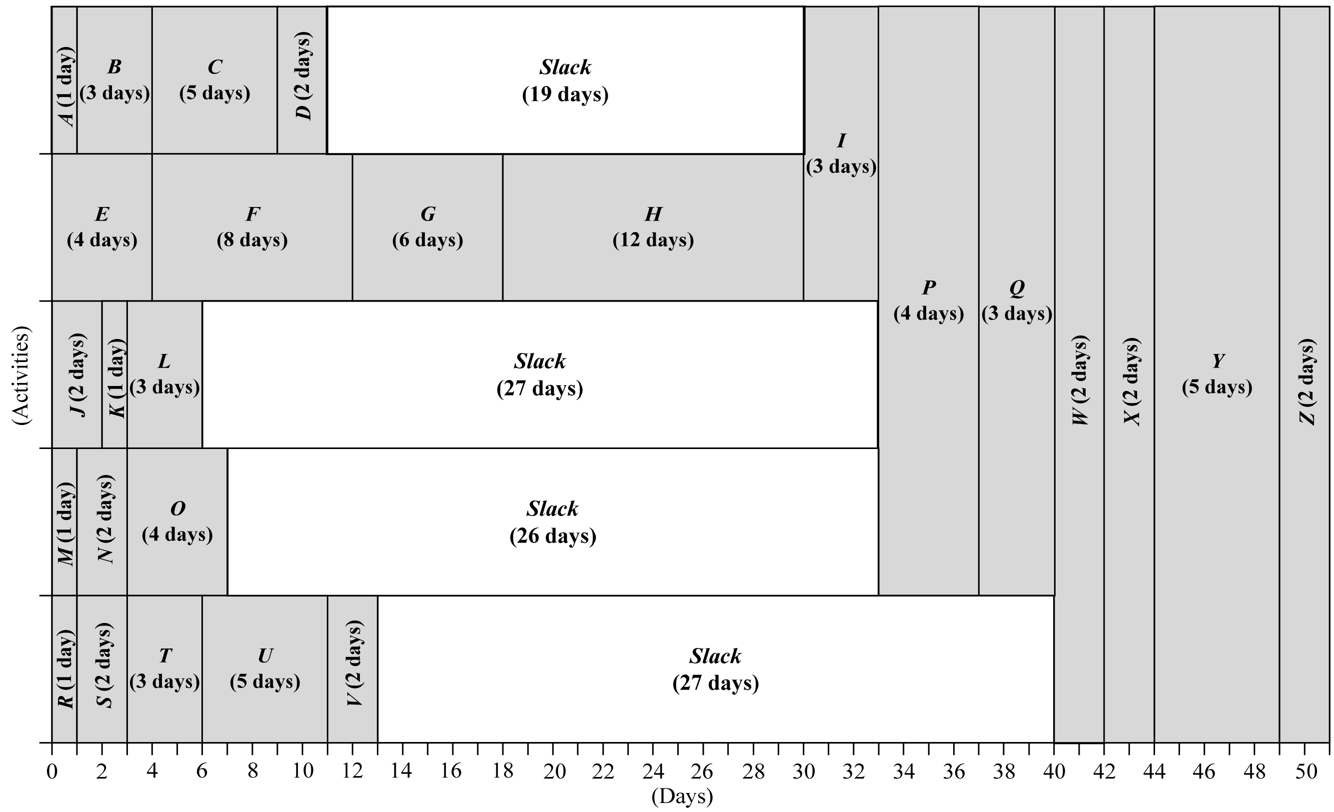


Figure 10.3.3: Completed Gantt chart for EST, EFT: Manufacturing a flight propulsion system

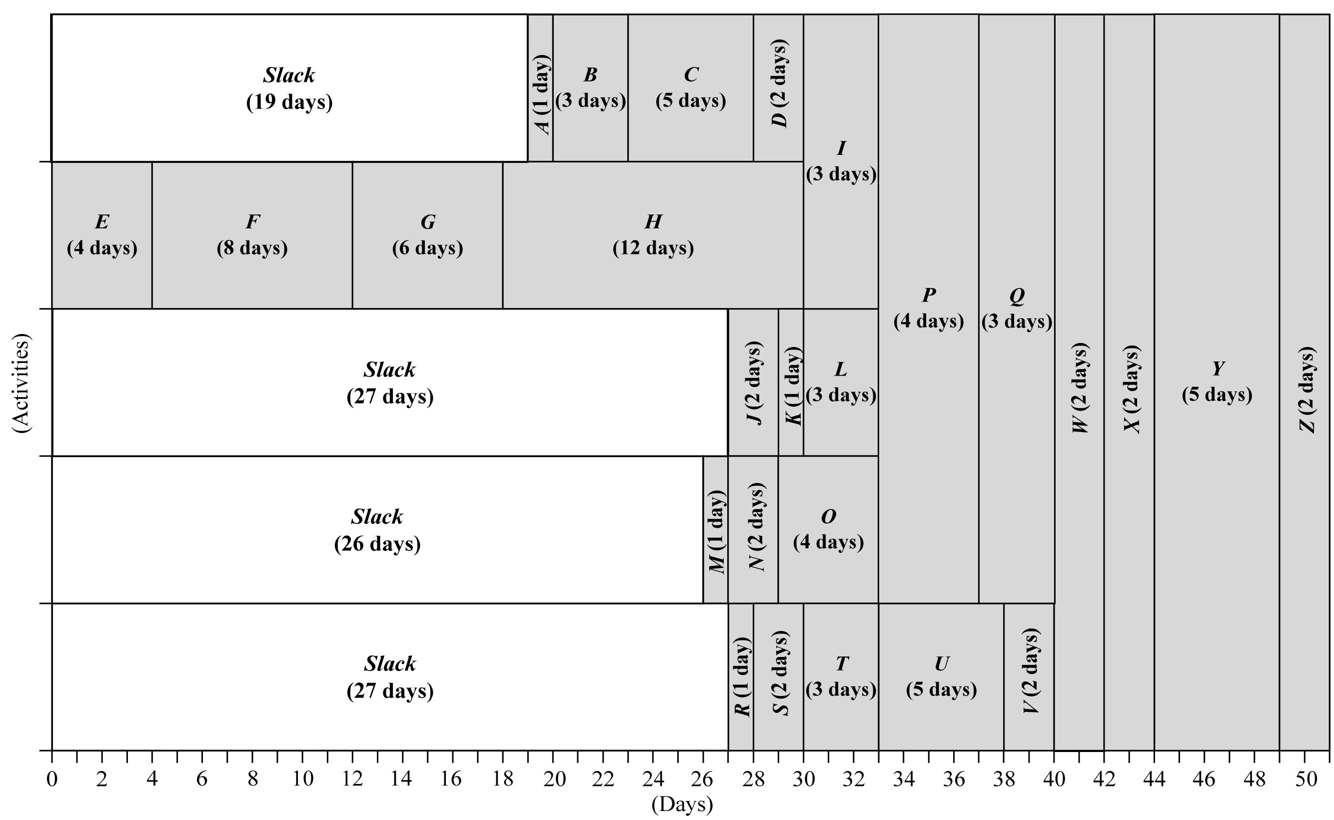


Figure 10.3.4: Completed Gantt chart for LST, LFT: Manufacturing a flight propulsion system

Q7. Suppose the engineers start preparing the structure (activity *J*) on day 0 but then took an extra 5 days to complete this activity. Explain how this would impact the rest of the activities in the project.

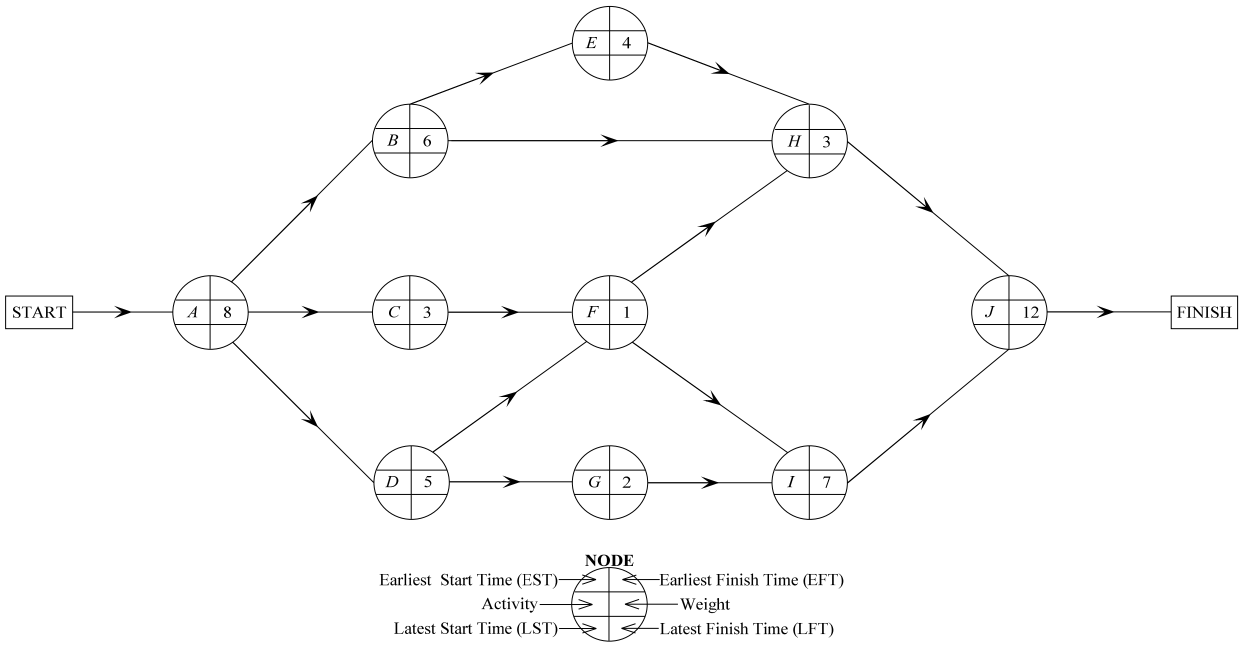
Q8. Suppose the engineers start accumulating the parts to assemble the rocket engines (activity *A*) on day 19 and took an extra day to complete this activity (for a total of 2 days). Explain how this would impact the rest of the activities in the project.

Q9. Suppose the engineers start accumulating the parts to assemble the rocket engines (activity *A*) on day 0 and took an extra day to complete this activity (for a total of 2 days). Then, they waited 10 days to being brazing the filters onto REAs (activity *B*) and took an extra day to complete this activity (for a total of 4 days). Explain how this would impact the rest of the activities in the project.

Engineers and project managers often utilize the CPM algorithm and Gantt charts when planning large projects, such as constructing a flight propulsion system. These tools allow them to determine how long the project will take, how much time flexibility they have, and how long the project will be delayed if an activity is delayed.

Section 10.4: Chapter 10 (CPM) Homework Questions

1. Consider the following project order-requirement digraph. Assume that the time required (in weeks) for each activity is a predictable constant.
   a. Find the EST, EFT, LST, LFT, and slack for each activity.
   b. Identify the critical path.



Activity	EST	LST	EFT	LFT	Slack (weeks)	Critical Activity
A						
B						
C						
D						
E						
F						
G						
H						
I						
J						

Critical Path: ____

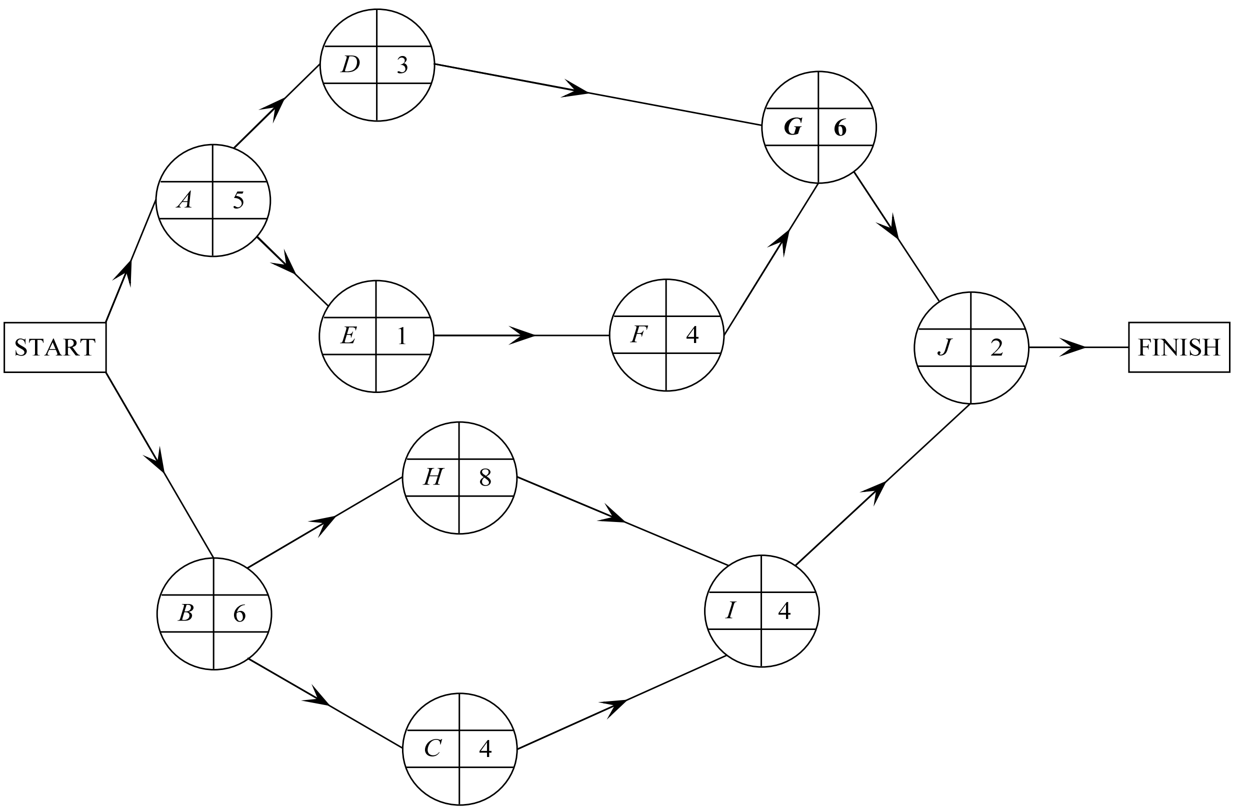
Project Completion Time is _________ weeks.

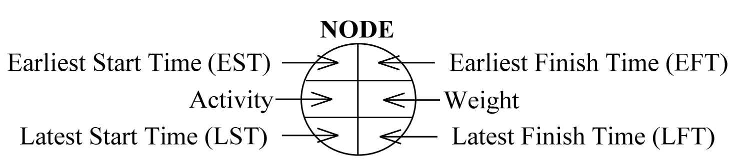
Extension on Preparing a Taco Dinner (Questions 2-4)

2. Change the activity *G*, brown hamburger and assemble tacos (including cheese), from 14 minutes to 6 minutes because you browned the ground beef in the microwave.

3. Re-calculate the EST, LST, EFT, LFT, and slack time for the new order-requirement digraph.

4. Is *A-E-F-G-J* still the critical path?





Activity	EST	LST	EFT	LFT	Slack (minutes)	Critical Activity
A						
B						

C						
D						
E						
F						
G						
H						
I						
J						
Critical Path: ______________________________						
Project Completion Time is _________ minutes.						

Building a Bookcase (Questions 5-14)

After reading an article in the April 2007 edition of *Fine Woodworking*, Tom the builder decides to build a new bookcase. He needs to determine how much it is going to cost him to build the bookcase. Besides the cost of parts, time is going to be another major cost factor because Tom will need to purchase food to feed his friends who help out with the project. In order to calculate the time he and his friends will spend building one bookcase, Tom will use the critical path method (CPM) algorithm. He first determines the activities he and his friends will need to complete, the order in which the activities will be completed, and how much time each activity will take. Tom creates Table 10.4.1 to organize the activities. He knows that since there are several paths to work through, he could be done much faster if he assigns his friends to accomplish different activities simultaneously. He needs to determine the best way to get these bookcases built in as short a time as possible.

Activity code	Activity description	Activity dependence	Completion Time (hours)
A	Study article in *Fine Woodworking*	—	2
B	Select type of shelving	*A*	1
C	Acquire drawings	*A*	4
D	Assemble tools and table saw	*A*	3
E	Purchase wood	*C*	2
F	Build jig for drilling pinholes	*B*, *C*	8
G	Cut wood for frame and shelves on table saw	*D, E*	7
H	Install pinholes in frames with jig	*F*, *G*	6
I	Construct bookcase	*H*	15
J	Install shelves for fit check	*I*	2
K	Disassemble and stain	*J*	6
L	Final assembly	*K*	8

Table 10.4.1: Activity chart for building a bookcase

5. Draw the order-requirement digraph with the activities shown in the table.
6. Find the EST, EFT, LST, LFT, and slack for each activity.
7. Identify the critical path.
8. Determine the critical path for the bookcase construction. Which friend should Tom use on this path?
9. Determine how much slack is in each of the other paths. What kind of helper would Tom ask to do these other paths?
10. Draw a Gantt chart illustrating the activity dependence, activity time completion, earliest start time (EST), and earliest finish time (EFT).
11. Draw a Gantt chart illustrating the activity dependence, activity time completion, latest start time (LST), and latest finish time (LFT).
12. What complications could Tom foresee? What other issues need to be considered when building the bookcase?
13. Look at the list and change one of the times to a significantly higher number.
    a. Predict how this will change the critical path.
    b. Modify the path and find out.
14. Repeat by changing one of the times on the critical path to zero hours.
    a. Predict how this will change the critical path.
    b. Modify the path and find out.

Activity	**EST**	**LST**	**EFT**	**LFT**	**Slack (hours)**	**Critical Activity**
A						
B						
C						
D						
E						
F						
G						
H						
I						
J						
K						
L						
Critical Path: ___________________________						

Project Completion Time is _________ hours.

Graduation Part (Questions 15-22)

Twins David and Molly want graduation to be a wonderful event. However, they are more excited to throw a *huge* party for family and friends after graduation. In order for their party to be a great success, David and Molly need to carefully plan long before the actual event. This means they need to consider what must be done before the party and construct a plan for accomplishing these tasks. For example, David and Molly need to hire a DJ, a task that cannot be done the day before. Therefore, they need to determine at what point in the planning process they should book the DJ and how much time it will take them to find the best DJ. David and Molly need to figure out what activities are required to prepare the party and how much time the whole process is going to take.

15. Make a list of at least 15 activities David, Molly, and their friends are going to have to accomplish before the graduation party and estimate the amount of time (in days) it will take each activity to complete.
16. Many activities will be going on simultaneously as preparations take place. However, some activities cannot start until other are completed. Determine the activity dependence for each activity.
17. Draw an order-requirement digraph with the earliest start time (EST), earliest finish time (EFT), latest start time (LST), and latest finish time (LFT) filled in.
18. Draw a Gantt chart illustrating the activity dependence, activity time completion, earliest start time (EST), and earliest finish time (EFT).
19. Draw a Gantt chart illustrating the activity dependence, activity time completion, latest start time (LST), and latest finish time (LFT).
20. Calculate the slack time for each activity.
21. What is the critical path for preparing a graduation party?
22. What is the length of the critical path? What does this mean in terms of the problem?

Group Project

Research all the activities that are required to host a basketball game at your school. Find out who is involved, what they must accomplish, and how long it takes them to accomplish the task. Start with how and when the game is schedule. End with the beginning of the game. A good place to begin is to interview the athletics director, an administrator, athletics-booster president, head custodian, athletics trainer, cheerleading coach, etc. to learn about the extensive tasks that must be accomplished to host a basketball game successfully. Other considerations include concessions, ticketing, referring, clock timer, scoreboard, statistician person to keep track of stats, local paper, etc.

Once your group has exhausted all of the possible activities and their accomplish time, use the critical path method (CPM) to determine the critical path of coordinating a basketball game and how long it takes to plan a basketball game.

Extension
Among the class, have some groups investigate a girls' basketball game and other groups investigate a boys' basketball game. Is there much difference in the activities that take place? Do the games require the same amount of planning time? If not, why are there differences?

Chapter 10 Summary

What have we learned?

Terms

Activity Dependence	The activities that must be completed before the next activity can begin
Critical Path	A path of critical activities between the start-event and the finish-event of a project; the critical path determines the overall duration of the project, representing the longest path through the order-requirement digraph, and there may be more than one critical path in the order-requirement digraph
Critical Path Activity	An activity that must be completed within the planned duration if the overall project is to finish at the planned time; it is on the critical path and has a slack time equal to zero
Critical Path Method (CPM)	A type of project management problem developed to balance the cost of reducing project duration against savings obtained by so doing; one of the original network-analysis techniques
Digraph	A graph in which the arcs are directed
Earliest Finish Time (EFT)	The earliest time at which an activity can be finished, assuming that it starts at the earliest start time
Earliest Start Time (EST)	The earliest time at which an event can begin; the preceding activities are assumed to start at their earliest start times
Gantt Chart	The representation of timings of activities by means of bars drawn against common time-scale, which can be relative or an absolute scale
Graph	Informally, a graph is a finite set of dots called nodes connected by links called arcs
Latest Finish Time (LFT)	The time by which an activity must finish, if the succeeding event is to occur by the latest event-time
Latest Start Time (LST)	The time by which an activity must start, if it is to finish by its latest finish time
Weight	Units such as time, distance, or other measures are assigned to the nodes a graph
Order-Requirement Digraph	A digraph that does not loop back onto itself
Path	A sequence of consecutive arcs in a graph
Slack	The difference between the earliest and latest event-time

Chapter 10 (Location Problems) Objectives

You should be able to:

- Given a contextual problem, find the critical path of a project using the critical path method using an order-requirement digraph

- Given a contextual problem, find the critical path of a project using the critical path method using a Gantt chart

- Analyze and interpret results; make decisions based on results

Chapter 10 Study Guide

1.

References

Henry Laurence Gantt's legacy to management is the Gantt chart. Retrieved October 8, 2008, from http://www.ganttchart.com/History.html.

Hillier, F.S., and Lieberman, G.J. (1980). *Introduction to operations research*. Oakland, CA: Holden-Day Inc. pp. 263-264.

Hoare, H.R. (1973). *Project management using network analysis*. Great Britain: McGraw-Hill Book Company. pp. 104-106.

Worksheet 1

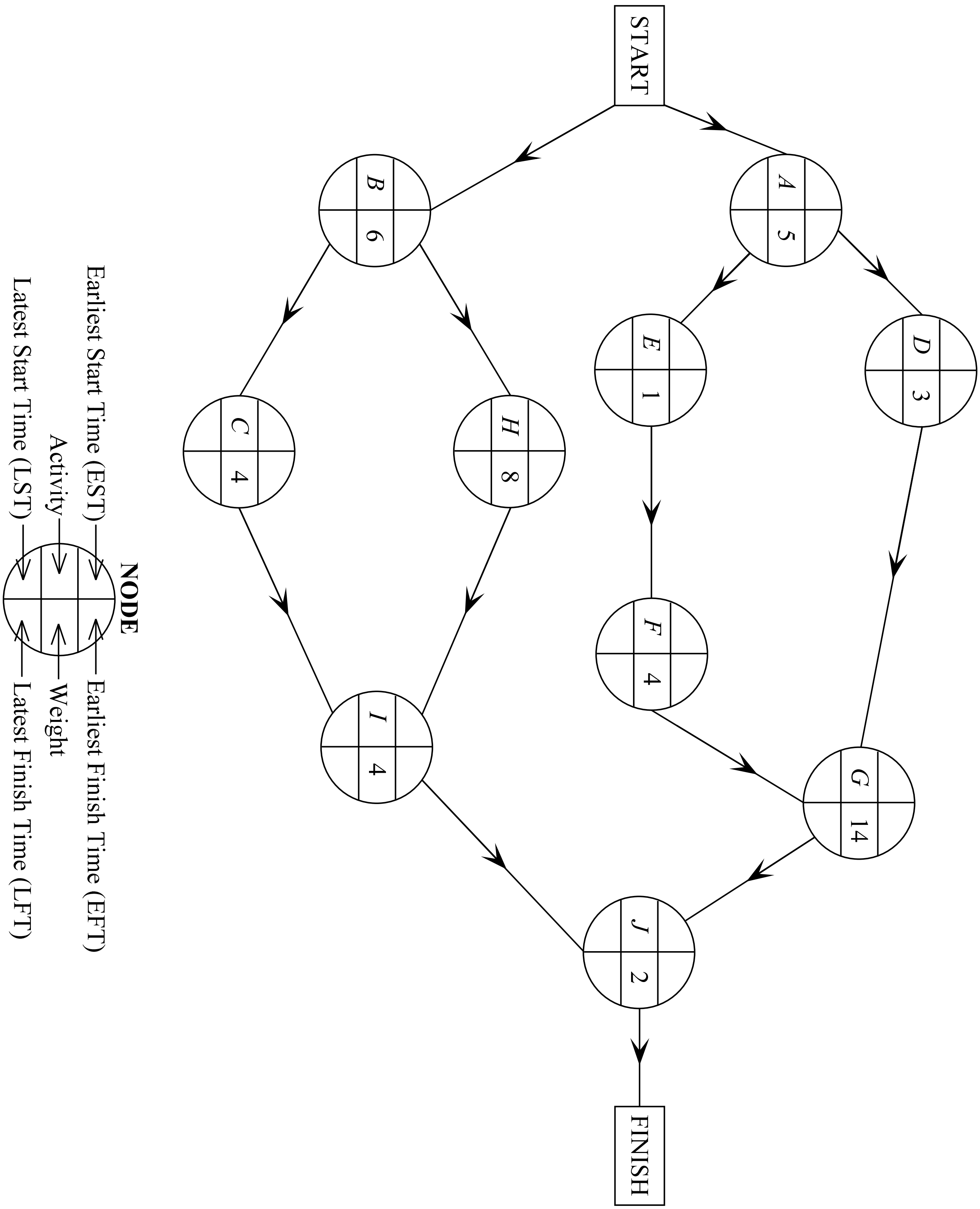
Table 10.1.1: Time estimates and activity dependence for activities accomplished before school

GETTING READY FOR SCHOOL

Activity Code	Activity description	Activity dependence	Completion time (minutes)
A			
B			
C			
D			
E			
F			
G			
H			
I			
J			
K			
L			
M			
N			
O			
P			
		TOTAL TIME	

Worksheet 2

Figure 10.3.3: Order-requirement digraph representing Preparing a Taco Dinner



Worksheet 3

Table 10.2.2: Application of CPM algorithm to determine the critical path

PREPARING A TACO DINNER

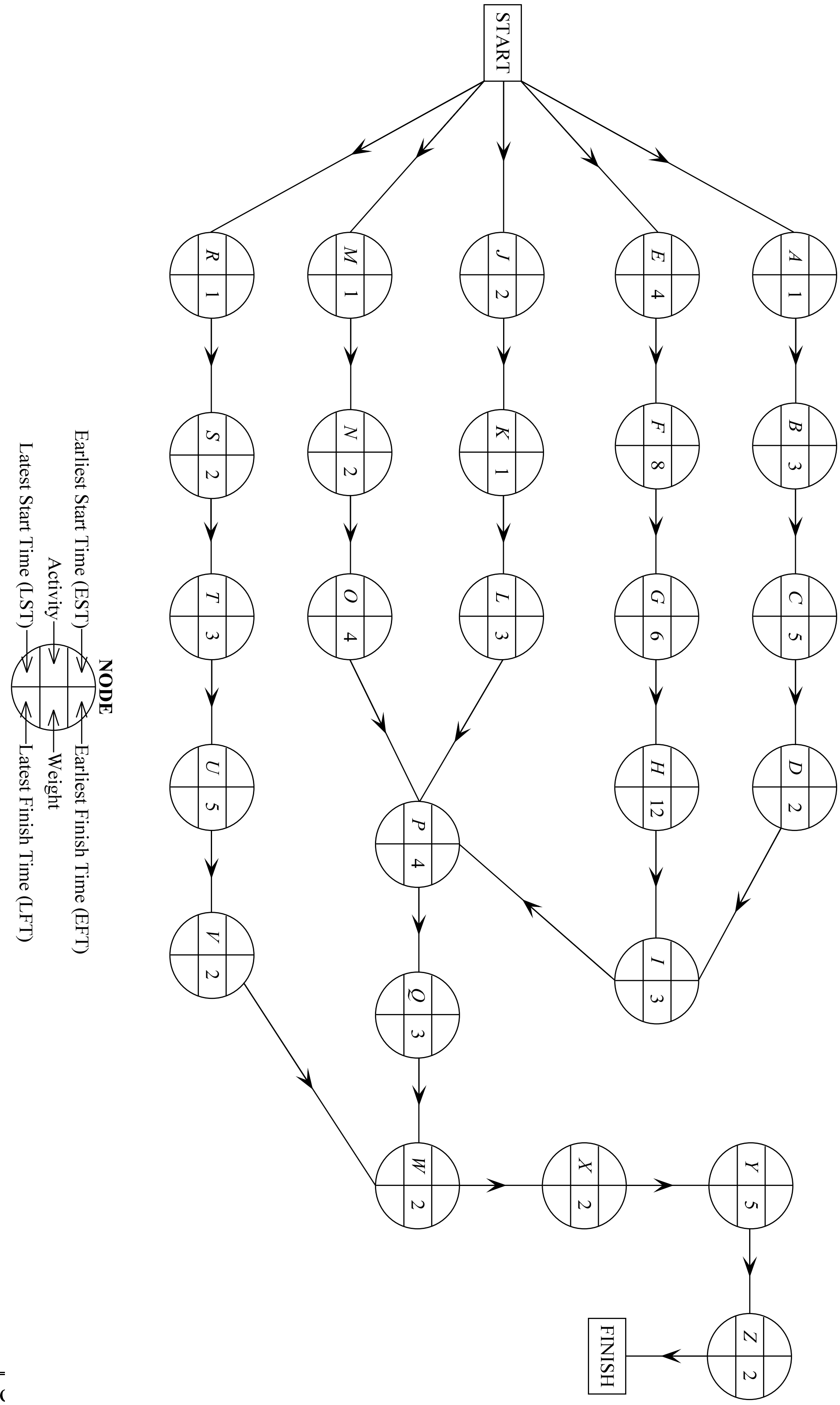
Activity	EST	LST	EFT	LFT	Slack (minutes)		Critical Activity
					LST – EST	LFT – EFT	
A							
B							
C							
D							
E							
F							
G							
H							
I							
J							

Critical Path: ____

Project Completion Time is ________ minutes.

Worksheet 4

Figure 10.4.1: CPM order-requirement digraph for the manufacturing of a flight propulsion system



Worksheet 5

MANAGING FLIGHT PROPULSION SYSTEM

Activity	EST	LST	EFT	LFT	Slack (days)	Critical Activity
A						
B						
C						
D						
E						
F						
G						
H						
I						
J						
K						
L						
M						
N						
O						
P						
Q						
R						
S						
T						
U						
V						
W						
X						
Y						
Z						

Critical Path: ____

Project Completion Time is ________ days

Table 10.4.2: Application of CPM algorithm to determine the critical path

