MATHEMATICS IS A WAY OF THINKING - HOW CAN WE BEST TEACH IT?

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Last month I described my own, half-century career in mathematics, the first twenty-five years as a pure mathematician, then, in the second half, working on problems for industry and various branches of the US Department of Defense.

I observed that at no stage did I make significant use of any of the specific techniques I learned in my entire educational arc, from elementary school through to receiving my bachelors degree in mathematics. Noting that my experience was not unusual—in fact, it is the norm among professional mathematicians—I started to set the stage for a promised discussion (it's coming up below!) as to what topics should be covered in high school mathematics. Here is what I said:

"First, *what* is taught is not, in itself, of any significance. The chances of anyone who finds they need to make use of mathematics at some point in their life or career being able to use any specific high school curriculum method is close to zero. In fact, by the time a student today graduates from university, the mathematics they may find themselves having to use may well have not been developed when they were at school. Such is the pace of change today.

Second, what is crucial to effective math learning is what is sometimes called "deep learning"; the ability to think fluidly and creatively, adapting definitions and techniques already mastered, reasoning by analogy with reasoning that has worked (or not worked) on similar problems in the past, and combining (in a creative fashion) known approaches to a novel situation."

The question for mathematics educators is, how best do we develop that way of thinking, and the understanding it depends upon?

For K-12 mathematics teachers, that's a question only those alive today have had to face. Until the early 1990s, when we acquired readily available technologies to execute ANY mathematical procedure, the first order of the day in the school math class was drill students in enough procedural skills to be able to get by in life and maybe get a leg up onto the mathematics ladder. If you did not have good mastery of basic arithmetic, you could find yourself disadvantaged in everyday life, and if you did not master arithmetic and some basic algebra (in particular), you could not get off the ground in mathematics. The sooner such mastery could be achieved, the better for all.

For a minority of students, the "drill the basics" approach worked. I was one of them. I became a professional mathematician. To do that, I had to make one critical transition. I had to learn how to go beyond my dependency on all those basic skills and develop the ability for mathematical thinking.

This was in the mid-1960s, long before procedural math tools like Wolfram Alpha came onto the scene. Electronic calculators were just coming onto the market, so I no longer had a need for my arithmetic skills, but by then the math I faced was well beyond arithmetic, so I still had to use those basic algebra skills. But as an early *Mathematica* adopter (I was on Wolfram's original Advisory Board), from around 1990 onwards, none of the basic skills I had mastered though my undergraduate degree were necessary in order to do my work. (For some years, I continued to use them, because I had them to hand; but over the years my fluency dropped as I made more and more use of technology, leaving me more time to focus on things machines could not do.)

But that time I spent mastering the basics was not totally wasted, even in the long run. It was the scaffolding that helped me learn to think mathematically.

So, the traditional, skills approach worked for me. And for many others. But it came at a huge societal cost. The majority of my fellow K-12 schoolmates not only never got to the stage of making that transition to mathematical thinking, they ended up hating math (in some cases being very afraid of it), and dropping it at the first opportunity. Even worse, they ended up with a perception of what mathematics is that is dangerously wrong—a perception they carried with them into parenthood and in many cases a career as a mathematics teacher, ensuring the continued consumption of a product that was long past its sell-by date.

The consequences have been devastating for generations of students. I wrote about this in Devlin's Angle in June 2010, in a post titled In Math You Have to Remember, In Other Subjects You Can Think About It. Please read that earlier post before you go any further with this essay. If it were not for the human carnage that results from teaching mathematics in a way that works only for a minority of students, there would be little point in my spending time writing this post or you reading it.

Make no mistake, the way our society teaches math, and has done so for generations, absolutely produces enough mathematicians to meet the national need.

On the other hand, because it discards so many, our approach also results in there always being a shortage of high school graduates who, while not mathematicians, have sufficiently adequate mathematical ability to succeed in a world where mathematics plays such a central role. But we sort of make up for that deficit with adult education. (Although "remedial math" courses for adults can be hampered by having to overcome various degrees of math phobia resulting from a flawed approach at K-12.)

The real problem is the collateral damage the approach inflicts on the majority of students. The ones who are turned off. The majority. For life. It's a national tragedy. One that, at least today (and in the past, had we proceeded differently, but that's for another time) CAN be avoided. That's where we need to go.

The goal is pretty clear. The purpose of education is to prepare the next generation for the lives they will lead. Agreed? So what might that entail for mathematics education? What do our students need to be able to do when they graduate?

Well, with today's technologies, no professional using mathematics in the world outside the math classroom executes standard procedures "by hand". Rather, we use all of the available technologies. They are faster than us, way more accurate, and can handle far more variables, and far greater datasets, than any human can. (Most real-world problems that require mathematics today typically have far too many variables to solve by hand.)

Being a mathematician (or a user of mathematics) today is all about using those tools effectively and efficiently. Our mathematics education system needs to produce people with that ability.

To re-use one of my favorite analogies, being a mathematician used to be like playing instruments in an orchestra (with all that entails), whereas today it is like conducting the orchestra. Mathematicians no longer solve problems by following rules in a step-by-step procedure. (Computers do that better.) They apply heuristics—flexible ways of thinking they acquire with repeated practice over time. (Computers cannot do that in any fashion worth discussing—unless you are a computer scientist, when it is a very interesting and challenging research problem encompassing AI, machine learning, and a bunch of other cool stuff.)

Note that word "acquire" in the above paragraph. To the best of my knowledge and experience, you cannot be taught heuristics; you acquire them over time. Mathematical thinking takes a long time to develop. The challenge facing today's math educators is finding the most efficient way to reach that goal. A way that does not fail, and alienate, the majority of our students. There is, I think, good reason to believe this can be done. What gives rise to my optimism is that another way to express the change in mathematical praxis that I outlined above is:

Today's mathematician *thinks like a human*, rather than *calculates like a computer*.

With mastery of computational skills no longer *an entry barrier*, mathematics learning starts to look very much like any other creative subject. (There has never been an educational problem of "English anxiety" or "Art phobia", right?)

For sure mathematical thinking certainly requires—or seems to require—some experience with computation. (Exactly how much, and to what degree, remains an open question.) In the past, a mathematician had to master both. But even then, the *focus* of the learning had to be on getting to the thinking. Or rather, it should have been. When done by hand, algorithmic calculation is, by its nature, purely routine; clearly not and end in itself. (Though some of us gained pleasure from the activity, and the associated feeling of achievement, while it was still relatively new to us.)

So what is the most efficient way (*today*) to get students to acquire that all important mathematical thinking ability? In my previous post, I continued the passage quoted above with these two paragraphs:

"But here's the rub. The mass of evidence from research in cognitive science* tells us that the only way to acquire that all important "deep learning" is by prolonged engagement with *some* specific mathematics topics. So, to be effective, any mathematics curriculum has to focus on a small number of specific topics.

Yet according to my first remark, there is no set of "most appropriate topics," at least in terms of subsequent "applicational utility". So what to do? How should we determine a curriculum?"

[* I will address some of that cognitive science research in a future post. Stay tuned.]

Absent any other overwhelming criteria, it makes sense to pick two or three curriculum topics that are most easily introduced, are at an abstraction level no more than two steps removed from the everyday physical world, and relate most closely to the daily life of the greatest number of students.

That puts arithmetic and geometry at the top of the list. And since the goal is to develop mathematical thinking, which involves handling abstractions and patterns, you need to throw in some elementary algebra as well. (There was a reason those parts of mathematics were developed first!)

Okay, perhaps also a bit about probability and statistical distributions (including creating and interpreting graphs and charts), since they play such a huge role in all our daily lives today. But that's it. Anything else is optional, and should be dispensed only in small doses. (So the main topics can be studied in depth.)

For sure, no calculus, which has no place in K-12 education. Not least because there is no way it can be done well at that stage. (In large part, because it operates at the third level of abstraction, with its fundamental objects being operations on functions, which are themselves operations on numbers. That's a huge cognitive leap that takes most of us several years to achieve.) The student who typically has the most trouble with university calculus is the one who has learned it poorly at high school, and comes to Calc 101 at university with a false belief they understand it and can do it, only to crash and burn in Calc 102. (Dealing with that crash scene was a large part of my life for over 25 years!)

Of course, it can be beneficial to *expose* students to various other parts of mathematics. The field of mathematics is one of the great accomplishments of human culture. There is benefit to be gained from showing the next generation some of that intellectual heritage (including calculus, in particular). Partly because it ***is*** part of human culture, and a major one at that. But also because it helps them appreciate the enormously wide scope of mathematical applications, providing them a meaningful context and purpose for devoting time to learning the stuff you are asking them to master (which will inescapably require a considerable amount or repetitive practice). But exposure is very different from achieving mastery, and requires a very different teaching approach.

In particular, there is certainly benefit to everyone in the classroom from the teacher showing their students some of the mathematics *they themselves enjoy*. It doesn't matter whether it is "useful" or not. In terms of the student's future lives, nothing you show them is likely to be of use to them (in the sense of applying it), as I indicated earlier. In fact it is highly likely that any math the teacher found of use in their life won't be very relevant to the student's life a generation later. Things change too quickly. But if the teacher likes it, that enthusiasm should shine through, to the great benefit of the class. It can never be an educational waste of time for a teacher to show the students something they are passionate about.

As to showing students the widespread utility of mathematics, in my experience the best way is to see what is in the news at the time, or to reflect on things going on in our daily lives, and ask the question, "What mathematics is (or might be), involved in that?" There is usually a lot. In the Google Era, it is generally not hard for a well-trained math teacher to find answers to that question.

For example, in a series of Devlin's Angle posts last year (beginning with this one), I wrote about one example of a short high-school mini-course based on the question, "How does UPS manage to get all those packages to their destination on time?" It involves some fascinating mathematics, well within the reach of a high school student.

Notice that to do this kind of thing does not require the teacher know any of the math involved. What is important is to be able to *find out* about that math! A task that is pretty easy given Google, Wikipedia, and YouTube. (The ability to master a new mathematical technique quickly is one of the most important skills for a mathematician today.)

Sure, you can't conduct that kind of investigation if you haven't mastered some mathematical topics well. As I noted earlier, there is no by-passing that step. (Many readers of my articles and social media posts mysteriously seem to skip over that part and get angry at the straw man that results.) But the three topics I listed above (arithmetic, algebra, geometry) do just fine for preparing the groundwork. (If you master the use of a paint brush by practicing with white paint, blue paint, and red paint, it's not that hard transferring your skill when you find yourself with can of yellow paint, or green, or any other color, including colors that have not yet been developed. Incidentally, this analogy is far more accurate than may appear to anyone who has not progressed sufficiently far in mathematics. When you get well into math, you realize that the entire field is really just color variations on a common theme.)

In the same vein of making educational use of real-life applications of mathematics, back in the 1990s I worked on a six-part PBS television series called Life by the Numbers, where we presented segments on professionals in all walks of life who described how their work used, or depended on, mathematics. Providing further testimony to my initial remark that the mathematics being used at any one time changes rapidly (so virtually nothing a student learns in school will be directly useful when they graduate), I should note that practically all the applications of mathematics we showed in the series—applications chosen because they were cutting-edge in the early 1990s when the series was being made—are no longer being used that way today. In our technology-rich world, mathematical obsolescence can be as rapid as the demise of a pop song. Nevertheless, the series can still provide a useful resource to show *how* mathematics typically gets used.

But to return to the main question, let's assume you decide to focus K-12 math education on arithmetic, geometry, and some elementary algebra, which is my suggestion (and that of a great many others in the mathematics education world). What is crucial is to teach it in a way that results in *understanding*. If it is approached as acquiring and achieving mastery of a toolkit of techniques, it ain't gonna work. It really won't. Teaching rote mastery of (some) tools was defensible in the days when there were no machines to do it. Today, the main benefit of engaging with a particular algorithm or procedure is *as a vehicle for developing mathematical thinking.* [Don't miss that point. It's important. That's part of the cognitive science stuff I promised earlier to cover in a later post.]

But if the focus goes beyond engaging with a small collection of methods, studied in depth for understanding how the mathematics works, and becomes a smorgasbord of techniques touted as a "universal toolkit", to be carried around and selected from each time a new problem arises, then you are in the tragic world Boaler studied and wrote about.

The absurdity of the "teach math as a toolkit" approach was highlighted to me recently in the tweet (shown below) from a math instructor that landed in my twitter-feed. It has two glaring errors.

> My point: the student having learned an effective, universally applicable and elegant procedure has a well-suited schema for understanding this and any task like it they will ever face. The child knowing "strategies" is stuck at the stage of deciding which strategy to use

Tweet directed to me by a teacher as the culmination of a thread comparing different approaches to teaching math

First, there are no procedures that are universally applicable. All procedures are designed to perform a particular task or set of tasks. As I remarked last month, in my entire fifty-years career as a professional mathematician, I used NONE of the procedures I mastered in my entire high-school career, and hardly any I mastered in my undergraduate career. That is typical. What I relied on, all the time, however, was my ability to think mathematically.

Second, a strategy is not something to be taught, it is something you develop as part of (mathematical) thinking. For sure, if you try to "teach strategies" as some sort of toolkit, the result will be confused students, as the tweeter reported.

(An earlier tweet in the thread I pulled the one above from, suggested to me that the instructor in question had indeed tried to teach mathematical problem solving "strategies" as a menu-accessed toolkit, with predictably disastrous results.)

Of course, you can show students particular strategies as illustrations, but they should be presented as just that: illustrations. The goal is to help the student acquire the ability to *think strategically*, not to "pick strategies from a menu."

What that particular teacher was missing is that deciding HOW to approach a particular problem is arguably the most critical ability in mathematical problem solving. The teacher's job is to help the student develop the ability to come up with a strategy. Yes, it may be a strategy they have already seen. Indeed, that is often the case. But it may require an *adaptation* of a known strategy, or the development of an entirely new approach.

In any event, since the list of potential strategies is effectively endless, it is hopeless trying to list them all in a select-from menu. If you do, the result will be, as another contributor to that thread noted, the students will be confused and put off by “the 1,578 different strategies they get pushed on them.”

No art teacher would provide students with a pull-down menu of specific ways to create paintings of different kinds—portraits of men, portraits of women, portraits of children, paintings showing buildings, paintings of rural scenes, paintings of skies, etc. No, they teach the student *how to paint* (which includes helping them learn how to *see*). From which grounding, the student can create and develop their own “strategies” to produce paintings of various kinds.

It’s a math ed twist to the old story about giving a starving person fish to eat as opposed to teaching them how to fish. One has limited value in the moment, the other is a valuable life skill. I suspect those contributors to that Twitter thread had little experience in real-life mathematical problem solving. (Which is why those of us who have should devote time and effort into keeping teachers informed about current praxis.)

At the most fundamental level, the issue is not algorithms versus strategies; it’s about approaching math as the *provision of a toolkit* (OUTDATED), as opposed to developing a *way of thinking* (CRUCIAL). The former, toolkit approach was defensible, and arguably unavoidable, in the millennia before we had tools for procedural math. But in today’s world, the crucial ability to be mastered is mathematical thinking. We need to get there as rapidly as possible, without losing the majority on the way.

To finish, I should note that many other mathematics educators have advocated that the main focus of K-12 mathematics education should be in-depth study of arithmetic, geometry, and a bit of algebra (and little else). For example, Liping Ma, whose approach I wrote about in the latter part of my Devlin’s Angle last October. You may find what she has to say of value. I certainly did.

In the November issue, I followed up on that post by taking the argument further, into the domain of systemic assessment of mathematics. I ended that November post (which was somewhat speculative, though I am part of a team conducting research in that area) with this sentence:

“Of course, I can keep repeating my message. In fact, you can count on me doing that. :) ”