SEGNALI nel DOMINIO della FREQUENZE

SVILUPPO in SERIE di FOURIER

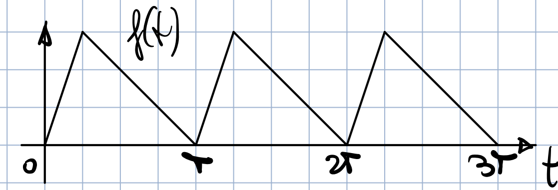
SERIE : es: Serie di POTENZE : $\sum_0^\infty n \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

SERIE di FUNZIONI : $S(x) = \sum_0^\infty n \; f_n(x)$

FOURIER (1822) :

Data una FUNZIONE PERIODICA, di PERIODO T :

$f: \quad f(t) = f(t+T), \quad \forall t, T \in \mathbb{R}$



se $f(t)$ rispetta opportune condizioni di "REGOLARITÀ" (condizioni di DIRICHLET)

ALLORA :

$f(t)$ è ESPRIMIBILE come SOMMA di SINUSOIDI con FREQUENZE MULTIPLE di $\frac{1}{T} = f_1$ della FREQUENZA FONDAMENTALE

$$f_n : \begin{cases} n = 0 \longrightarrow f_0 = 0 \text{ (COSTANTE)} & \text{COMPONENTE CONTINUA} \\ n = 1 \longrightarrow f_1 = \frac{1}{T} & \text{FREQUENZA FONDAMENTALE} \\ n > 1 \longrightarrow f_n = \frac{n}{T} & n\text{-ESIMA ARMONICA} \end{cases}$$

In forma ESPONENZIALE : (per $f(t) \in \mathbb{C}$)

$$f(t) = \sum_{-\infty}^{+\infty} n \; c_n \underbrace{e^{j2\pi \frac{n}{T} t}}_{\text{FASORE}}$$

$$\left\{ e^{j2\pi \frac{n}{T} t} = \cos\left(2\pi \frac{n}{T} t\right) + j \sin\left(2\pi \frac{n}{T} t\right) \right\}$$

In forma TRIGONOMETRICA : (per : $f(t) \in \mathbb{R}$)

$$f(t) = c_0 + 2 \sum_1^\infty n \left[a_n \cos\left(2\pi \frac{n}{T} t\right) + b_n \sin\left(2\pi \frac{n}{T} t\right) \right]$$

$a_n = \mathrm{Re}[c_n]$
$b_n = -\mathrm{Im}[c_n]$

$$= c_0 + 2\sum_{1}^{\infty} \rho_n \cos\left(2\pi \frac{n}{T} t + \vartheta_n\right), \qquad \rho_n = |c_n|,\ \vartheta_n = \measuredangle c_n$$

EQUAZIONI di SINTESI : da $\{c_n\} \longrightarrow f(t)$

dove : $$c_n = \frac{1}{T}\int_T f(t)\, e^{-j2\pi \frac{n}{T} t}\, dt$$

EQUAZIONE di ANALISI : da $f(t) \longrightarrow \{c_n\}$

Dalla forma ESPONENZIALE alla TRIGONOMETRICA :

Ipotesi : $f(t) \in \mathbb{R},\ \forall t \in \mathbb{R}$

$$f(t) = \sum_{-\infty}^{+\infty} c_n\, e^{j2\pi \frac{n}{T} t} = c_0 + \sum_{1}^{\infty}\left[c_n e^{j2\pi \frac{n}{T} t} + c_{-n} e^{-j2\pi \frac{n}{T} t}\right] =$$

considero c_{-n} :

$$c_{-n} = \frac{1}{T}\int_T f(t)\, e^{j2\pi \frac{n}{T} t}\, dt = (f(t) \in \mathbb{R}) = \frac{1}{T}\int_T \overline{f(t) \cdot e^{-j2\pi \frac{n}{T} t}}\, dt = \overline{c_n}$$

$$f(t) = c_0 + \sum_{1}^{\infty}\left[c_n e^{j2\pi \frac{n}{T} t} + \overline{c_n}\, \underbrace{\overline{e^{-j2\pi \frac{n}{T} t}}}_{e^{j2\pi \frac{n}{T} t}}\right] = c_0 + 2\sum_{1}^{\infty} \mathrm{Re}\left[c_n e^{j2\pi \frac{n}{T} t}\right] =$$

(definisco : $c_n = \rho_n e^{j\vartheta_n} \longrightarrow \rho_n = |c_n|\ ;\ \vartheta_n = \measuredangle c_n$)

$$f(t) = c_0 + 2\sum_{1}^{\infty} \mathrm{Re}\left[\rho_n e^{j\vartheta_n} \cdot e^{j2\pi \frac{n}{T} t}\right] = c_0 + 2\sum_{1}^{\infty} \mathrm{Re}\left[\rho_n e^{j\left(2\pi \frac{n}{T} t + \vartheta_n\right)}\right] =$$

$$f(t) = c_0 + 2\sum_{1}^{\infty} \rho_n \cos\left(2\pi \frac{n}{T} t + \vartheta_n\right)$$

$$\rho_n = |c_n|\ ;\ \vartheta_n = \measuredangle c_n$$

I FORMA TRIGONOMETRICA

Definisco : $c_n = a_n - jb_n \longrightarrow a_n = \mathrm{Re}[c_n]\ ;\ b_n = -\mathrm{Im}[c_n]$

$$f(t) = c_0 + 2\sum_{1}^{\infty} \mathrm{Re}\left[(a_n - jb_n)\, e^{j2\pi \frac{n}{T} t}\right] =$$

$j = e^{j\frac{\pi}{2}}$

$$= c_0 + 2\sum_{1}^{\infty} \mathrm{Re}\left[a_n e^{j2\pi \frac{n}{T} t} - jb_n e^{j2\pi \frac{n}{T} t}\right] =$$

$$= c_0 + 2\sum_{1}^{\infty} \mathrm{Re}\left[a_n e^{j2\pi \frac{n}{T} t} - b_n e^{j\left(2\pi \frac{n}{T} t + \frac{\pi}{2}\right)}\right] =$$

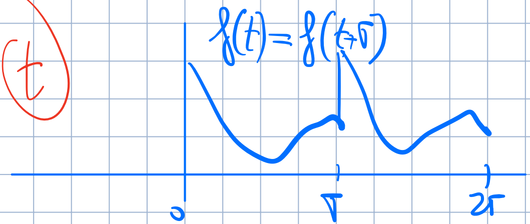
$$= c_0 + 2\sum_{n=1}^{\infty} \left[a_n \cos\left(2\pi \frac{n}{T} t\right) - b_n \cos\left(2\pi \frac{n}{T} t + \frac{\pi}{2}\right)\right] = \qquad \cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha$$

$$f(t) = c_0 + 2\sum_{n=1}^{\infty} \left[a_n \cos\left(2\pi \frac{n}{T} t\right) + b_n \sin\left(2\pi \frac{n}{T} t\right)\right] \qquad \text{EQ. di SINTESI}$$

$$\text{dove: } \begin{cases} a_n = \mathrm{Re}[c_n] = \mathrm{Re}\left[\frac{1}{T}\int_T f(t) e^{-j2\pi \frac{n}{T} t} dt\right] = \frac{1}{T}\int_T f(t)\cos\left(2\pi\frac{n}{T}t\right) dt \\ b_n = -\mathrm{Im}[c_n] = -\mathrm{Im}\left[\frac{1}{T}\int_T f(t) e^{-j2\pi \frac{n}{T} t} dt\right] = \frac{1}{T}\int_T f(t)\sin\left(2\pi\frac{n}{T}t\right) dt \end{cases} \quad \text{EQ. di ANALISI}$$

II FORMA TRIGONOMETRICA

(t)



ANALISI →

← SINTESI

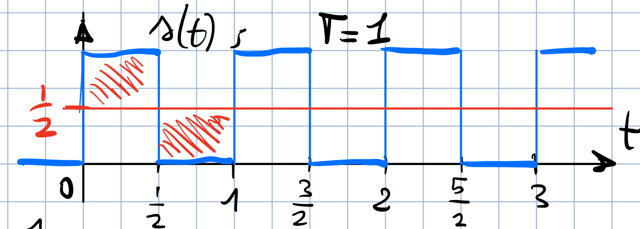
$\{c_n\}, \ n \in \mathbb{Z}$

$\{a_n, b_n\}, \ n \in \mathbb{N}$

(n)

ESEMPIO: calcolo SERIE di FOURIER (FS) di un'onda QUADRA:

$$s(t) = \begin{cases} 1 & k \le t < k + \frac{1}{2} \\ 0 & k + \frac{1}{2} \le t < k+1 \end{cases}, \quad k \in \mathbb{Z}$$



$$c_0 = \frac{1}{T}\int_T s(t)\,dt = \left[\begin{matrix}\text{VALOR} \\ \text{MEDIO}\end{matrix}\right] = \frac{1}{1}\int_0^1 s(t)\,dt = \int_0^{1/2} 1\,dt + \cancel{\int_{1/2}^{1} 0\,dt} = [t]_0^{1/2} - 0 = \frac{1}{2}$$

Per $n \neq 0$:

$$c_n = \frac{1}{T}\int_T s(t) e^{-j2\pi\frac{n}{T}t} dt = \int_0^1 s(t) e^{-j2\pi n t} dt = \int_0^{1/2} e^{-j2\pi n t} dt = \qquad \left(\int e^{at} dt = \frac{1}{a} e^{at} + c\right)$$

$$= \frac{-1}{j2\pi n}\left[e^{-j2\pi n t}\right]_0^{1/2} = \frac{j}{2\pi n}\left[\underbrace{e^{-j\pi n}}_{(-1)^n} - \underbrace{e^0}_{1}\right] = \frac{j}{2\pi n}\left[(-1)^n - 1\right] = \begin{cases} -\frac{j}{\pi n}, & n \text{ DISPARI} \\ 0, & n \text{ PARI} \end{cases}$$

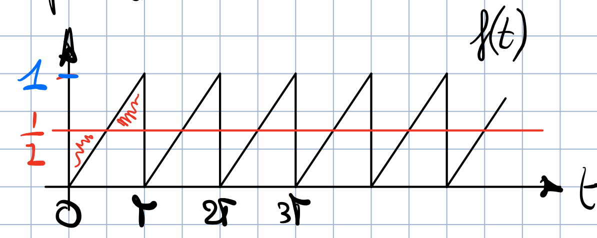
$$\text{Quindi: } s(t) = \frac{1}{2} + 2\sum_{\substack{n=1 \\ (n \text{ DISPARI})}}^{\infty} \frac{-j}{\pi n} e^{j2\pi n t} = \frac{1}{2} + 2\sum_{n=0}^{\infty} \frac{-j}{\pi(2n+1)} e^{j2\pi(2n+1)t}$$

In forma TRIGONOMETRICA (II):

$$a_n = Re[c_n] = 0 \,, \forall n \quad ; \quad b_n = -Im[c_n] = \begin{cases} 0 & n \text{ PARI} \\ \frac{j}{\pi n} & n \text{ DISPARI} \end{cases}$$

$$\rightarrow s(t) = \frac{1}{2} + 2 \sum_{\substack{n=1 \\ \text{DISPARI}}}^{\infty} b_n \sin(2\pi n t) = \frac{1}{2} + 2 \sum_{n=0}^{\infty} \frac{j}{\pi(2n+1)} \sin\left(2\pi(2n+1)t\right)$$

ESEMPIO : FS del "DENTE DI SEGA" di periodo T:



$$\begin{cases} f(t) = \dfrac{t}{T} \,, & 0 \leq t < T \\ f(t+T) = f(t) \,, & \forall t \in \mathbb{R} \end{cases}$$

$$c_0 = \frac{1}{T}\int_T f(t)\, dt = \frac{1}{T}\int_0^T \frac{t}{T}\, dt = \frac{1}{T^2}\int_0^T t\, dt = \frac{1}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{1}{\cancel{T^2}}\left[\frac{\cancel{T^2}}{2} \cancel{- 0}\right] = \frac{1}{2}$$

$$c_n = \frac{1}{T}\int_T f(t)\, e^{-j2\pi\frac{n}{T}t}\, dt = \frac{1}{T}\int_0^T \frac{t}{T}\, e^{-j2\pi\frac{n}{T}t}\, dt = \frac{1}{T^2}\int_0^T \underbrace{t}_{f(t) \to f'(t) = 1} \cdot \underbrace{e^{-j2\pi\frac{n}{T}t}}_{g'(t) \to g(t) = \frac{e^{-j2\pi\frac{n}{T}t}}{-\frac{j2\pi n}{T}}}\, dt =$$

[Integraz. per PARTI : $\int f(t)\cdot g'(t)\, dt = [f(t)\cdot g(t)] - \int f'(t)\, g(t)\, dt$]

$$c_n = \frac{1}{T^2}\left\{\left[t \cdot \frac{-T}{j2\pi n} e^{-j2\pi\frac{n}{T}t}\right]_0^T - \int_0^T \cancel{1} \frac{-T}{j2\pi n} e^{-j2\pi\frac{n}{T}t}\, dt\right\} =$$

$$= \frac{1}{T^2}\left\{\left[\frac{-T^2}{j2\pi n} \underbrace{\cancel{e^{-j2\pi n}}}_{1} \cancel{- 0}\right] - \left(\frac{-T}{j2\pi n}\right)^2 \left[e^{-j2\pi\frac{n}{T}t}\right]_0^T\right\} =$$

$$= \frac{1}{\cancel{T^2}}\left\{\frac{j\cancel{T^2}}{2\pi n} \cancel{- \left(\frac{-T}{j2\pi n}\right)^2 \left[\underbrace{e^{-j2\pi n}}_{1} - \underbrace{e^0}_{1}\right]}\right\} = \frac{j}{2\pi n} \,, \quad n \neq 0$$

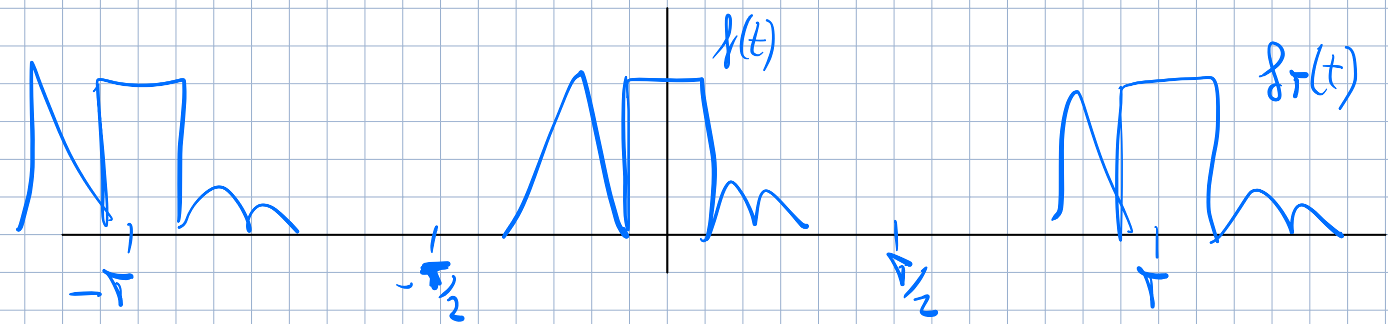
$$\boxed{\begin{aligned} c_0 &= \frac{1}{2} \\ c_n &= \frac{j}{2\pi n} \,, \; n \neq 0 \end{aligned}}$$

Quindi :

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{j}{2\pi n} e^{j2\pi\frac{n}{T}t} \qquad \text{forma ESPONENZIALE}$$

$$\longrightarrow \quad a_n = Re[c_n] = 0 \quad ; \quad b_n = -Im[c_n] = \frac{-1}{2\pi n}$$

$$f(t) = \frac{1}{2} + \cancel{2}\sum_{n=1}^{\infty} \frac{-1}{\cancel{2}\pi n} \sin\left(2\pi\frac{n}{T}t\right) \qquad \text{forma TRIGONOMETRICA}$$



Data $f(t)$ NON PERIODICA

Considero $f_T(t) = \begin{cases} f(t) & \text{per } -\frac{T}{2} \le t \le \frac{T}{2} \\ f(t) = f(t+kT), & k \in \mathbb{Z} \end{cases}$ → PERIODICA

→ $\boxed{f(t) = \lim_{T\to\infty} f_T(t)}$

Data $f(t)$ NON PERIODICA,

→ def: $f_T(t) = \begin{cases} f(t), & -\frac{T}{2} \le t < \frac{T}{2} \\ f(t+kT) & \text{altrove}, \quad k \in \mathbb{Z} \end{cases}$ → PERIODICA, periodo T

Posso scrivere: $f(t) = \lim_{T\to\infty} f_T(t)$ → PERIODICA, periodo $T \to \infty$

Dato che $f_T(t)$ è PERIODICA → posso applicare SF:

$$f_T(t) = \sum_{-\infty}^{\infty} c_n e^{j2\pi \frac{n}{T} t} = \left\{ \text{def: } f_n = \frac{n}{T} \right\} = \sum_{-\infty}^{+\infty} c_n e^{j2\pi f_n t}$$

$$f(t) = \lim_{T\to\infty} f_T(t) = \lim_{T\to\infty} \sum_{-\infty}^{+\infty} c_n e^{j2\pi f_n t} = \text{(sostituisco } c_n \text{:)}$$

$$\text{dove} \quad c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} f_T(t) e^{-j2\pi f_n t} dt = \left(\text{def: } \Delta f = \frac{1}{T}\right) = \underbrace{\Delta f \int_{-\frac{T}{2}}^{+\frac{T}{2}} f_T(t) e^{-j2\pi f_n t} dt}_{c_n}$$

$$f(t) = \lim_{T\to\infty} \sum_{-\infty}^{+\infty} \left[\Delta f \int_{-\frac{T}{2}}^{+\frac{T}{2}} f_T(t) e^{-j2\pi f_n t} dt \right] e^{j2\pi f_n t} =$$

$$= \lim_{T\to\infty} \sum_{-\infty}^{\infty} \underbrace{\int_{-T/2}^{+T/2} f(\tau) e^{-j2\pi f_n \tau} d\tau}_{F(f_n)} \cdot e^{j2\pi f_n t} \cdot \Delta f$$

Definisco : $F(f) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi f t} dt$ $\qquad f_n = \frac{n}{T}$

$$f(t) = \lim_{\substack{T\to\infty \\ \Delta f \to 0}} \sum_{-\infty}^{+\infty} F(f_n) e^{j2\pi f_n t} \cdot \Delta f = \int_{-\infty}^{+\infty} F(f) e^{j2\pi f t} df$$

In SINTESI :

SPETTRO di $f(t)$

$$F(f) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi f t} dt$$ TRASFORMATA di FOURIER di $f(t)$ [eq. di ANALISI]

$$F(f) = \mathcal{F}\{f(t)\} \quad ; \quad f(t) \xrightarrow{\mathcal{F}} F(f)$$

$$f(t) = \int_{-\infty}^{+\infty} F(f) e^{j2\pi f t} df$$ (TRASFORM. INVERSA) ANTITRASFORMATA di FOURIER [eq. di SINTESI]

$$f(t) = \mathcal{F}^{-1}\{F(f)\} ; \quad F(f) \xrightarrow{\mathcal{F}^{-1}} f(t)$$

$F(f)$ è lo SPETTRO di $f(t)$: è una FUNZIONE COMPLESSA di f

$$F(f) : \mathbb{R} \longrightarrow \mathbb{C} \quad ; \quad F(f) \in \mathbb{C}$$

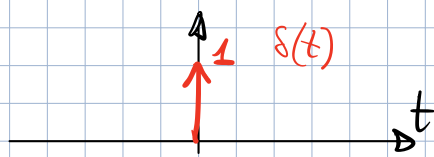
$|F(f)|$ è lo SPETTRO di AMPIEZZA di $f(t)$

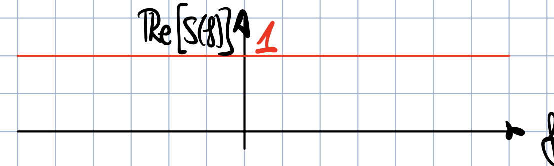
$\measuredangle F(f)$ è lo SPETTRO di FASE di $f(t)$

ESEMPI NOTEVOLI di TRASFORMATA

DELTA di DIRAC : $\delta(t)$

$$s(t) = \delta(t) \longrightarrow S(f) = \mathcal{F}\{s(t)\} = \int_{-\infty}^{+\infty} \delta(t) e^{-j2\pi f t} dt = e^{-j2\pi f 0} = 1$$





$\mathcal{F}\{\delta(t)\} = 1$; $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$ FOURIER PAIR

$$s(t) = \delta(t-t_0) \longrightarrow S(f) = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j2\pi ft}\, dt = e^{-j2\pi f t_0}$$

$$\delta(t-t_0) \longleftrightarrow e^{-j2\pi f t_0} \left(= \cos(2\pi t_0 \cdot f) - j \sin(2\pi t_0 f) \right)$$

RETTANGOLO : $\mathrm{rect}(t) = \begin{cases} 1 & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{altrove} \end{cases}$

$$S(f) = \int_{-\infty}^{+\infty} \mathrm{rect}(t)\, e^{-j2\pi ft}\, dt = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cancel{1} \cdot e^{-j2\pi ft}\, dt = \left[\frac{-1}{j2\pi f} e^{-j2\pi ft} \right]_{-\frac{1}{2}}^{\frac{1}{2}} =$$

$$= \frac{-1}{j2\pi f} \left[e^{-j\pi f} - e^{+j\pi f} \right] =$$ (Eulero : $e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$)

$$= \frac{-1}{j2\pi f} \left[\cancel{\cos(\pi f)} - j\sin(\pi f) - \left(\cancel{\cos(\pi f)} + j\sin(\pi f) \right) \right] =$$

$$= \frac{\cancel{-}1}{\cancel{j2}\pi f} \left[\cancel{-2j} \sin(\pi f) \right] = \frac{\sin \pi f}{\pi f}$$

SENO CARDINALE

$$\boxed{\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\sin \pi f}{\pi f} = \mathrm{sinc}(f) \quad \text{FOURIER PAIR}}$$

TRASFORMATA di FOURIER di SEGNALI REALI

Ipotesi : $s(t) \in \mathbb{R},\ \forall t \in \mathbb{R}$

$$s(t) \xrightarrow{\mathcal{F}} S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty} s(t) \left[\cos(2\pi ft) - j\sin(2\pi ft) \right] dt =$$

EULERO

$$S(f) = \underbrace{\int_{-\infty}^{+\infty} s(t) \cos(2\pi ft)\, dt}_{\in \mathbb{R}} - j \underbrace{\int_{-\infty}^{+\infty} s(t) \sin(2\pi ft)\, dt}_{\in \mathbb{R}} = \mathrm{Re}[S(f)] - j\, \mathrm{Im}[S(f)]$$

$$\mathrm{Re}[S(f)] = \int_{-\infty}^{+\infty} s(t) \cos(2\pi ft)\, dt = \int_{-\infty}^{\infty} s(t) \cos(2\pi(-f)t)\, dt = \mathrm{Re}[S(-f)]$$

$\hookrightarrow$ $Re[S(f)]$ ha SIMMETRIA PARI : $Re[S(f)] = Re[S(-f)]$

$$Im[S(f)] = \int_{-\infty}^{+\infty} s(t)\sin(2\pi f t)\,dt = -\int_{-\infty}^{\infty} s(t)\sin(2\pi(-f)t)\,dt = -Im[S(-f)]$$

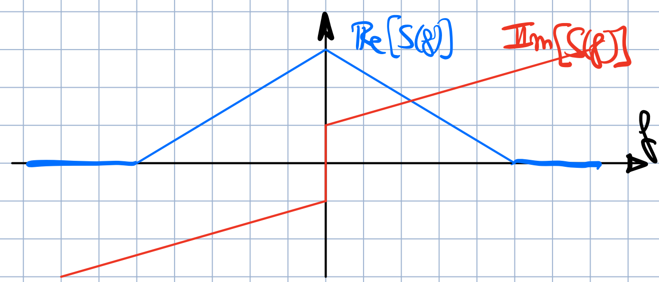
$\hookrightarrow$ $Im[S(f)]$ ha SIMMETRIA DISPARI : $Im[S(f)] = -Im[S(-f)]$

MODULO : $|S(f)| = |S(-f)|$ $\longrightarrow$ MODULO ha SIMMETRIA PARI

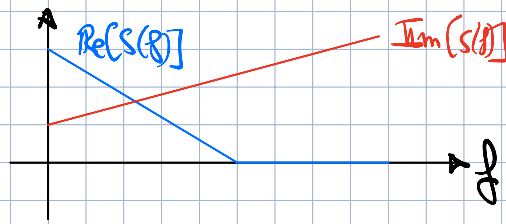
FASE : $\measuredangle S(f) = -\measuredangle S(-f)$ $\longrightarrow$ FASE ha SIMMETRIA DISPARI

Se $s(t)$ è REALE : $s(t) \in \mathbb{R}, \forall t \longleftrightarrow S(f) = \overline{S(-f)}$

SIMMETRIA "HERMITIANA"



TRASF. di FOURIER "BILATERA"



TRASF. di FOURIER MONOLATERA

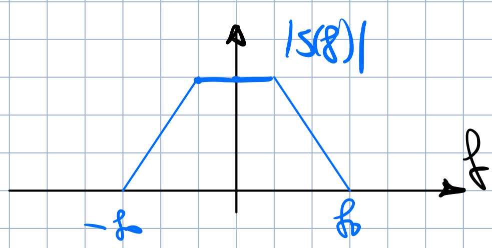
BANDA di un SEGNALE

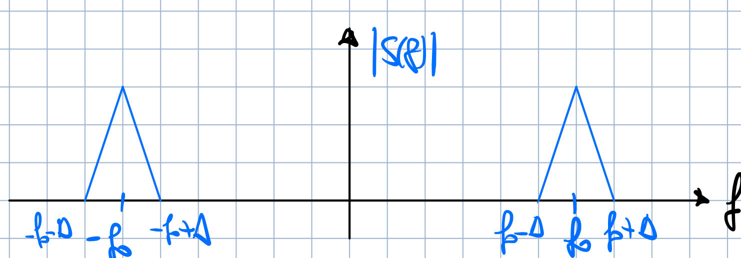
BANDA : il SUPPORTO dello SPETTRO di un SEGNALE

Dato $s(t) \xrightarrow{\mathcal{F}} S(f)$: BANDA di $s(t) := B = \{ f \in \mathbb{R} : S(f) \neq 0 \}$

LARGHEZZA di BANDA (BANDWIDTH) : ESTENSIONE della BANDA B

- ★ quando B CONTIENE $f=0$ $\longrightarrow$ $s(t)$ è un SEGNALE in BANDA BASE
- ★ " B NON CONTIENE $f=0$ $\longrightarrow$ $s(t)$ è un SEGNALE in BANDA PASSANTE





LA LARGHEZZA di BANDA di un segnale viene DEFINITA sulla RAPPRESENTAZIONE MONOLATERA del suo SPETTRO

PROPRIETÀ della TRASFORMATA di FOURIER

LINEARITÀ

Se: $x(t) \xleftrightarrow{\mathcal{F}} X(f)$
$y(t) \xleftrightarrow{\mathcal{F}} Y(f)$ $\longrightarrow$ $a x(t) + b y(t) \xleftrightarrow{\mathcal{F}} a X(f) + b Y(f)$

DIM: $\mathcal{F}\{a x(t) + b y(t)\} = \int_{-\infty}^{+\infty} (a x(t) + b y(t)) e^{-j2\pi f t} dt = \int_{-\infty}^{+\infty} a x(t) e^{-j2\pi f t} dt + \int_{-\infty}^{\infty} b y(t) e^{-j2\pi f t} dt =$

$$= a \underbrace{\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt}_{X(f)} + b \underbrace{\int_{-\infty} y(t) e^{-j2\pi f t} dt}_{Y(f)} = a X(f) + b Y(f)$$ c.v.d.

→ la trasf. di Fourier è LINEARE

DUALITÀ (o SIMMETRIA)

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow S(t) \xleftrightarrow{\mathcal{F}} s(-f)$$

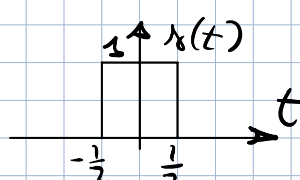
DIM: $\mathcal{F}\{S(t)\} = \int_{-\infty}^{\infty} S(t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} S(t) e^{j2\pi(-f)t} dt = s(-f)$ c.v.d.

Per $s(t)$ con SIMMETRIA PARI: $s(t) = s(-t)$, $\forall t$

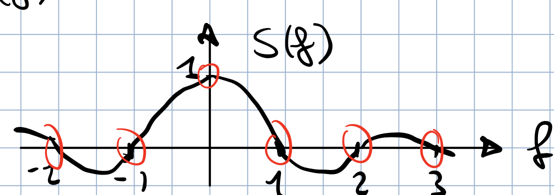
se $s(t) = s(-t)$, $\forall t \in \mathbb{R}$ $\longrightarrow$ $S(t) \xleftrightarrow{\mathcal{F}} s(-f) = s(f)$

$s(t) \xleftrightarrow{\mathcal{F}} S(f) \Longrightarrow S(t) \xleftrightarrow{\mathcal{F}} s(f)$

ESEMPIO: $s(t) = \mathrm{rect}(t) \xrightarrow{\mathcal{F}} S(f) = \mathrm{sinc}(f)$



$\xleftrightarrow{\mathcal{F}}$



→ per DUALITÀ: $s(t) = \mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} S(f) = \mathrm{rect}(f)$

TRASLAZIONE nei TEMPI

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow s(t-t_0) \xleftrightarrow{\mathcal{F}} S(f)\,e^{-j2\pi f t_0}$$

DIM: $s(t-t_0) \xrightarrow{\mathcal{F}} S'(f) = \int_{-\infty}^{\infty} s(t-t_0)\, e^{-j2\pi f t}\, dt = \begin{Bmatrix} t'=t-t_0 \to t=t'+t_0 \\ dt'=dt \end{Bmatrix} =$

$$= \int_{-\infty}^{\infty} s(t')\, e^{-j2\pi f (t'+t_0)}\, dt' = \int_{-\infty}^{\infty} s(t')\, e^{-j2\pi f t'} \cdot \underbrace{e^{-j2\pi f t_0}}_{\text{costante}}\, dt' =$$

$$= e^{-j2\pi f t_0} \cdot \underbrace{\int_{-\infty}^{\infty} s(t')\, e^{-j2\pi f t'}\, dt'}_{S(f)} = S(f)\cdot e^{-j2\pi f t_0} \quad \text{CVD}$$

TRASLAZIONE nelle FREQUENZE (per DUALITÀ):

TRASL. TEMPI: $s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow s(t-t_0) \xleftrightarrow{\mathcal{F}} S(f)\,e^{-j2\pi f t_0}$

per DUALITÀ: $s(t)\,e^{-j2\pi f t_0} \xleftrightarrow{\mathcal{F}} S(f+f_0)$

MODULAZIONE di $s(t)$:

$$s_M(t) = s(t)\cos(2\pi f_c t)$$

$\to$ freq. PORTANTE

$$s_M(t) = s(t)\cdot \frac{1}{2}\left(e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right) =$$

$$= \frac{1}{2}\left[s(t)\, e^{j2\pi f_c t} + s(t)\, e^{-j2\pi f_c t}\right] =$$

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow s_M(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[S(f-f_c) + S(f+f_c)\right] = S_M(f)$$

$e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$

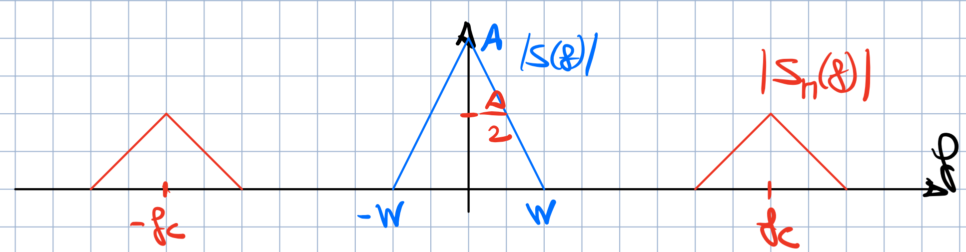
$e^{-j\vartheta} = \cos\vartheta - j\sin\vartheta$

$e^{j\vartheta} + e^{-j\vartheta} = 2\cos\vartheta$

$e^{j\vartheta} - e^{-j\vartheta} = 2j\sin\vartheta$

$\cos\vartheta = \frac{1}{2}\left(e^{j\vartheta} + e^{-j\vartheta}\right)$

$\sin\vartheta = \frac{1}{2j}\left(e^{j\vartheta} - e^{-j\vartheta}\right)$



SCALATURA

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow s(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}\, S\!\left(\frac{f}{a}\right)$$

DIM: 1) $a > 0$:

$$\mathcal{F}\{s(at)\} = \int_{-\infty}^{\infty} s(at)\, e^{-j2\pi f t}\, dt = \begin{cases} \tau = at \\ d\tau = a\,dt \end{cases} = \int_{-\infty}^{\infty} s(\tau)\, e^{-j2\pi f \frac{\tau}{a}}\, \frac{d\tau}{a} =$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} s(\tau)\, e^{-j2\pi \frac{f}{a}\tau}\, d\tau = \frac{1}{a} S\left(\frac{f}{a}\right) \checkmark$$

2) $a<0$: $$\mathcal{F}\{s(at)\} = \begin{cases} \tau = at \\ d\tau = a\,dt \end{cases} = \int_{+\infty}^{-\infty} s(\tau)\, e^{-j2\pi f \frac{\tau}{a}}\, \frac{d\tau}{a} = -\int_{-\infty}^{\infty} s(\tau)\, e^{-j2\pi \frac{f}{a}\tau}\, \frac{d\tau}{a} =$$

$$= \frac{1}{-a}\int_{-\infty}^{\infty} s(\tau)\, e^{-j2\pi \frac{f}{a}\tau}\, d\tau = \frac{1}{-a} S\left(\frac{f}{a}\right) \checkmark$$

Di conseguenza : $s(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} S\left(\frac{f}{a}\right)$ CVD

CONVOLUZIONE

$$\begin{matrix} x(t) \xleftrightarrow{\mathcal{F}} X(f) \\ y(t) \xleftrightarrow{\mathcal{F}} Y(f) \end{matrix} \longrightarrow x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(f)\cdot Y(f)$$

DIM:

$$\mathcal{F}\{x(t)*y(t)\} = \mathcal{F}\left\{\int_{-\infty}^{+\infty} x(\tau)y(t-\tau)\,d\tau\right\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau\right] e^{-j2\pi f t}\, dt =$$

$t \to t-\tau+\tau$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, e^{-j2\pi(t-\tau)} \cdot e^{-j2\pi f\tau} \cdot dt\, d\tau =$$

$$= \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f\tau}\, d\tau}_{X(f)} \cdot y(t-\tau)\, e^{-j2\pi f(t-\tau)}\, dt =$$

DIPENDE SOLO da τ

$$= \int_{-\infty}^{\infty} X(f)\, y(t-\tau)\, e^{-j2\pi f(t-\tau)}\, dt = X(f)\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j2\pi f(t-\tau)}\, dt = \begin{cases} t' = t-\tau \\ dt' = dt \end{cases}$$

$$= X(f)\int_{-\infty}^{\infty} \underbrace{y(t')\, e^{-j2\pi f t'}\, dt'}_{Y(f)} = X(f)\cdot Y(f) \quad \text{C.V.D}$$

$$\boxed{\begin{matrix} x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(f)\cdot Y(f) \\ x(t)\cdot y(t) \xleftrightarrow{\mathcal{F}} X(f) * Y(f) \end{matrix}}$$

per DUALITÀ :

DERIVAZIONE

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow s'(t) = \frac{d}{dt} s(t) \xleftrightarrow{\mathcal{F}} j2\pi f \cdot S(f)$$

DIM:

$$s'(t) = \frac{d}{dt} s(t) = \frac{d}{dt} \int_{-\infty}^{\infty} \underbrace{S(f) e^{j2\pi f t}}_{f(t)} df = \int_{-\infty}^{\infty} \underbrace{S(f) \cdot j2\pi f}_{} \cdot e^{j2\pi f t} df = \underbrace{s'(t)}_{}$$

$$s'(t) \xleftrightarrow{\mathcal{F}} j2\pi f \cdot S(f) \quad \text{CVD}$$

INTEGRAZIONE (INDEFINITA)

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow \int_{-\infty}^{t} s(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j2\pi f} S(f) + \frac{S(0)}{2} \delta(f)$$

DIM: $s(t) * u(t) = \int_{-\infty}^{+\infty} s(\tau)\, u(t-\tau)\, d\tau = \int_{-\infty}^{t} s(\tau) \cdot 1\, d\tau = \int_{-\infty}^{t} s(\tau) d\tau$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow u(t-\tau) = \begin{cases} 1 & t-\tau \geq 0 \rightarrow \tau \leq t \\ 0 & t-\tau < 0 \rightarrow \tau > t \end{cases}$$

$$\longrightarrow \int_{-\infty}^{t} s(\tau) d\tau = s(t) * u(t) \xleftrightarrow{\mathcal{F}} S(f) \cdot U(f) \; : \; \left[U(f) = \frac{1}{j2\pi f} + \frac{1}{2} \delta(f) \right]$$

$$\int_{-\infty}^{t} s(\tau) d\tau \xleftrightarrow{\mathcal{F}} S(f)\, U(f) = S(f) \left[\frac{1}{j2\pi f} + \frac{\delta(f)}{2} \right] = \frac{1}{j2\pi f} S(f) + \frac{S(0)}{2} \delta(f)$$

CVD

POTENZA ed ENERGIA di SEGNALI nel dominio delle FREQUENZE

REMINDER: dato un segnale $s(t)$:

POTENZA ISTANTANEA di $s(t)$: $P(t) = s(t) \cdot \overline{s(t)} = |s(t)|^2 \quad (= s^2(t),\ s(t) \in \mathbb{R})$

ENERGIA di $s(t)$: $E_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt$

POTENZA MEDIA di $s(t)$: $P_s = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} |s(t)|^2 dt$

$s(t)$ è SEGNALE - ENERGIA sse: $0 < E_s < \infty$ (energia FINITA)
$\hookrightarrow$ $P_s = 0$ (pot. media NULLA)

$s(t)$ è SEGNALE - POTENZA sse: $0 < P_s < \infty$ (pot. media FINITA)
$\hookrightarrow$ $E_s = \infty$ (energia INFINITA)

TEOREMA di PARSEVAL (per SEGNALI ENERGIA)

Sia $s(t)$ un SEGNALE ENERGIA $\rightarrow$ sia $S(f)$ il suo SPETTRO

$$\boxed{E_s = \int_{-\infty}^{+\infty} |s(t)|^2 \, dt = \int_{-\infty}^{\infty} |S(f)|^2 \, df}$$

"L'ENERGIA di $s(t)$ COINCIDE con l'ENERGIA di $S(f)$"

DIM: $$E_s = \int_{-\infty}^{+\infty} |s(t)|^2 \, dt = \int_{-\infty}^{\infty} s(t) \cdot \overline{s(t)} \, dt = \int_{-\infty}^{\infty} \overline{s(t)} \underbrace{\int_{-\infty}^{\infty} S(f) e^{j2\pi ft} \, df}_{s(t)} \cdot dt =$$

$$= \int_{-\infty}^{\infty} S(f) \left\{ \int_{-\infty}^{\infty} \underbrace{\overline{s(t)} \, e^{j2\pi ft}} \, dt \right\} df = \left(\bar{a} \cdot b = \overline{a \cdot \bar{b}} \right) = \int_{-\infty}^{\infty} S(f) \, \overline{\underbrace{\int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} \, dt}_{\overline{S(f)}}} \cdot df =$$

$$= \int_{-\infty}^{\infty} S(f) \cdot \overline{S(f)} \, df = \int_{-\infty}^{\infty} |S(f)|^2 \, df \quad \text{C.V.D.}$$

DEF: $|S(f)|^2$: DENSITÀ SPETTRALE di ENERGIA

TEOREMA di PARSEVAL per SEGNALI POTENZA:

Dato $s(t)$ SEGNALE POTENZA, considero la POTENZA MEDIA:

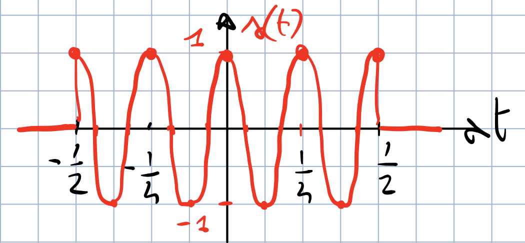
$$P_s = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |s(t)|^2 \, dt = \int_{-\infty}^{\infty} \underbrace{|S(f)|^2}_{\text{DENSITÀ SPETTRALE di POTENZA}} \, df$$

ESEMPI di CALCOLO di TRASFORMATA di FOURIER

ES: calcolare la trasf. di Fourier di $s(t) = \mathrm{rect}(t)\cos(8\pi t)$

$$s(t) = \mathrm{rect}(t)\cos(\underset{2\pi f t}{8\pi t}) = \begin{cases} \cos(8\pi t) & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{altrove} \end{cases}$$

$f = 4 \rightarrow T = \frac{1}{4}$

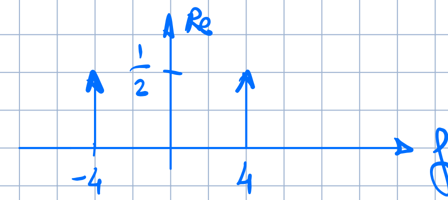


$$s(t) = \mathrm{rect}(t) \cdot \cos(8\pi t)$$

$$\downarrow \mathcal{F} \qquad \downarrow \mathcal{F}$$

$$\mathrm{sinc}(f) \qquad \frac{1}{2}[\delta(f-4)+\delta(f+4)]$$

$$\cos(8\pi t) = \frac{1}{2}\left[e^{j8\pi t} + e^{-j8\pi t}\right] \xrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta(f-4) + \delta(f+4)\right]$$



$$S(f) = \mathrm{sinc}(f) * \frac{1}{2}\left[\delta(f-4)+\delta(f+4)\right] = \frac{1}{2}\mathrm{sinc}(f)*\delta(f-4) + \frac{1}{2}\mathrm{sinc}(f)*\delta(f+4) =$$

$$= \frac{1}{2}\left[\mathrm{sinc}(f-4) + \mathrm{sinc}(f+4)\right] \quad \checkmark$$

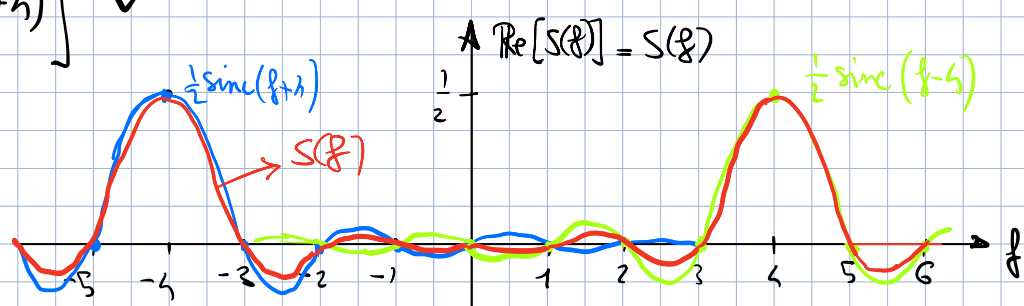
Applicando la definizione:

$$s(t) = \mathrm{rect}(t)\cos(8\pi t) \longrightarrow S(f) = \int_{-\infty}^{\infty} \mathrm{rect}(t)\cos(8\pi t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(8\pi t)\, e^{-j2\pi f t}\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}\left[e^{j8\pi t} + e^{-j8\pi t}\right] e^{-j2\pi f t}\, dt =$$

$$= \frac{1}{2}\left\{ \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t + j8\pi t}\, dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t - j8\pi t}\, dt \right\} = \frac{1}{2}\left\{ \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi(f-4)t}\, dt}_{\mathrm{sinc}(f-4)} + \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi(f+4)t}\, dt}_{\mathrm{sinc}(f+4)} \right\} =$$

$$= \frac{1}{2}\left[\mathrm{sinc}(f-4) + \mathrm{sinc}(f+4)\right] \quad \checkmark$$



ES: calcolare la FT di: $s(t) = \text{sinc}\left(\frac{t-4}{2}\right)$

e rappresentarne: $\text{Re}[\,]$, $\text{Im}[\,]$, $|\,|$, $\measuredangle$.

$$s'(t) = \text{sinc}(t) \xrightarrow{\mathcal{F}} S'(f) = \text{rect}(f)$$

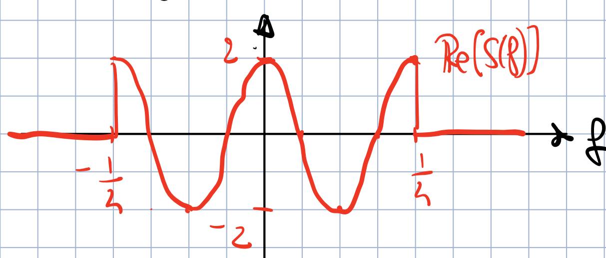
SCALATURA: $$s''(t) = \text{sinc}\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} S''(f) = 2\,\text{rect}(2f)$$

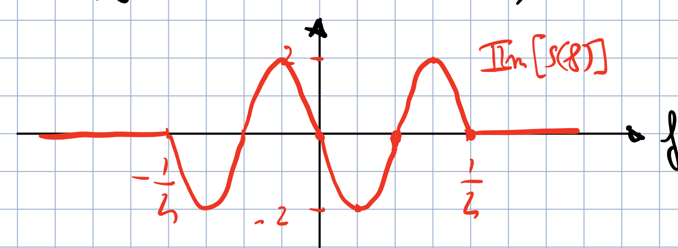
TRASLAZ: $$s(t) = \text{sinc}\left(\frac{t-4}{2}\right) \xrightarrow{\mathcal{F}} S(f) = 2\,\text{rect}(2f)\, e^{-j2\pi f 4}$$

$$S(f) = 2\,\text{rect}(2f)\, e^{-j8\pi f}$$

$$\rightarrow S(f) = 2\,\text{rect}(2f)\left[\cos(-8\pi f) + j\sin(-8\pi f)\right] = \underbrace{2\,\text{rect}(2f)\cos(8\pi f)}_{\text{Re}} - j\,\underbrace{2\,\text{rect}(2f)\sin(8\pi f)}_{\text{Im}}$$

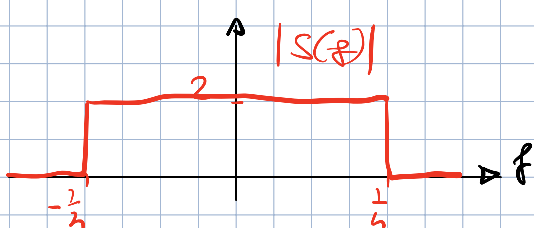
$$\rightarrow \text{Re}[S(f)] = 2\,\text{rect}(2f)\cos(8\pi f)\ ;\quad \text{Im}[S(f)] = -2\,\text{rect}(2f)\sin(8\pi f)$$

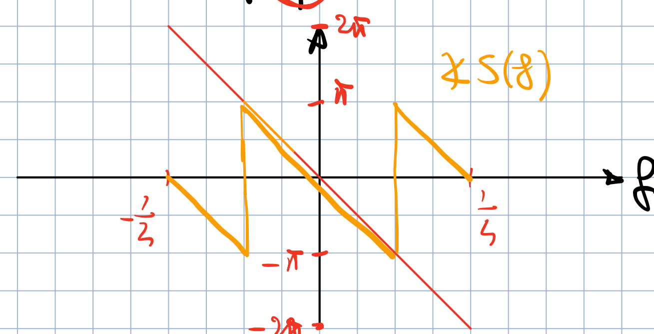




$$|S(f)| = \left|2\,\text{rect}(2f)\, e^{-j8\pi f}\right| = |2|\cdot|\text{rect}(2f)|\,\underbrace{\left|e^{-j8\pi f}\right|}_{=1} = 2\,\text{rect}(2f) = \begin{cases} 2 & -\frac{1}{4} \le f \le \frac{1}{4} \\ 0 & \text{altrove} \end{cases}$$

$$\measuredangle S(f) = \measuredangle\left[2\,\text{rect}(2f)\, e^{-j8\pi f}\right] = \begin{cases} \measuredangle\left(2e^{-j8\pi f}\right) & -\frac{1}{4} \le f \le \frac{1}{4} \\ \measuredangle\, 0 & \text{altrove} \end{cases} = \begin{cases} -8\pi f & -\frac{1}{4} \le f \le \frac{1}{4} \\ \nexists & \end{cases}$$





Es 1 FEB 2020

Calcolare la FT di: $s(t) = 8\,\text{sinc}^2\left(\frac{t-3}{2}\right)$ e rappresentare $|S(f)|$, $\measuredangle S(f)$

$$s(t) = 8\,\text{sinc}\left(\frac{t-3}{2}\right)\text{sinc}\left(\frac{t-3}{2}\right)$$

$$\text{sinc}(t) \xrightarrow{\mathcal{F}} \text{rect}(f)$$

$$\text{sinc}^2(t) = \text{sinc}(t)\cdot\text{sinc}(t) \xrightarrow{\mathcal{F}} \text{rect}(f) * \text{rect}(f) = \text{tri}(f)$$

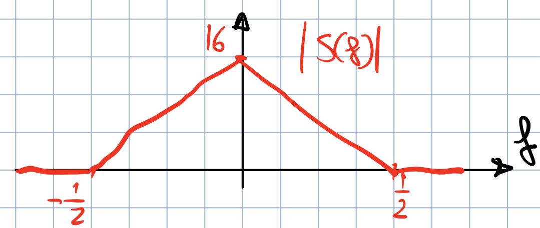
scalatura: $$\text{sinc}^2\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\,\text{tri}(2f)$$

traslazione: $$s(t) = \text{sinc}^2\left(\frac{t-3}{2}\right) \xrightarrow{\mathcal{F}} S(f) = 2\,\text{tri}(2f)\,e^{-j2\pi f 3}$$

$$\boxed{S(f) = 16\,\text{tri}(2f)\,e^{-j6\pi f}}$$

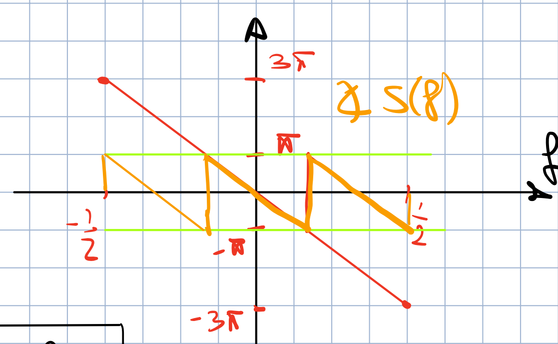
MODULO:

$$|S(f)| = 16\,|\text{tri}(2f)|\cdot\underbrace{|e^{-j6\pi f}|}_{=1} = 16\,\text{tri}(2f)$$



FASE:

$$\measuredangle S(f) = \measuredangle\{16\,\text{tri}(2f)\,e^{-j6\pi f}\} = \underbrace{\measuredangle(16\,\text{tri}(2f))}_{\begin{cases} 0, & |f|\le\frac{1}{2} \\ \nexists & \text{altrove}\end{cases}} + \underbrace{\measuredangle e^{-j6\pi f}}_{-6\pi f} = \begin{cases} -6\pi f & -\frac{1}{2}\le f\le\frac{1}{2} \\ \nexists & \text{altrove}\end{cases}$$



SISTEMI LTI nel DOMINIO della FREQUENZA

Considero un sistema LTI:

$x(t) \longrightarrow$ SISTEMA LTI $h(t)$ $\longrightarrow$ $\boxed{y(t) = x(t) * h(t)}$

Nel DOMINIO delle FREQUENZE:

$X(f)$ $\qquad H(f)$ $\qquad \boxed{Y(f) = X(f)\cdot H(f)}$

RISPOSTA in FREQUENZA

RISPOSTA in FREQUENZA: $H(f)$: TRASF. di FOURIER della dato sistema LTI: RISPOSTA all'IMPULSO $h(t)$

$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(f) = \int_{-\infty}^{+\infty} h(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \longrightarrow y(t) = x(t) * h(t) \qquad \text{CONVOLUZIONE}$$

$$X(f) \qquad H(f) \longrightarrow Y(f) = X(f) \cdot H(f) \qquad \text{PRODOTTO}$$

Es. ANALISI IN FREQUENZA di SISTEMI

$$\begin{array}{ccc} \text{da } x(t) & \xrightarrow[h(t)]{*} & y(t) = x(t) * h(t) \\ \mathcal{F} \downarrow & & \uparrow \mathcal{F}^{-1} \\ X(f) & \xrightarrow[\cdot]{H(f)} & Y(f) = X(f) \cdot H(f) \end{array}$$

SIGNIFICATO della RISPOSTA in FREQUENZA

Consideriamo la RISPOSTA a un FASORE a frequenza f_0:

$$x_{f_0}(t) = e^{j2\pi f_0 t} \qquad x_{f_0}(t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t) \,/\, H(f)\end{matrix}} \longrightarrow y_{f_0}(t)$$

$$\downarrow \mathcal{F}$$

$$X_{f_0}(f) = \delta(f - f_0) \longrightarrow Y_{f_0}(f) = X_{f_0}(f) \cdot H(f) = \delta(f-f_0)\, H(f) = H(f_0)\, \delta(f - f_0)$$

$$\xrightarrow{\mathcal{F}^{-1}} y_{f_0}(t) = H(f_0)\, e^{j2\pi f_0 t}$$

- L'USCITA è un FASORE alla STESSA FREQUENZA f_0!

MODULO: $|y_{f_0}(t)| = |H(f_0)| \cdot |x_{f_0}(t)|$

FASE: $\measuredangle y_{f_0}(t) = \measuredangle x_{f_0}(t) + \measuredangle H(f_0)$

$\longrightarrow \begin{cases} |H(f)| : \text{RISPOSTA in AMPIEZZA} \\ \measuredangle H(f) : \text{RISPOSTA in FASE} \end{cases}$

RISPOSTA in FREQUENZA : "RISPOSTA al FASORE"

"La componente SINUSOIDALE a frequenza f dell'INGRESSO $X(f)$ viene MOLTIPLICATA per $H(f)$ a dare l'uscita $Y(f)$:

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = H(f)X(f)$$

$$Y(f) = X(f) \cdot H(f)$$

In particolare : $Y(f) = H(f)X(f)$ viene : $\begin{cases} \text{AMPLIFICATA del fattore } |H(f)| \\ \text{SFASATA dell'angolo } \measuredangle H(f) \end{cases}$

Per questo i SISTEMI LTI vengono chiamati **FILTRI** (LINEARI)

FILTRI IDEALI

SISTEMI SENZA DISTORSIONE (DISTORTIONLESS)

- Un sistema LTI è detto SENZA DISTORSIONE SSE:

l'USCITA è UGUALE all'INGRESSO, A MENO di : $\begin{cases} \text{- un FATTORE COSTANTE} \\ \text{- un RITARDO} \end{cases}$

Formalmente :

$$x(t) \longrightarrow \underset{S[\cdot]}{\boxed{h(t) \mid H(f)}} \longrightarrow y(t)$$

$S[\cdot]$ è SENZA DISTORSIONE sse : $y(t) = A\,x(t-t_0)$

→ RISPOSTA all'IMPULSO : $h(t) = A\,\delta(t-t_0)$ di un sistema S. DISTORSIONE

$\downarrow \mathcal{F}$ $\qquad\qquad \downarrow \mathcal{F}$

RISPOSTA in FREQUENZA : $H(f) = A\,e^{-j2\pi f t_0}$ " " "

Osserviamo $H(f)$: $\begin{cases} |H(f)| = A \;:\; \text{amplificazione COSTANTE}, \ \forall f \\ \measuredangle H(f) = -2\pi t_0 f \;\; \text{FASE proporzionale a } f \rightarrow \text{RITARDO ?} \end{cases}$

Es: $s(t) = \cos(2\pi f t - \varphi)$, $\varphi = kf \rightarrow s(t) = \cos(2\pi f t - kf) \overset{!}{=}$

$$\overset{!}{=} \cos\left[2\pi f\left(t - \frac{k}{2\pi}\right)\right]$$

RITARDO

FILTRO IDEALE $:= \begin{cases} \text{SISTEMA SENZA DISTORSIONE}, & f \in B \\ H(f) = 0, & f \notin B \end{cases}$

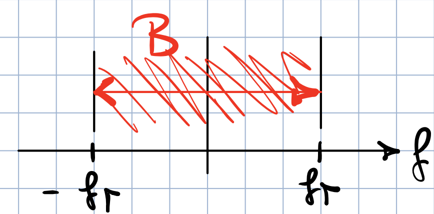
B: BANDA PASSANTE del FILTRO

$$\boxed{H(f) = \begin{cases} A e^{-j2\pi f t_0} & \text{per } f \in B \\ 0 & \text{per } f \notin B \end{cases}}$$

In funzione di come definisco B ottengo varie TIPOLOGIE di FILTRO:

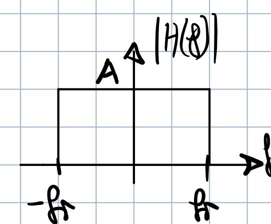
1) B: intervallo SIMMETRICO intorno a $f=0$

B: $-f_T \le f \le f_T$ $\quad f_T$: FREQUENZA di TAGLIO



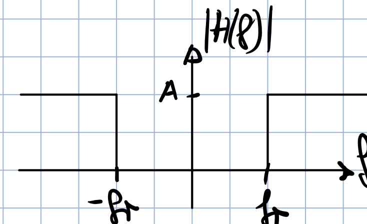
→ FILTRO PASSA-BASSO IDEALE: (LOW PASS)

$$H(f) = \begin{cases} A\, e^{-j2\pi f t_0} & -f_T \le f \le f_T \\ 0 & \text{altrove} \end{cases} = A e^{-j2\pi f t_0} \operatorname{rect}\left(\frac{f}{2f_T}\right)$$



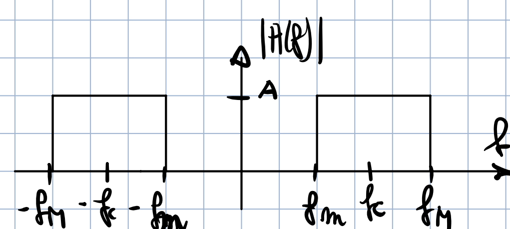
FILTRO PASSA-ALTO IDEALE: (HIGH-PASS)

$$H(f) = \begin{cases} A e^{-j2\pi f t_0} & |f| \ge f_T \\ 0 & \text{altrove} \end{cases} = A e^{-j2\pi f t_0} \left[1 - \operatorname{rect}\left(\frac{f}{2f_T}\right)\right]$$



FILTRO PASSA-BANDA IDEALE: (BAND-PASS)

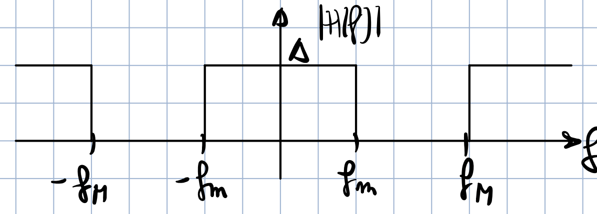
$$H(f) = \begin{cases} A e^{-j2\pi f t_0} & \text{per } f_m \le |f| < f_M \\ 0 & \text{altrove} \end{cases} =$$



$$= \left\{ \begin{array}{l} f_c = \frac{f_m + f_M}{2} \\ B = f_M - f_m \end{array} \right\} = \left[\operatorname{rect}\left(\frac{f - f_c}{B}\right) + \operatorname{rect}\left(\frac{f + f_c}{B}\right)\right] A e^{-j2\pi f t_0}$$

FILTRO ARRESTA-BANDA IDEALE: (STOP-BAND, NOTCH)

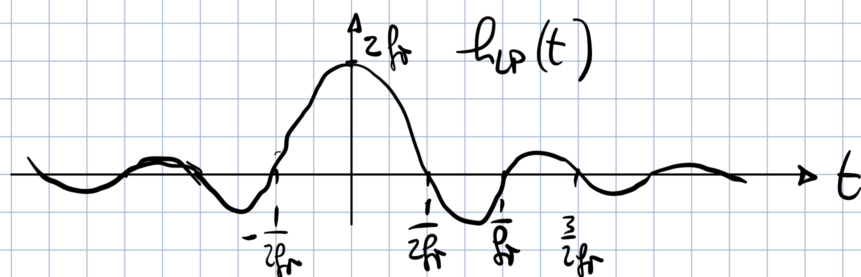
$$H(f) = \begin{cases} A\, e^{-j2\pi f t_0} & \text{per } |f|<f_m \vee |f|>f_M \\ 0 & \text{altrove} \end{cases}$$



Nel DOMINIO dei TEMPI : RISPOSTA all' IMPULSO dei FILTRI IDEALI

PASSA BASSO

$$H_{LP}(f) = \begin{cases} 1 & |f|\leq f_T \\ 0 & \text{altrove} \end{cases} = \text{rect}\left(\frac{f}{2f_T}\right) \xrightarrow{\mathcal{F}^{-1}} h_{LP}(t) = 2f_T\, \text{sinc}(2f_T \cdot t)$$



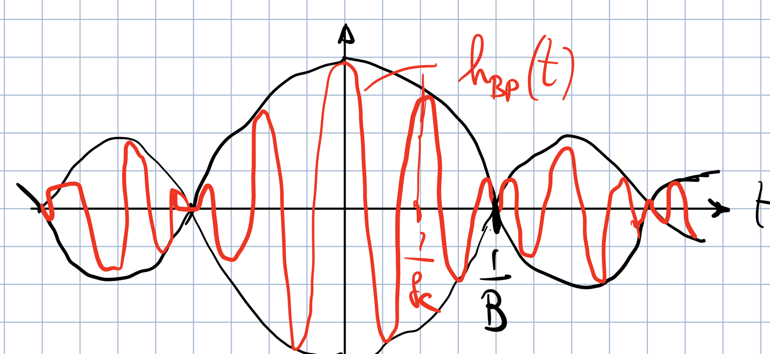
PASSA-ALTO :

$$H_{HP}(f) = \begin{cases} 0 & |f|\leq f_T \\ 1 & \text{altrove} \end{cases} = 1-\text{rect}\left(\frac{f}{2f_T}\right) \xrightarrow{\mathcal{F}^{-1}} h_{HP}(t) = \delta(t)-2f_T\, \text{sinc}(2f_T t)$$

PASSA-BANDA :

$$H_{BP}(f) = \begin{cases} 1 & f_m\leq|f|\leq f_M \\ 0 & \text{altrove} \end{cases} = \text{rect}\left(\frac{f-f_c}{B}\right)+\text{rect}\left(\frac{f+f_c}{B}\right) = \text{rect}\left(\frac{f}{B}\right) * \left[\delta(f-f_c)+\delta(f+f_c)\right]$$

$$\xrightarrow{\mathcal{F}^{-1}} h_{BP}(t) = B\,\text{sinc}(Bt)\cdot 2\cos(2\pi f_c t) = 2B\,\text{sinc}(Bt)\cos(2\pi f_c t)$$



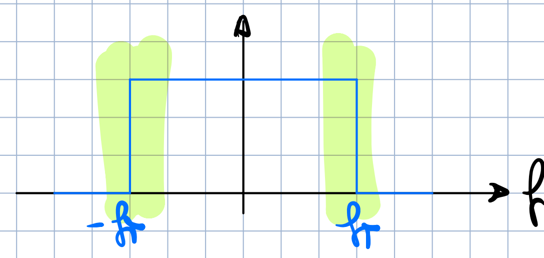
ARRESTA-BANDA

$$H_{SB}(f) = 1 - H_{BP}(f)$$

$$\xrightarrow{\mathcal{F}^{-1}} h_{SB}(t) = \delta(t) - 2B\,\text{sinc}(Bt)\cos(2\pi f_c t)$$

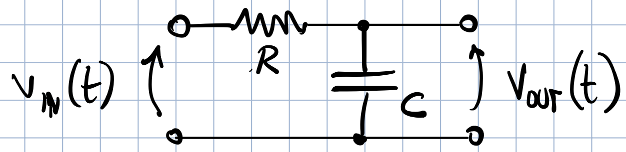
FILTRI REALI

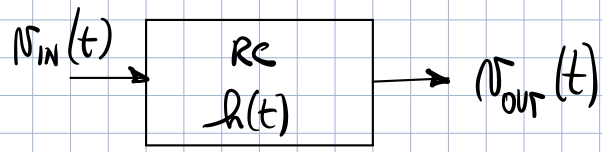
→ la TRANSIZIONE è GRADUALE!



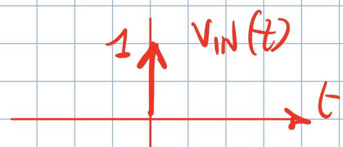
ESEMPIO di PASSA-BASSO REALE :

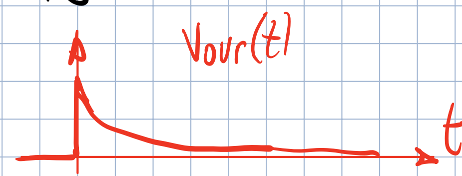
CIRCUITO RC





$$V_{IN}(t) = \delta(t) \longrightarrow V_{OUT}(t) = \frac{1}{RC}\, u(t)\, e^{-t/RC} = h(t)$$





$$h(t) = \frac{1}{RC}\, u(t)\, e^{-\frac{t}{RC}}$$

FOURIER PAIR : $e^{-at}u(t) \longleftrightarrow \dfrac{1}{a+j2\pi f}$

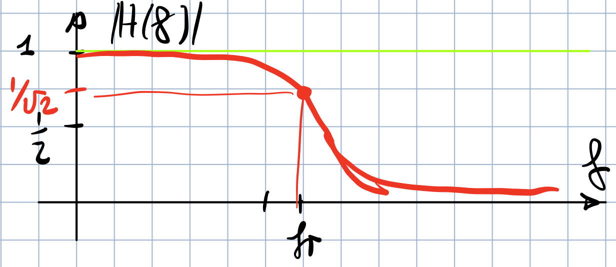
$$\left[a = \frac{1}{RC}\right] \longrightarrow H(f) = \frac{1}{RC}\cdot\frac{1}{\frac{1}{RC}+j2\pi f} = \frac{1}{1+j2\pi RC f}$$

Definisco $\dfrac{1}{2\pi RC} = f_T \longrightarrow \boxed{H(f) = \dfrac{1}{1+j\,f/f_T}}$

Risposte in AMPIEZZA e FASE :

$$|H(f)| = \frac{|1|}{\left|1+j\,f/f_T\right|} = \frac{1}{\sqrt{1^2+\left(\frac{f}{f_T}\right)^2}} = \begin{cases} f \ll f_T \rightarrow |H(f)| \approx 1 \\ f = f_T \rightarrow |H(f)| = 1/\sqrt{2} \\ f \gg f_T \rightarrow |H(f)| \approx \frac{f_T}{f} \end{cases}$$

$$\measuredangle H(f) = \cancel{\measuredangle 1} - \measuredangle\left(1+j\frac{f}{f_T}\right) = -\operatorname{atan2}\left(\frac{f}{f_T}\right) = \begin{cases} f \ll f_T \longrightarrow \measuredangle H(f) \approx 0 \\ f = f_T \longrightarrow \measuredangle H(f) = -\pi/4 \\ f \gg f_T \longrightarrow \measuredangle H(f) = -\pi/2 \end{cases}$$



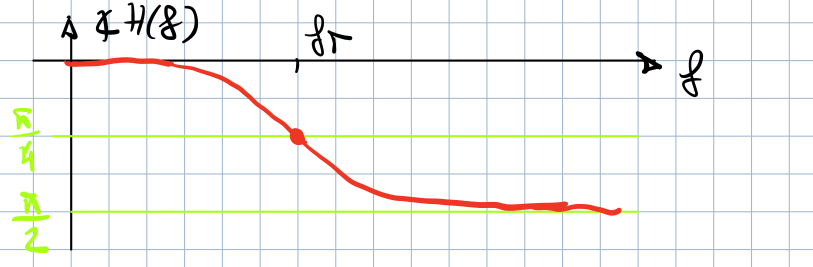
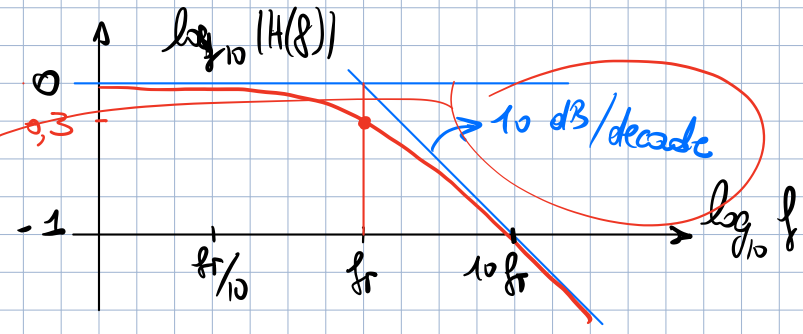


DIAGRAMMA di BODE : GRAFICO "LOG-LOG" di $|H(f)|$

$\log_{10}|H(f)|$ vs $\log_{10} f$

FILTRO del 1° ORDINE



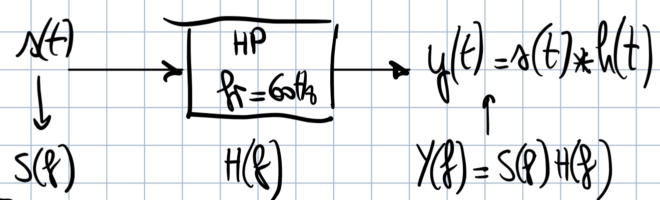
FAMIGLIE di FILTRI REALI :

- filtri di BUTTERWORTH
- filtri di CHEBYSHEV
- filtri di BESSEL

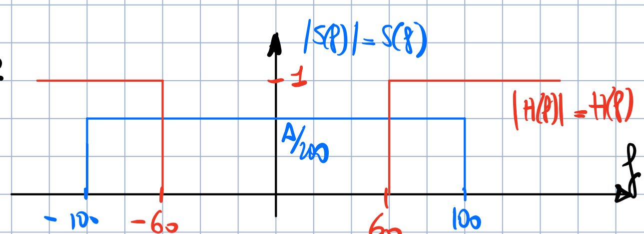
ESERCIZIO di RIEPILOGO :

Dato un filtro PASSA-ALTO IDEALE con $f_T = 60$ Hz in cui entra il segnale $s(t) = A\,\mathrm{sinc}(200t)$, calcolare il segnale in uscita dal filtro.

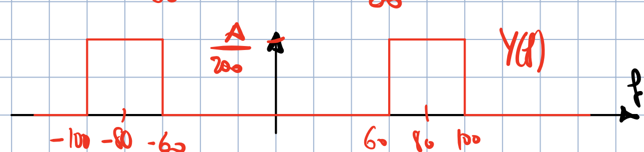
$$H(f) = \begin{cases} 1 & |f| > 60 \\ 0 & \text{altrove} \end{cases} = 1 - \mathrm{rect}\left(\frac{f}{120}\right)$$



Calcolo $S(f)$: $S(f) = A\frac{1}{200}\mathrm{rect}\left(\frac{f}{200}\right) = \begin{cases} \frac{A}{200} & |f| < 100 \\ 0 & \text{altrove} \end{cases}$



$\rightarrow Y(f) = S(f) \cdot H(f) = \frac{A}{200}\left[\mathrm{rect}\left(\frac{f-80}{40}\right) + \mathrm{rect}\left(\frac{f+80}{40}\right)\right]$



Antitrasformo $Y(f)$:

$$\mathrm{rect}(f) \longleftrightarrow \mathrm{sinc}(t)$$

scalatura : $\mathrm{rect}\left(\frac{f}{40}\right) \longleftrightarrow 40\,\mathrm{sinc}(40t)$

Traslazione: $\mathrm{rect}\left(\frac{f\pm 80}{40}\right) \longleftrightarrow 40\,\mathrm{sinc}(40t)\,e^{\pm j2\pi 80t}$

$$\rightarrow y(t) = \frac{A}{200}\left[40\,\mathrm{sinc}(40t)\,e^{-j160\pi t} + 40\,\mathrm{sinc}(40t)\,e^{j160\pi t}\right] =$$

$$= \frac{A}{5}\,\mathrm{sinc}(40t)\underbrace{\left(e^{-j160\pi t} + e^{j160\pi t}\right)}_{2\cos(160\pi t)} = \frac{2}{5}A\,\mathrm{sinc}(40t)\cos(160\pi t)$$

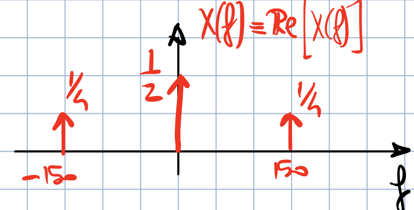
ES 2 Tema marzo 2024

Calcolare la $H(f)$ di un filtro IDEALE PASSA-ALTO con $f_T = 100$ Hz, $G = 8$ e RITARDO $\tau = 0{,}5$ s.

$$\rightarrow H(f) = \begin{cases} 8\,e^{-j2\pi f \cdot 0,5} & |f| > 100 \\ 0 & \text{altrove} \end{cases} = 8\left[1 - \mathrm{rect}\left(\frac{f}{200}\right)\right]e^{-j\pi f}$$

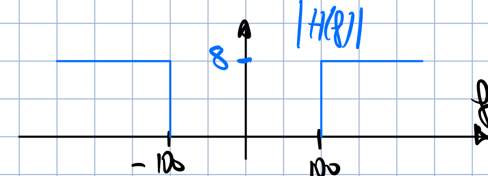
... e calcolare l'uscita del filtro quando l'ingresso sia:

$$x(t) = \cos^2(150\pi t) = \cos(150\pi t)\cdot\cos(150\pi t)$$



$$X(f) = \frac{1}{2}\left[\delta(f-75)+\delta(f+75)\right] * \frac{1}{2}\left[\delta(f-75)+\delta(f+75)\right] =$$

$$= \frac{1}{4}\left[\delta(f-150) + \underbrace{\delta(f) + \delta(f)}_{2\delta(f)} + \delta(f+150)\right] = \frac{1}{4}\left[\delta(f-150)+\delta(f+150)\right] + \cancel{\frac{\delta(f)}{2}}$$



$$\rightarrow Y(f) = X(f)\cdot H(f) = \frac{1}{4}\left[\delta(f-150)+\delta(f+150)\right]8\,e^{-j\pi f} =$$

$$= 2\left[\delta(f-150)e^{-j\pi f} + \delta(f+150)e^{-j\pi f}\right] = 2\left[\delta(f-150)\cancelto{1}{e^{-j\pi 150}} + \delta(f+150)\cancelto{1}{e^{j\pi 150}}\right] =$$

$$= 2\left(\delta(f-150)+\delta(f+150)\right)$$

$$\xrightarrow{\mathcal{F}^{-1}} \boxed{y(t) = 4\cos(300\pi t)}$$