SEGNALI nel DOMINIO della FREQUENZA

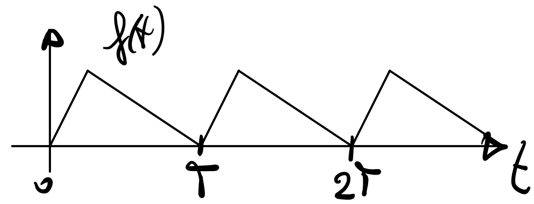
SVILUPPO in SERIE di FOURIER

[SERIE : $\sum_{0}^{\infty} {}_n f_n(x)$ (serie di POTENZE : $\sum_{0}^{\infty} {}_n \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$]

FOURIER (1822) :

Data una funzione PERIODICA di PERIODO T:

$$f: \ f(t) = f(t+T), \ \forall t, T \in \mathbb{R}$$



se $f(t)$ rispetta opportune condizioni di "REGOLARITÀ" (condizioni di DIRICHLET)

ALLORA :

$f(t)$ è ESPRIMIBILE come SOMMA di SINUSOIDI con FREQUENZE MULTIPLE di $\frac{1}{T} = f_1$ della FREQUENZA FONDAMENTALE

$$f_n = \begin{cases} n = 0 \longrightarrow f = 0 \ (\text{COSTANTE}) \\ n = 1 \longrightarrow f = \frac{1}{T} \\ n > 1 \longrightarrow f_n = \frac{n}{T} \end{cases}$$

COMPONENTE CONTINUA
FREQ. FONDAMENTALE
n-ESIMA ARMONICA

In FORMA ESPONENZIALE : (per $f(t) \in \mathbb{C}$)

$$f(t) = \sum_{-\infty}^{+\infty} {}_n c_n \underbrace{e^{j2\pi \frac{n}{T} t}}_{\text{FASORE}}$$

$$\left\{ e^{j2\pi \frac{n}{T} t} = \cos\left(2\pi \frac{n}{T} t\right) + j \sin\left(2\pi \frac{n}{T} t\right) \right\}$$

In FORMA TRIGONOMETRICA : (per : $f(t) \in \mathbb{R}$)

$$f(t) = c_0 + 2 \sum_{1}^{\infty} {}_n \left[a_n \cos\left(2\pi \frac{n}{T} t\right) + b_n \sin\left(2\pi \frac{n}{T} t\right) \right] =$$

$$= c_0 + 2\sum_{1}^{\infty}{}_n\, \rho_n \cos\left(2\pi \frac{n}{T} t + \vartheta_n\right)$$

EQUAZIONI di SINTESI: da $\{c_n\}$ a $f(t)$

dove: $$c_n = \frac{1}{T}\int_T f(t)\, e^{-j2\pi \frac{n}{T} t}\, dt$$

→ EQUAZIONE di ANALISI: da $f(t) \to$ a $\{c_n\}$

Dalla forma ESPONENZIALE alla TRIGONOMETRICA:

$$f(t) = \sum_{-\infty}^{+\infty}{}_n\, c_n e^{j2\pi \frac{n}{T} t} = c_0 + \sum_{1}^{\infty}{}_n \left[c_n e^{j2\pi \frac{n}{T} t} + c_{-n} e^{-j2\pi \frac{n}{T} t} \right] =$$

Considero c_{-n}:

$$c_{-n} = \frac{1}{T}\int_T f(t)\, e^{j2\pi \frac{n}{T} t}\, dt = (f(t) \in \mathbb{R}) = \frac{1}{T}\int_T \overline{f(t)\, e^{-j2\pi \frac{n}{T} t}}\, dt = \overline{c_n}$$

$$f(t) = c_0 + \sum_{1}^{\infty}{}_n \left[\underbrace{c_n e^{j2\pi \frac{n}{T} t}} + \underbrace{\overline{c_n}\, e^{-j2\pi \frac{n}{T} t}}_{\overline{e^{j2\pi \frac{n}{T} t}}} \right] = c_0 + 2\sum_{1}^{\infty}{}_n \operatorname{Re}\left[c_n e^{j2\pi \frac{n}{T} t} \right] =$$

Definisco: $c_n = \rho_n e^{j\vartheta_n}$ $\qquad \{ \rho_n = |c_n| \,;\ \vartheta_n = \angle c_n \}$

$$f(t) = c_0 + 2\sum_{1}^{\infty}{}_n \operatorname{Re}\left[\rho_n e^{j\vartheta_n} \cdot e^{j2\pi \frac{n}{T} t} \right] = c_0 + 2\sum_{1}^{\infty}{}_n \operatorname{Re}\left[\rho_n e^{j\left(2\pi \frac{n}{T} t + \vartheta_n\right)} \right]$$

$$f(t) = c_0 + 2\sum_{1}^{\infty}{}_n\, \rho_n \cos\left(2\pi \frac{n}{T} t + \vartheta_n\right)$$

I FORMA TRIGONOMETRICA

$$\rho_n = |c_n| \,;\quad \vartheta_n = \angle c_n$$

DEFINISCO: $\underbrace{c_n = a_n - j b_n}$ $\to \left(a_n = \operatorname{Re}[c_n] \,;\ b_n = -\operatorname{Im}[c_n] \right)$

$$f(t) = c_0 + 2\sum_{n=1}^{\infty} \mathrm{Re}\left[(a_n - j b_n)\, e^{j2\pi\frac{n}{T}t}\right] =$$

$j = e^{j\frac{\pi}{2}}$

$$= c_0 + 2\sum_{n=1}^{\infty} \mathrm{Re}\left[a_n e^{j2\pi\frac{n}{T}t} - j b_n e^{j2\pi\frac{n}{T}t}\right] =$$

$e^{j2\pi\frac{n}{T}t} \cdot e^{j\frac{\pi}{2}} = e^{j\left(2\pi\frac{n}{T}t + \frac{\pi}{2}\right)}$

$$= c_0 + 2\sum_{n=1}^{\infty} \left[a_n \cos\left(2\pi\frac{n}{T}t\right) - b_n \cos\left(2\pi\frac{n}{T}t + \frac{\pi}{2}\right)\right] =$$

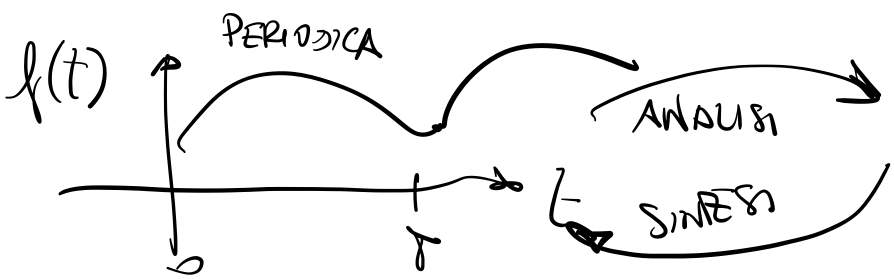
$\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin(\alpha)$

$$f(t) = c_0 + 2\sum_{n=1}^{\infty} \left[a_n \cos\left(2\pi\frac{n}{T}t\right) + b_n \sin\left(2\pi\frac{n}{T}t\right)\right]$$

II FORMA TRIGONOMETRICA

dove:
$$\begin{cases} a_n = \mathrm{Re}[c_n] = \mathrm{Re}\left[\frac{1}{T}\int_T f(t)\, e^{-j2\pi\frac{n}{T}t}\, dt\right] = \frac{1}{T}\int_T f(t)\cos\left(2\pi\frac{n}{T}t\right) dt \\ b_n = -\mathrm{Im}[c_n] = -\mathrm{Im}\left[\quad \text{''} \quad\right] = \frac{1}{T}\int_T f(t)\sin\left(2\pi\frac{n}{T}t\right) dt \end{cases}$$

EQQ. di ANALISI in forma TRIGONOMETRICA

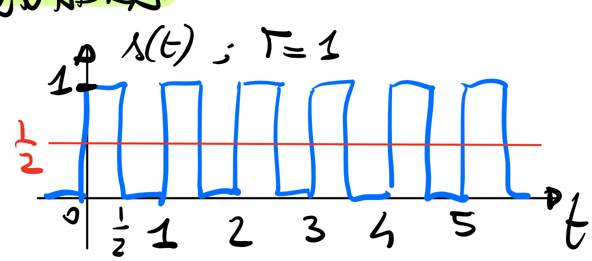


$\{c_n\}$, $n \in \mathbb{Z}$
$\Downarrow$
$\{a_n, b_n\}$ $n \in \mathbb{N}$
$\{\rho_n, \vartheta_n\}$ $n \in \mathbb{N}$

$$\boxed{c_n = a_n - j b_n = \rho_n e^{j\vartheta_n}}$$

ESEMPIO: calcolo SF dell'ONDA QUADRA

$$s(t) = \begin{cases} 1 & k \le t \le k+\frac{1}{2} \\ 0 & k+\frac{1}{2} < t < k+1 \end{cases} \quad k \in \mathbb{Z}$$



$$c_0 = \frac{1}{T}\int_T s(t)\, dt = \int_0^1 s(t)\, dt = \int_0^{1/2} 1 \cdot dt + \cancel{\int_{1/2}^{1} 0\, dt} = [t]_0^{1/2} = \frac{1}{2} - 0 = \frac{1}{2}$$

Per $n \neq 0$:

$$c_n = \frac{1}{T}\int_T x(t)\, e^{-j2\pi \frac{n}{T} t}\, dt = \int_0^1 x(t)\, e^{-j2\pi n t}\, dt = \int_0^{1/2} e^{-j2\pi n t}\, dt = \qquad \left(\int e^{at}\,dt = \frac{1}{a} e^{at}\right)$$

$$= \frac{-1}{j2\pi n}\left[e^{-j2\pi n t}\right]_0^{1/2} = \frac{j}{2\pi n}\left[\underbrace{e^{-j\pi n}}_{(-1)^n} - \underbrace{e^{0}}_{1}\right] = \frac{j}{2\pi n}\left[(-1)^n - 1\right] =$$

$$= \begin{cases} -\dfrac{j}{\pi n}, & n \text{ DISPARI} \\ 0, & n \text{ PARI} \end{cases}$$

Quindi: $$x(t) = \frac{1}{2} + 2\sum_{\substack{1 \\ (n\ \text{DISPARI})}}^{\infty} \frac{-j}{\pi n}\, e^{j2\pi n t} = \frac{1}{2} + \sum_{\cancel{1}\,0}^{\infty} \frac{-2j}{\pi(2n+1)}\, e^{j2\pi(2n+1)t}$$

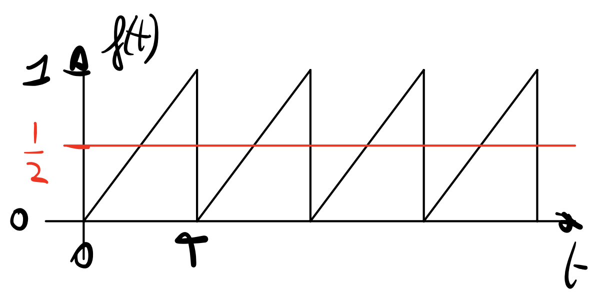
In FORMA TRIGONOMETRICA (II):

$$a_n = Re[c_n] = 0,\ \forall n \ ; \quad b_n = -Im(c_n) = \begin{cases} \dfrac{1}{\pi n} & n \text{ DISPARI} \\ 0 & n \text{ PARI} \end{cases}$$

Quindi: $$x(t) = \frac{1}{2} + 2\sum_{\substack{1 \\ (n\ \text{DISPARI})}}^{\infty} b_n \sin(2\pi n t) = \frac{1}{2} + \sum_{\substack{1 \\ (n\ \text{DISPARI})}}^{\infty} \frac{2}{\pi n} \sin(2\pi n t) =$$

$$= \frac{1}{2} + \sum_{0}^{\infty} \frac{2}{\pi(2n+1)} \sin\left(2\pi(2n+1)t\right)$$

FS del "DENTE di SEGA"



$$f(t) = \frac{t}{T}, \quad 0 \le t < T$$

$$f(t+T) = f(t), \quad \forall t \in \mathbb{R}$$

$$c_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{T}\int_0^T \frac{t}{T}\,dt = \frac{1}{T^2}\int_0^T t\,dt = \frac{1}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{1}{\cancel{T^2}}\left(\frac{\cancel{T^2}}{2} - 0\right) = \frac{1}{2}$$

$$c_n = \frac{1}{T}\int_T f(t)\, e^{-j2\pi \frac{n}{T} t}\,dt = \frac{1}{T}\int_0^T \frac{t}{T}\, e^{-j2\pi \frac{n}{T} t}\,dt = \frac{1}{T^2}\int_0^T t \cdot e^{-j2\pi \frac{n}{T} t}\,dt =$$

$\to g'(t)$

$f(t) \to f'(t) = 1$

[Integr. per PARTI: $\int f(t)\,g'(t)\,dt = [f(t)g(t)] - \int f'(t)g(t)\,dt$]

$$c_n = \frac{1}{T^2}\left\{\left[t\cdot\frac{-T}{j2\pi n}e^{-j2\pi\frac{n}{T}t}\right]_0^T - \int_0^T \cancel{1}\cdot\underline{\frac{-T}{j2\pi n}}e^{-j2\pi\frac{n}{T}t}\,dt\right\} =$$

$$= \frac{1}{T^2}\left\{\left[\frac{-T^2}{j2\pi n}e^{\overset{1}{-j2\pi n}} - 0\right] - \left(\frac{-T}{j2\pi n}\right)^2\left[e^{-j2\pi\frac{n}{T}t}\right]_0^T\right\} =$$

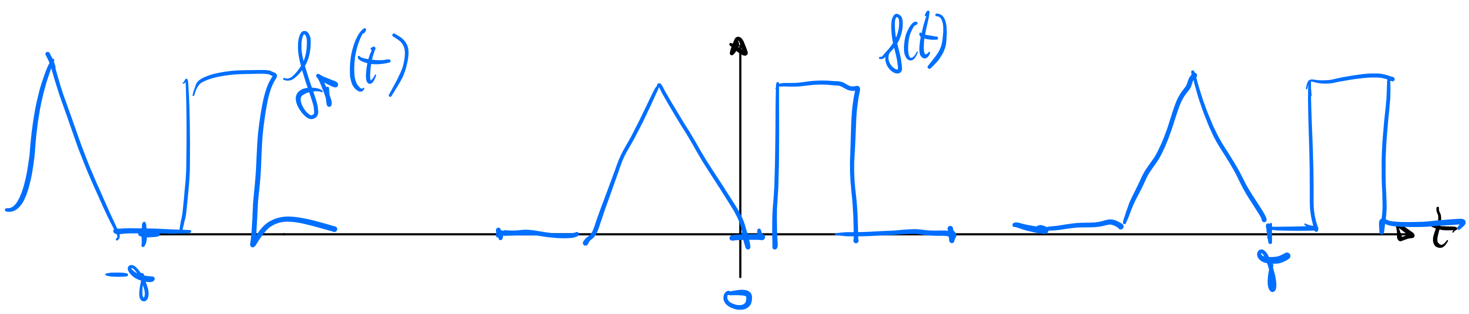
$$= \underbrace{\frac{1}{\cancel{T^2}}\left\{\frac{j\cancel{T^2}}{2\pi n}\right.}\cancel{-\left(\frac{-T}{j2\pi n}\right)^2\left[\underset{1}{e^{-j2\pi n}} - \underset{1}{e^0}\right]}\Big\}_{0} = \frac{j}{2\pi n}, \quad n\neq 0$$

$c_0 = \frac{1}{2}$ $\qquad c_n = \frac{j}{2\pi n},\ n\neq 0$

Quindi:
$$\boxed{f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{j}{2\pi n}e^{j2\pi\frac{n}{T}t}}$$

$$\boxed{\begin{aligned} &a_n = \mathrm{Re}[c_n] = 0 \quad ; \quad b_n = -\mathrm{Im}[c_n] = \frac{-1}{2\pi n} \\ &f(t) = \frac{1}{2} + \cancel{2}\sum_{n=1}^{\infty}\frac{-1}{\cancel{2}\pi n}\sin\left(2\pi\frac{n}{T}t\right)\end{aligned}}$$

ESISTE un ANALOGO per FUNZIONI NON PERIODICHE ?



$$\boxed{\text{Data } f(t) \longrightarrow f_T(t) = f(t) \text{ per } -\frac{T}{2}\leq t\leq\frac{T}{2},\ f_T(t+kT) = f_T(t)}$$

Dato che $f_T(t)$ è PERIODICA: posso usare SF:

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi\frac{n}{T}t} = \left\{\text{def: } \frac{n}{T} = f_n\right\} = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_n t}$$

$$f(t) = \lim_{T\to\infty} f_T(t)$$

$$f(t) = \lim_{T\to\infty} f_T(t) = \lim_{T\to\infty} \sum_{-\infty}^{\infty} c_n e^{j2\pi f_n t} = (\text{sostituisco } c_n):$$

$$\left(\text{dove } c_n : \; c_n = \frac{1}{T}\int_{-T/2}^{+T/2} f_T(t) e^{-j2\pi f_n t}\, dt = \left(\Delta f = \frac{1}{T}\right) = \underbrace{\Delta f \cdot \int_{-T/2}^{+T/2} f_T(t) e^{-j2\pi f_n t}\, dt} \right)$$

$$f(t) = \lim_{T\to\infty} \sum_{-\infty}^{\infty} \left[\Delta f \cdot \int_{-T/2}^{+T/2} f_T(\tau) e^{-j2\pi f_n \tau} d\tau \right] e^{j2\pi f_n t} = \left(\begin{matrix} T \to \infty \\ \Delta f \to 0 \end{matrix} \right)$$

$$= \lim_{T\to\infty} \sum_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{+\infty} f(\tau) e^{-j2\pi f_n \tau} d\tau}_{F(f_n)} \cdot e^{j2\pi f_n t} \cdot \Delta f =$$

Definisco: $F(f) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi f t} \cdot dt$

$$f(t) = \lim_{\substack{T\to\infty \\ \Delta f \to 0}} \left[\sum_{-\infty}^{+\infty} F(f_n) e^{j2\pi f_n t} \cdot \Delta f \right] = \int_{-\infty}^{+\infty} F(f) e^{j2\pi f t} \cdot df$$

In SINTESI:

$$F(f) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi f t} \cdot dt \qquad (\text{da } f(t) \longrightarrow F(f))$$

SPETTRO di $f(t)$

TRASFORMATA di FOURIER di $f(t)$ [ANALISI]

$$F(f) = \mathcal{F}\{f(t)\} \; ; \quad f(t) \xrightarrow{\mathcal{F}} F(f)$$

$$f(t) = \int_{-\infty}^{+\infty} F(f) e^{j2\pi f t} \cdot df \qquad [\text{da}: F(f) \to \text{a } f(t)$$

ANTI-TRASFORMATA di FOURIER di $F(f)$ [SINTESI]

$$f(t) = \mathcal{F}^{-1}\{F(f)\} \; ; \quad F(f) \xrightarrow{\mathcal{F}^{-1}} f(t)$$

$F(f)$ è lo SPETTRO di $f(t)$ è una funzione COMPLESSA:

$$f(t) \in \mathbb{R} \quad ; \quad F(f) \in \mathbb{C}$$

$|F(f)|$: SPETTRO di AMPIEZZA di $f(t)$

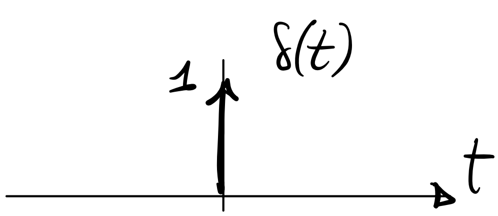
$\measuredangle F(f)$: SPETTRO di FASE di $f(t)$

ESEMPI NOTEVOLI di TRASFORMATA

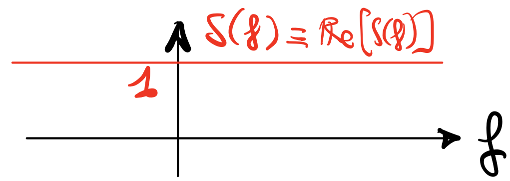
DELTA di DIRAC : $\delta(t)$

$$s(t) = \delta(t) \longrightarrow S(f) = \mathcal{F}\{s(t)\} = \int_{-\infty}^{+\infty} s(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} \delta(t)\, e^{-j2\pi f t}\, dt = e^{-j2\pi f 0} = e^{0} = 1$$



$\longrightarrow$



$$\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 1 \qquad \text{FOURIER PAIR}$$

DUALITÀ

$$s(t) = \delta(t-t_0) \longrightarrow S(f) = \int_{-\infty}^{+\infty} \delta(t-t_0)\, e^{-j2\pi f t}\, dt = e^{-j2\pi f t_0}$$

$$\delta(t-t_0) \longleftrightarrow e^{-j2\pi f t_0}$$

Dato: $S(f) = \delta(f) \longrightarrow s(t) = \mathcal{F}^{-1}\{S(f)\} = \int_{-\infty}^{\infty} \delta(f) \cdot e^{j2\pi f t}\, df = 1$

$$1 \longleftrightarrow \delta(f)$$

$$S(f) = \delta(f-f_0) \longrightarrow s(t) = \int_{-\infty}^{\infty} \delta(f-f_0)\, e^{j2\pi f t}\, df = e^{j2\pi f_0 t}$$

$$e^{j2\pi f_0 t} \longleftrightarrow \delta(f-f_0)$$

RETTANGOLO : $\mathrm{rect}(t)$

$$s(t) = \mathrm{rect}(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{altrove} \end{cases}$$

$$S(f) = \int_{-\infty}^{\infty} \mathrm{rect}(t)\, e^{-j2\pi f t}\, dt = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cancel{1} \cdot e^{-j2\pi f t}\, dt = \left[\frac{-1}{j2\pi f} e^{-j2\pi f t} \right]_{-\frac{1}{2}}^{+\frac{1}{2}} =$$

$$= \frac{-1}{j2\pi f} \left[e^{-j\pi f} - e^{+j\pi f} \right] = \quad \left[\text{Eulero}: e^{j\vartheta} = \cos\vartheta + j\sin\vartheta \right]$$

$$= \frac{-1}{j2\pi f} \left[\cancel{\cos(\pi f)} + j\sin(-\pi f) - \left(\cancel{\cos(\pi f)} + j\sin(\pi f) \right) \right] =$$

$$= \frac{-1}{j2\pi f} \left(-j\sin(\pi f) - j\sin(\pi f) \right) = \frac{1}{\cancel{j2}\pi f} \cancel{2j} \sin(\pi f) = \frac{\sin(\pi f)}{\pi f}$$

$$\boxed{s(t) = \mathrm{rect}(t) \longrightarrow S(f) = \frac{\sin \pi f}{\pi f} = \mathrm{sinc}(f)}$$

FOURIER PAIR

TRASFORMATA di FOURIER di SEGNALI REALI

Ipotesi: $s(t) \in \mathbb{R}, \ \forall t \in \mathbb{R}$

$$s(t) \xrightarrow{\mathcal{F}} S(f) = \int_{-\infty}^{+\infty} s(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} s(t) \left[\cos(2\pi f t) - j\sin(2\pi f t) \right] dt =$$

EULERO: $e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$

$$S(f) = \underbrace{\int_{-\infty}^{+\infty} \underbrace{s(t)\cos(2\pi f t)}_{\in \mathbb{R}}\, dt}_{Re[S(f)]} - j \underbrace{\int_{-\infty}^{\infty} \underbrace{s(t)\sin(2\pi f t)}_{\in \mathbb{R}}\, dt}_{Im[S(f)]} = \mathrm{Re}[S(f)] - j\,\mathrm{Im}[S(f)]$$

$$\mathrm{Re}[S(f)] = \int_{-\infty}^{+\infty} s(t)\cos(2\pi f t)\, dt = \int_{-\infty}^{\infty} s(t)\cos(2\pi(-f)t)\, dt = \mathrm{Re}[S(-f)]$$

ha SIMMETRIA PARI: $\mathrm{Re}[S(f)] = \mathrm{Re}[S(-f)]$

$$\mathrm{Im}[S(f)] = \int_{-\infty}^{\infty} s(t)\sin(2\pi f t)\, dt = -\int_{-\infty}^{\infty} s(t)\sin(2\pi(-f)t)\, dt = -\mathrm{Im}[S(-f)]$$

ha SIMMETRIA DISPARI: $\mathrm{Im}[S(f)] = -\mathrm{Im}[S(-f)]$

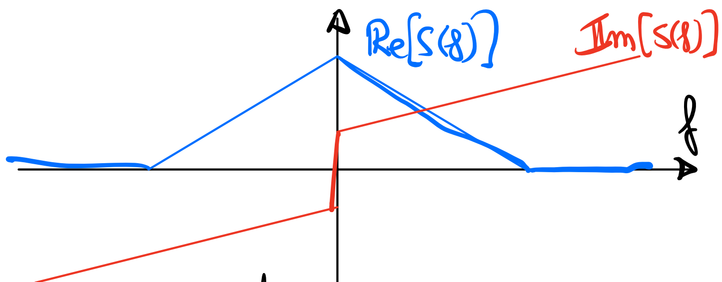
MODULO: $|S(f)| = |S(-f)|$ → SIMMETRIA PARI

FASE: $\measuredangle S(f) = -\measuredangle S(-f)$ → SIMMETRIA DISPARI

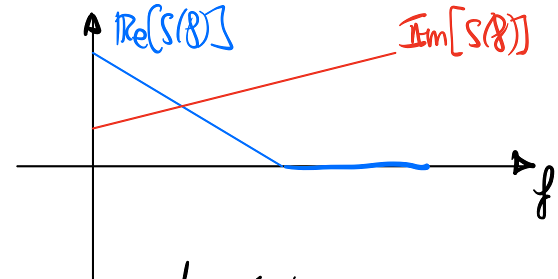
- MODULO dello SPETTRO : SPETTRO di AMPIEZZA
- FASE dello SPETTRO : SPETTRO di FASE

Se $s(t)$ è REALE: $\boxed{s(t) \in \mathbb{R},\ \forall t \longrightarrow S(f) = \overline{S(-f)}}$

SIMMETRIA "HERMITIANA"



TRASF. di FOURIER "BILATERA"



TR. di FOURIER "MONOLATERA"

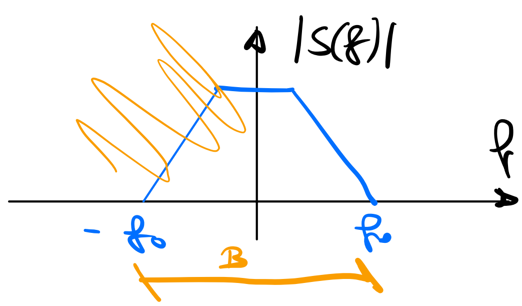
BANDA di un SEGNALE

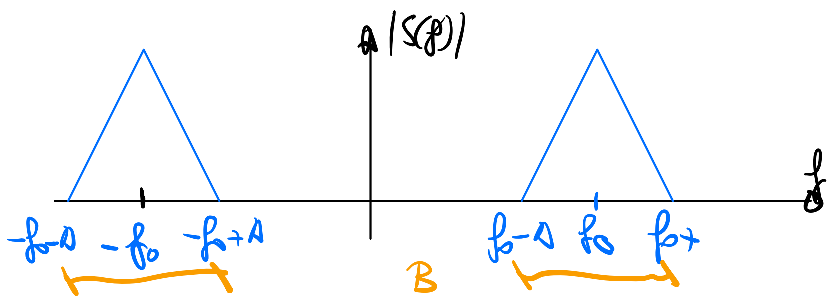
BANDA: il SUPPORTO dello SPETTRO di un SEGNALE

Dato $s(t) \xrightarrow{\mathcal{F}} S(f)$; BANDA di $s(t)$ $:= B = \{ f \in \mathbb{R} : S(f) \neq 0 \}$

LARGHEZZA di BANDA (BANDWIDTH): ESTENSIONE della Banda: B

- ✱ quando B CONTIENE: $f = 0$ → $s(t)$ è un SEGNALE in BANDA BASE
- ✱ " B NON CONTIENE: $f = 0$ → $s(t)$ è un segnale IN BANDA PASSANTE





La LARGHEZZA di BANDA viene DEFINITA sulla RAPPRESENTAZIONE MONOLATERA dello SPETTRO

PROPRIETÀ della TRASFORMATA di FOURIER

LINEARITÀ

Se $x(t) \xrightarrow{\mathcal{F}} X(f)$, $y(t) \xrightarrow{\mathcal{F}} Y(f)$ $\Longrightarrow$ $ax(t)+by(t) \xrightarrow{\mathcal{F}} aX(f)+bY(f)$

DIM: $\mathcal{F}\{ax(t)+by(t)\} = \int_{-\infty}^{+\infty} (ax(t)+by(t))\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} a x(t) e^{-j2\pi ft} dt + \int_{-\infty}^{\infty} b y(t) e^{-j2\pi ft} dt =$

$$= a \underbrace{\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt}_{=X(f)} + b \underbrace{\int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt}_{=Y(f)} = aX(f) + bY(f) \quad \text{c.v.d.}$$

→ La Trasf. di FOURIER È LINEARE.

DUALITÀ (o SIMMETRIA)

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \Longrightarrow S(t) \xleftrightarrow{\mathcal{F}} s(-f)$$

DIM: $\mathcal{F}\{S(t)\} = \int_{-\infty}^{\infty} S(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} S(t) e^{j2\pi(-f)t} dt = s(-f)$ CVD

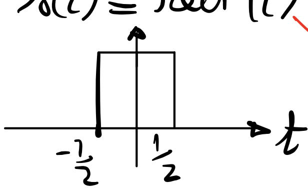
Per $s(t)$ con SIMMETRIA PARI : $s(t) = s(-t)$:

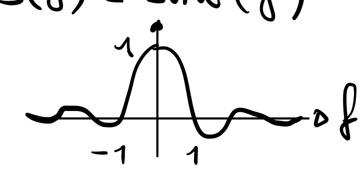
Se $s(t) = s(-t),\ \forall t \in \mathbb{R} \longrightarrow S(t) \xleftrightarrow{\mathcal{F}} s(-f) = s(f)$

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \Longrightarrow S(t) \xleftrightarrow{\mathcal{F}} s(f)$$

ESEMPIO : $s(t) = \text{rect}(t) \xleftrightarrow{\mathcal{F}} S(f) = \text{sinc}(f)$

PARI!





$\longrightarrow$ Per DUALITÀ $\longrightarrow$ $s(t) = \text{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) = \text{rect}(f)$

TRASLAZIONE nei TEMPI

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \longrightarrow s(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} S(f)\, e^{-j2\pi f t_0}$$

DIM : $s(t-t_0) \xrightarrow{\mathcal{F}} S'(f) = \int_{-\infty}^{\infty} s(t-t_0)\, e^{-j2\pi f t}\, dt = \begin{Bmatrix} t' = t - t_0 \\ dt' = dt \end{Bmatrix} =$

$$= \int_{-\infty}^{\infty} s(t')\, e^{-j2\pi f (t'+t_0)}\, dt' = \int_{-\infty}^{\infty} s(t')\, e^{-j2\pi f t'} \cdot \underbrace{e^{-j2\pi f t_0}}_{\text{COSTANTE}} \cdot dt' =$$

$$= e^{-j2\pi f t_0} \underbrace{\int_{-\infty}^{\infty} s(t')\, e^{-j2\pi f t'}\, dt'}_{S(f)} = S(f) \cdot e^{-j2\pi f t_0} \qquad \text{C.V.D.}$$

TRASLAZIONE in FREQUENZA (per DUALITÀ):

$$s(t) \longleftrightarrow S(f) \longrightarrow s(t-t_0) \longleftrightarrow S(f)\, e^{-j2\pi f t_0}$$

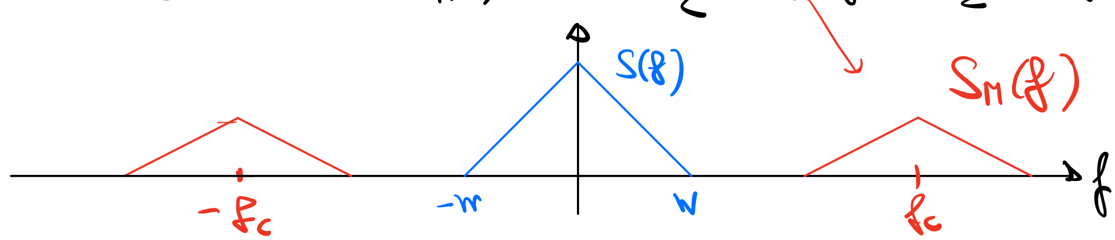
DUALITÀ: $\Longrightarrow$ $s(t)\, e^{-j2\pi f_0 t} \longleftrightarrow S(f+f_0)$

MODULAZIONE di $s(t)$: $s_M(t) = s(t) \cos(2\pi f_c t)$

f_c ← freq. PORTANTE

$$s_M(t) = s(t)\, \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2} = \frac{1}{2}\left[s(t)\, e^{j2\pi f_c t} + s(t)\, e^{-j2\pi f_c t} \right] =$$

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \longrightarrow s_M(t) \xrightarrow{\mathcal{F}} \frac{1}{2} S(f-f_c) + \frac{1}{2} S(f+f_c) = S_M(f)$$



SCALATURA

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \longrightarrow s(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} S\left(\frac{f}{a}\right)$$

DIM: 1) $a > 0$;

$$\mathcal{F}\{s(at)\} = \int_{-\infty}^{\infty} s(at)\, e^{-j2\pi f t}\, dt = \left\{\begin{matrix}\tau = at \\ d\tau = a\, dt\end{matrix}\right\} = \int_{-\infty}^{\infty} s(\tau)\, e^{-j2\pi f \frac{\tau}{a}}\, \frac{d\tau}{a} =$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} s(\tau)\, e^{-j2\pi \frac{f}{a}\tau}\, d\tau = \frac{1}{a}\, S\left(\frac{f}{a}\right) \quad ✓$$

2): $a < 0$, $\mathcal{F}\{s(at)\} = \left\{\begin{matrix}\tau = at \\ d\tau = a\, dt\end{matrix}\right\} = \int_{+\infty}^{-\infty} s(\tau)\, e^{-j2\pi f \frac{\tau}{a}}\, \frac{d\tau}{a} = \int_{-\infty}^{+\infty} s(\tau)\, e^{-j2\pi \frac{f}{a}\tau}\, \frac{d\tau}{-a} =$

$$= \frac{1}{-a}\int_{-\infty}^{\infty} s(\tau)\, e^{-j2\pi \frac{f}{a}\tau}\, d\tau = \frac{1}{-a}\, S\left(\frac{f}{a}\right)$$

Di conseguenza: $\quad s(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}\, S\left(\frac{f}{a}\right) \quad$ C.V.D.

DERIVAZIONE

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow \frac{d}{dt}s(t) \xleftrightarrow{\mathcal{F}} S(f)\cdot j2\pi f$$

DIM: $\frac{d}{dt}s(t) = \frac{d}{dt}\left(\int_{-\infty}^{\infty} \underbrace{S(f)\, e^{j2\pi f t}}_{f(t)}\, df\right) =$ DERIVO. int sotto l'INTEGRALE $= \int_{-\infty}^{+\infty} S(f)\, \frac{d}{dt}\left(e^{j2\pi f t}\right) df =$

$$= \int_{-\infty}^{\infty} S(f)\, (j2\pi f)\, e^{j2\pi f t}\, df = j2\pi f \underbrace{\int_{-\infty}^{\infty} S(f)\, e^{j2\pi f t}\, df}_{s(t)}$$

CONVOLUZIONE

$$\begin{matrix} x(t) \xleftrightarrow{\mathcal{F}} X(f) \\ y(t) \xleftrightarrow{\mathcal{F}} Y(f) \end{matrix} \longrightarrow x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(f)\cdot Y(f)$$

$t-\tau, \tau$

$$\mathcal{F}\{x(t) * y(t)\} = \mathcal{F}\left\{\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\right\} = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\right) e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, e^{-j2\pi f (t-\tau)}\cdot e^{-j2\pi f \tau}\, dt\, d\tau =$$

$$= \int_{-\infty}^{\infty} \overbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f \tau}\, d\tau}^{X(f)} \cdot y(t-\tau)\, e^{-j2\pi f (t-\tau)}\, dt =$$

$$= \int_{-\infty}^{\infty} X(f)\, y(t-\tau)\, e^{-j2\pi f(t-\tau)}\, dt = X(f) \int_{-\infty}^{\infty} y(t-\tau)\, e^{-j2\pi f(t-\tau)}\, dt = \begin{cases} t' = t-\tau \\ dt' = dt \end{cases}$$

$$= X(f) \underbrace{\int_{-\infty}^{\infty} y(t')\, e^{-j2\pi f t'}\, dt'}_{Y(f)} = X(f)\cdot Y(f) \quad \text{c.v.d}$$

$$\boxed{\begin{array}{ll} & x(t) * y(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)\cdot Y(f) \\ \text{per DUALITÀ:} & x(t)\cdot y(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) * Y(f) \end{array}}$$

DERIVAZIONE

$$\underbrace{s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f)}\ \longrightarrow\ s'(t) = \frac{d\,s(t)}{dt} \longleftrightarrow j2\pi f \cdot S(f)$$

$$s'(t) = \frac{d}{dt} s(t) = \frac{d}{dt} \int_{-\infty}^{+\infty} S(f)\, e^{j2\pi f t}\, df = \int_{-\infty}^{\infty} S(f)\cdot j2\pi f\, e^{j2\pi f t}\, df$$

$$\rightarrow\ s'(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f)\, j2\pi f \quad \text{c.v.d.}$$

INTEGRAZIONE (INTEGRALE INDEFINITO)

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \longrightarrow \int_{-\infty}^{t} s(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j2\pi f} S(f) + \frac{S(0)}{2}\delta(f)$$

DIM:

$$s(t) * u(t) = \int_{-\infty}^{+\infty} s(\tau)\, u(\underbrace{t-\tau}_{\geq 0})\, d\tau = \int_{-\infty}^{t} s(\tau)\cdot 1\, d\tau = \int_{-\infty}^{t} s(\tau)\, d\tau$$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \longrightarrow u(t-\tau) = \begin{cases} 1 & t-\tau \geq 0 \longrightarrow t \geq \tau \\ 0 & t-\tau < 0 \end{cases}$$

$$\longrightarrow \int_{-\infty}^{t} s(\tau)\, d\tau = s(t) * u(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f)\cdot U(f) \qquad \left[U(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \right]$$

$$\int_{-\infty}^{t} s(\tau)\, d\tau \longleftrightarrow S(f)\cdot U(f) = S(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] = \frac{1}{j2\pi f} S(f) + \frac{1}{2} S(0)\, \delta(f)$$

c.v.d

POTENZA ed ENERGIA di SEGNALI, nel DOMINIO delle FREQUENZE

Dato un segnale $s(t)$:

POTENZA ISTANTANEA : $P(t) = s(t)\cdot\overline{s(t)} = |s(t)|^2 \quad (= s^2(t),\ s(t)\in\mathbb{R})$

ENERGIA di $s(t)$: $E_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt$

POTENZA MEDIA di $s(t)$: $P_s = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |s(t)|^2 dt$

SEGNALE - ENERGIA $s(t)$: sse : $0 < E_s < \infty$ (Energia FINITA)
$\hookrightarrow$ $P_s = 0$ (Pot. MEDIA NULLA)

SEGNALE - POTENZA $s(t)$ sse : $0 < P_s < \infty$ (Pot. MEDIA FINITA)
$\hookrightarrow$ $E_s = \infty$ (ENERGIA INFINITA)

TEOREMA di PARSEVAL (per SEGNALI ENERGIA)

Sia $s(t)$ un SEGNALE ENERGIA $\rightarrow$ sia $S(f)$ il suo SPETTRO

$$\boxed{E_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt = \int_{-\infty}^{+\infty} |S(f)|^2 df}$$

"L'ENERGIA di $s(t)$ COINCIDE con l'ENERGIA di $S(f)$"

DIM: $E_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt = \int_{-\infty}^{+\infty} s(t)\cdot\overline{s(t)}\, dt = \int_{-\infty}^{+\infty} \overline{s(t)} \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df \cdot dt =$

$= \int_{-\infty}^{\infty} S(f) \left\{ \int_{-\infty}^{\infty} \overline{s(t)}\, e^{j2\pi ft} dt \right\} df = \left(\overline{a\cdot b} = \overline{a}\cdot\overline{b} \right) = \int_{-\infty}^{\infty} S(f) \overline{\underbrace{\int_{-\infty}^{\infty} s(t)\cdot e^{-j2\pi ft} dt}_{S(f)}}\, df =$

$= \int_{-\infty}^{\infty} S(f)\cdot\overline{S(f)}\, df = \int_{-\infty}^{\infty} |S(f)|^2 df$ C.V.D.

$|S(f)|^2$: DENSITÀ SPETTRALE di ENERGIA

TEOREMA di PARSEVAL per SEGNALI POTENZA :

Dato $s(t)$ SEGNALE POTENZA, considero la POTENZA MEDIA P_s :

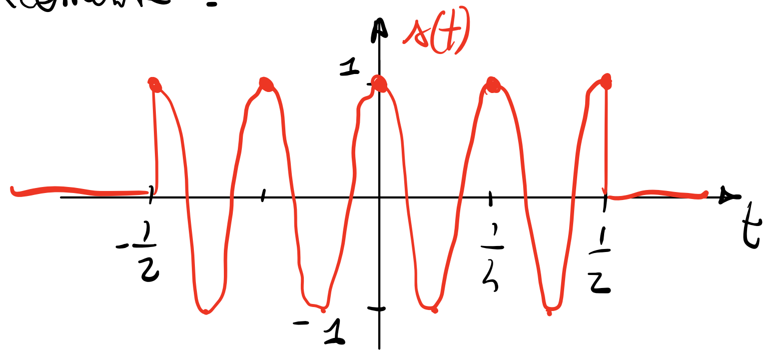
$$P_s = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |s(t)|^2 dt = \int_{-\infty}^{\infty} \underbrace{|S(f)|^2}_{\text{DENSITÀ SPETTRALE di POTENZA}} df$$

ESEMPI CALCOLO TRASFORMATA di FOURIER

ES: rappresentare $s(t) = \text{rect}(t)\cos(8\pi t)$ e calcolarne la TR. di Fourier, rappresentandola graficamente.

$2\pi f t$ $\quad f = 4 \quad T = \frac{1}{4}$

$$s(t) = \text{rect}(t)\cos(8\pi t) = \begin{cases} \cos(8\pi t) & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{altrove} \end{cases}$$



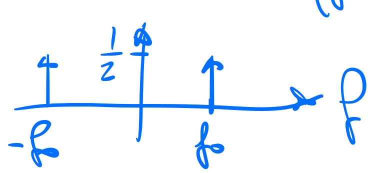
$$s(t) = \underset{\downarrow \mathcal{F}}{\text{rect}(t)} \cdot \underset{\downarrow \mathcal{F}}{\cos(8\pi t)} \xrightarrow{\mathcal{F}} S(f) = \underbrace{\text{sinc}(f) * \frac{1}{2}[\delta(f-4)+\delta(f+4)]}$$

$$\text{sinc}(f) \qquad \frac{1}{2}[\delta(f-4)+\delta(f+4)]$$

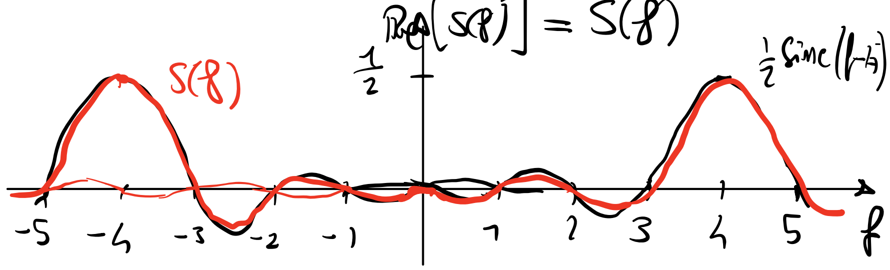
$$\cos 2\pi f_0 t = \frac{1}{2}[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}] \qquad e^{j2\pi f_0 t} \longleftrightarrow \delta(f-f_0)$$

$$\downarrow$$

$$= \frac{1}{2}[\delta(f+f_0)+\delta(f-f_0)]$$



$$S(f) = \frac{1}{2}\{\text{sinc}(f-4) + \text{sinc}(f+4)\}$$



Applicando la DEFINIZIONE:

$$s(t) = rect(t)\cos(8\pi t) \longrightarrow S(f) = \int_{-\infty}^{\infty} rect(t)\cos(8\pi t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cancel{1}\cos(8\pi t)\, e^{-j2\pi f t}\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}\left(e^{j8\pi t} + e^{-j8\pi t}\right) e^{-j2\pi f t}\, dt =$$

$$= \frac{1}{2}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t + j8\pi t}\, dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t - 8\pi t}\, dt\right] = \frac{1}{2}\left[\underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi(f-4)t}\, dt}_{sinc(f-4)} + \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi(f+4)t}\, dt}_{sinc(f+4)}\right] =$$

$$= \frac{1}{2}\left[sinc(f-4) + sinc(f+4)\right]$$

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Calcolare la FT di $s(t) = 8\, sinc^2\left(\frac{t-3}{2}\right)$ $\quad |S(f)|$; $\measuredangle S(f)$

$$s(t) = 8\, sinc\left(\frac{t-3}{2}\right) \cdot sinc\left(\frac{t-3}{2}\right)$$

$$sinc(t) \longrightarrow rect(f)$$

SCALATURA: $sinc\left(\frac{t}{2}\right) \longrightarrow 2\, rect(2f)$

TRASLAZIONE: $sinc\left(\frac{t-3}{2}\right) \longrightarrow 2\, rect(2f) \cdot e^{-j2\pi f \cdot 3} = 2\, rect(2f)\, e^{-j6\pi f}$

$$sinc^2(t) = sinc(t) \cdot sinc(t) \longrightarrow rect(f) * rect(f) = tri(f)$$

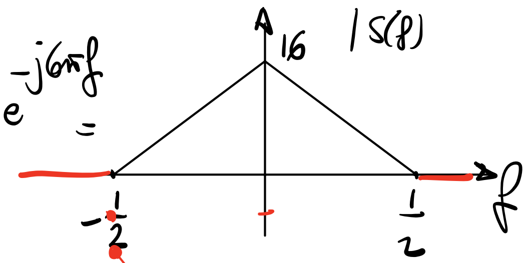
$$sinc^2\left(\frac{t}{2}\right) \longrightarrow 2\, tri(2f)$$

$$sinc^2\left(\frac{t-3}{2}\right) \longrightarrow 2\, tri(2f)\, e^{-j6\pi f}$$

$$s(t) = 8\,\text{sinc}^2\left(\frac{t-3}{2}\right) \xrightarrow{\mathcal{F}} \boxed{S(f) = 16\,\text{tri}(2f)\,e^{-j6\pi f}}$$

MODULO:

$$|S(f)| = \left|16\,\text{tri}(2f)\,e^{-j6\pi f}\right| = |16|\cdot|\text{tri}(2f)|\,\left|e^{-j6\pi f}\right| = 16\cdot\text{tri}(2f)\cdot\cancel{1} = 16\,\text{tri}(2f)$$

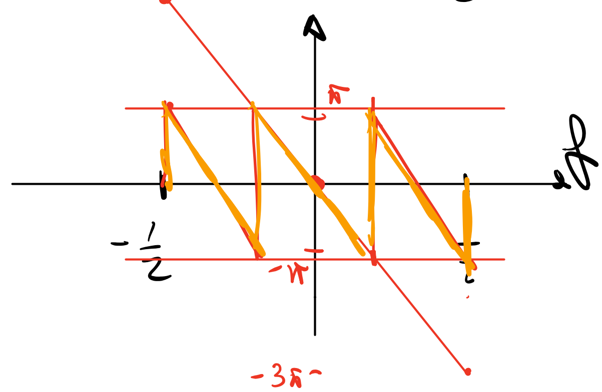


FASE:

$$\measuredangle S(f) = \measuredangle\left\{16\,\text{tri}(2f)\,e^{-j6\pi f}\right\} = \measuredangle 16 + \measuredangle\,\text{tri}(2f) + \measuredangle e^{-j6\pi f} =$$

$$= 0 + 0 \cancel{+} - 6\pi f = \qquad |f| \le \frac{1}{2}$$

$$\boxed{\measuredangle S(f) = \begin{cases} -6\pi f, & |f| \le \frac{1}{2} \\ \nexists & \text{altrove} \end{cases}}$$



Calcolare la FT di: $s(t) = \text{sinc}\left(\frac{t-4}{2}\right)$

$$s(t) = \text{sinc}(t) \longrightarrow \text{rect}(f) = S(f)$$

$$s'(t) = \text{sinc}\left(\frac{t}{2}\right) \longrightarrow S'(f) = 2\,\text{rect}(2f)$$

$$\left(a = \tfrac{1}{2}\right)$$

$$s''(t) = \text{sinc}\left(\frac{t-4}{2}\right) \longrightarrow \boxed{S''(f) = 2\,\text{rect}(2f)\cdot e^{-j2\pi f\cdot 4} = 2\,\text{rect}(2f)\,e^{-j8\pi f}}$$

$$= 2\,\text{rect}(2f)\left[\cos(8\pi f) - j\sin(8\pi f)\right]$$

$$\text{Re}[S(f)] = 2\,\text{rect}(2f)\cos(8\pi f)$$

$$\text{Im}[S(f)] = -2\,\text{rect}(2f)\sin(8\pi f)$$

$$|S(f)| = |2\,\text{rect}(2f)|\cdot\underbrace{\left|e^{-j8\pi f}\right|}_{1} = 2\,\text{rect}(2f) = \begin{cases} 2 & -\frac{1}{2} \le 2f \le \frac{1}{2} \\ 0 & \text{altrove} \end{cases} \rightarrow |f| \le \frac{1}{4}$$

$$\measuredangle S(f) = \underbrace{\measuredangle(2\,\text{rect}(2f))}_{0} + \measuredangle\left(e^{-j8\pi f}\right) = \begin{cases} -8\pi f, & \cancel{-\frac{1}{2} \le f \le \frac{1}{2}}\ |f| \le \frac{1}{4} \\ \nexists & \text{altrove} \end{cases}$$

SISTEMI LTI nel DOMINIO della FREQUENZA

Considero un sistema LTI:

$$x(t) \longrightarrow \boxed{\text{SISTEMA LTI} \quad h(t)} \longrightarrow \boxed{y(t) = x(t) * h(t)}$$

Nel DOMINIO delle FREQUENZE: $\big\Downarrow \mathcal{F}$

$$X(f) \qquad H(f) \qquad \boxed{Y(f) = X(f) \cdot H(f)}$$

$H(f)$: RISPOSTA IN FREQUENZA

ANALISI in FREQUENZA:

$$x(t) \xrightarrow{\;*\;} y(t) = x(t) * h(t)$$

$$x(t) \xrightarrow{\mathcal{F}\{\}} X(f) \xrightarrow[\times]{} Y(f) = H(f)\,X(f) \xrightarrow{\mathcal{F}^{-1}\{\}} y(t)$$

RISPOSTA in FREQUENZA $H(f)$:

$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(f) = \int_{-\infty}^{+\infty} h(t)\, e^{-j2\pi f t}\, dt$$

: TRASF. di FOURIER della RISPOSTA all'IMPULSO $h(t)$

$$x(t) \longrightarrow \boxed{\text{LTI} \quad h(t)} \longrightarrow y(t) = x(t) * h(t) \quad \text{(CONVOLUZIONE)}$$

$$X(f) \qquad H(f) \longrightarrow Y(f) = X(f) \cdot H(f) \quad \text{(PRODOTTO!)}$$

SIGNIFICATO della RISPOSTA in FREQUENZA

"Le componente sinusoidale a frequenza f dell'ingresso $X(f)$ viene MOLTIPLICATA per $H(f)$ all'uscita $Y(f)$: $\quad Y(f) = X(f) \cdot H(f)$"

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = H(f) \cdot X(f)$$

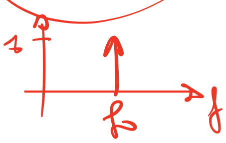
In particolare: $Y(f) = H(f) X(f)$ viene: $\begin{cases} \text{- AMPLIFICATA del fattore } |H(f)| \\ \text{- SFASATA dell'angolo } \measuredangle H(f) \end{cases}$

RISPOSTA in FREQUENZA := "RISPOSTA al FASORE"

$x_{f_0}(t) = e^{j2\pi f_0 t}$:

$$x_{f_0}(t) \longrightarrow \boxed{H(f)} \longrightarrow y_{f_0}(t)$$

$\downarrow \mathcal{F}$



$X_{f_0}(f) = \delta(f - f_0) \longrightarrow Y_{f_0}(f) = X_{f_0}(f) \cdot H(f) = H(f)\,\delta(f-f_0) = H(f_0)\,\delta(f-f_0)$

$\mathcal{F}^{-1} \Big\{ \; y_{f_0}(t) = H(f_0) \cdot e^{j2\pi f_0 t} = H(f_0) \cdot x_{f_0}(t)$

$\downarrow$

MODULO: $\quad |y_{f_0}(t)| = |x_{f_0}(t)| \cdot |H(f_0)|$

FASE: $\quad \measuredangle y_{f_0}(t) = \measuredangle x_{f_0}(t) + \measuredangle H(f_0)$

$\rightarrow \begin{cases} |H(f)| : \text{ RISPOSTA in AMPIEZZA} \\ \measuredangle H(f) : \text{ RISPOSTA in FASE} \end{cases}$

Per questo i SISTEMI LTI vengono detti: **FILTRI LINEARI**

FILTRI IDEALI

SISTEMA SENZA DISTORSIONE (DISTORTION LESS)

- Un SISTEMA LTI è detto SENZA DISTORSIONE SSE:

l'USCITA è UGUALE all'INGRESSO, A MENO DI: $\begin{cases} \text{- un FATTORE COSTANTE} \\ \text{- un RITARDO} \end{cases}$

Formalmente :

$$x(t) \longrightarrow \boxed{\begin{matrix} \text{LTI} \\ h(t)/H(f) \end{matrix}} \longrightarrow y(t)$$

senza DISTORSIONE sse : $y(t) = A\,x(t-t_0)$ $\longrightarrow$ $h(t)$?

$$y(t) = x(t) * h(t) = x(t) * \left[A\,\delta(t-t_0)\right] = \longrightarrow = A\,x(t-t_0)$$

$$\longrightarrow \boxed{h(t) = A\,\delta(t-t_0)} \xleftrightarrow{\mathcal{F}}$$

Nelle frequenze :

$$Y(f) = A\,X(f)\,e^{-j2\pi f t_0} = H(f)\,X(f) \longrightarrow \boxed{H(f) = A\,e^{-j2\pi f t_0}}$$

Osserviamo $H(f)$: $\begin{cases} |H(f)| = A & : \text{Amplificazione COSTANTE}, \forall f \\ \measuredangle H(f) = -2\pi t_0 \cdot f & : \text{PROPORZIONALE a } f \ (= \text{RITARDO ?}) \end{cases}$

$$s(t) = \cos(2\pi f t) \xrightarrow{\text{SFASAMENTO}} \Delta\varphi = k \cdot f \longrightarrow s'(t) = \cos(2\pi f t + k f) \overset{!}{=} \cos\left[2\pi f\left(t + \frac{k}{2\pi}\right)\right]$$

FILTRO IDEALE $:=$ $\begin{cases} \text{SISTEMA SENZA DISTORSIONE, per } f \in B \\ H(f) = 0 \text{, altrove } (f \notin B) \end{cases}$

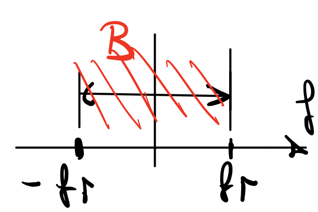
B : BANDA PASSANTE del FILTRO

$$\boxed{H(f) = \begin{cases} A\,e^{-j2\pi f t_0} & \text{per } f \in B \\ 0 & \text{per } f \notin B \end{cases}}$$

In funzione di B definisco diverse TIPOLOGIE DI FILTRO :

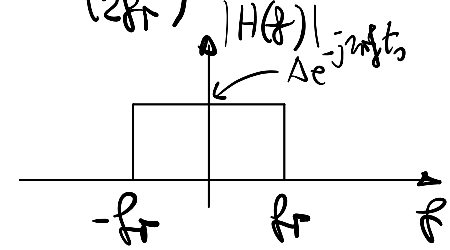
1) : B : INTERVALLO SIMMETRICO, INTORNO a $f = 0$:

$B: -f_T \le f \le f_T$: f_T : FREQUENZA di TAGLIO



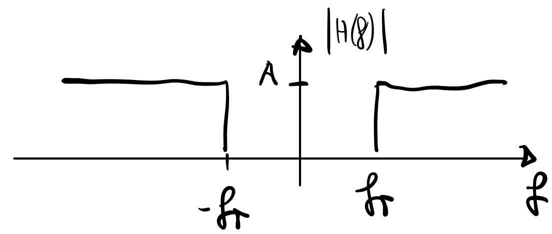
→ FILTRO PASSA-BASSO IDEALE:

$$H(f) = \begin{cases} A e^{-j2\pi f t_0} & -f_T \le f \le f_T \\ 0 & \text{altrove} \end{cases} = A e^{-j2\pi f t_0} \, \mathrm{rect}\left(\frac{f}{2f_T}\right)$$



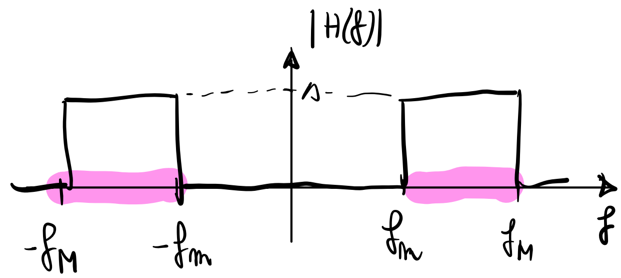
FILTRO PASSA-ALTO IDEALE:

$$H(f) = \begin{cases} A e^{-j2\pi f t_0} & \text{per } |f| > f_T \\ 0 & \text{altrove} \end{cases}$$



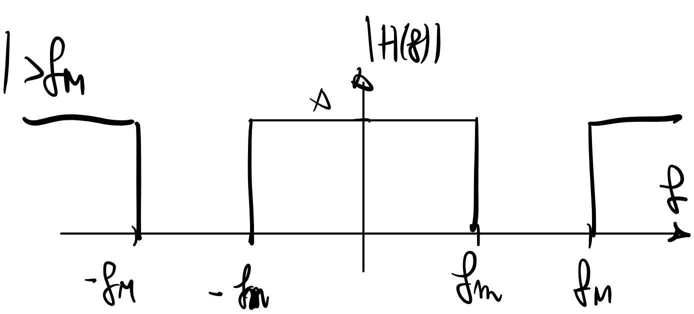
FILTRO PASSA-BANDA IDEALE:

$$H(f) = \begin{cases} A e^{-j2\pi f t_0} & \text{per } f_m < |f| < f_M \\ 0 & \text{altrove} \end{cases}$$



FILTRO ARRESTA-BANDA IDEALE:

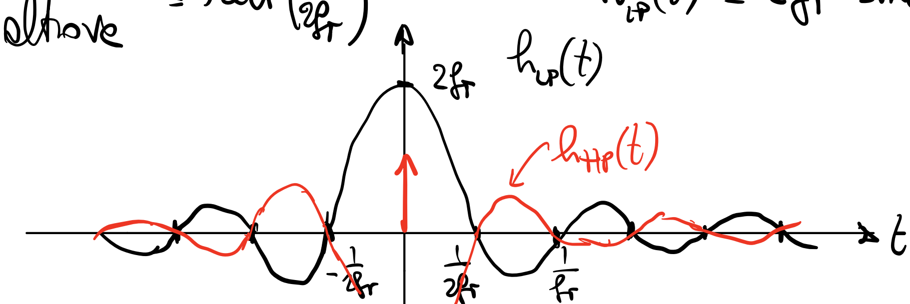
$$H(f) = \begin{cases} A e^{-j2\pi f t_0} & \text{per } |f| < f_m \vee |f| > f_M \\ 0 & \text{altrove} \end{cases}$$



Nel dominio dei TEMPI: RISPOSTA ALL'IMPULSO dei filtri IDEALI

PASSA-BASSO (LOW-PASS)

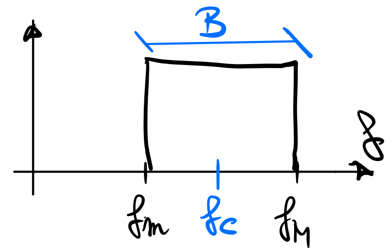
$$H_{LP}(f) = \begin{cases} 1 & |f| < f_T \\ 0 & \text{altrove} \end{cases} = \mathrm{rect}\left(\frac{f}{2f_T}\right) \xrightarrow{\mathcal{F}^{-1}} h_{LP}(t) = 2f_T \,\mathrm{sinc}(2f_T \cdot t)$$



PASSA-ALTO (HIGH-PASS)

$$H_{HP}(f) = \begin{cases} 0 & \text{per } |f| < f_T \\ 1 & \text{altrove} \end{cases} = 1 - \text{rect}\left(\frac{f}{2f_T}\right) \xrightarrow{\mathcal{F}^{-1}} h_{HP}(t) = \delta(t) - 2f_T \,\text{sinc}\left(\frac{f}{2f_T}\right)$$

PASSA-BANDA (BAND-PASS)

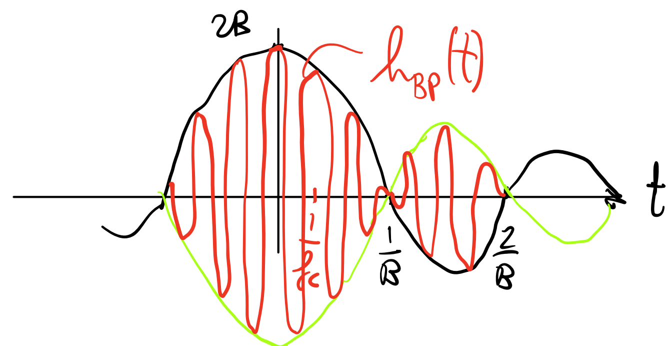


$$H_{BP}(f) = \begin{cases} 1 & f_m < |f| < f_M \\ 0 & \text{altrove} \end{cases} = \left\{ \begin{aligned} f_c &= \frac{f_m + f_M}{2} \\ B &= f_M - f_m \end{aligned} \right\} =$$

$$= \text{rect}\left(\frac{f - f_c}{B}\right) + \text{rect}\left(\frac{f + f_c}{B}\right) = \text{rect}\left(\frac{f}{B}\right) * \left[\delta(f - f_c) + \delta(f + f_c)\right]$$

$$\xrightarrow{\mathcal{F}^{-1}} h_{BP}(t) = B\,\text{sinc}(Bt) \cdot 2\cos(2\pi f_c t) = 2B\,\text{sinc}(Bt)\cos(2\pi f_c t)$$

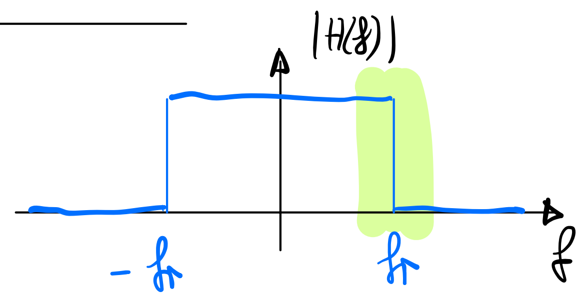
ARRESTA-BANDA (STOP-BAND)



$$H_{SB}(f) = 1 - H_{BP}(f)$$

$$h_{SB}(t) = \delta(t) - h_{BP}(t) = \delta(t) - 2B\,\text{sinc}(Bt)\cos(2\pi f_c t)$$

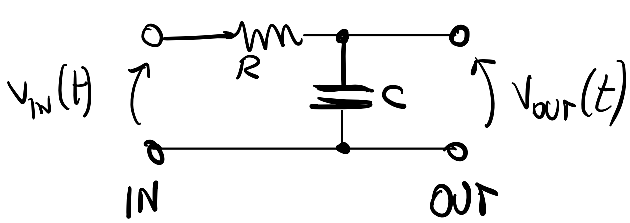
FILTRI REALI

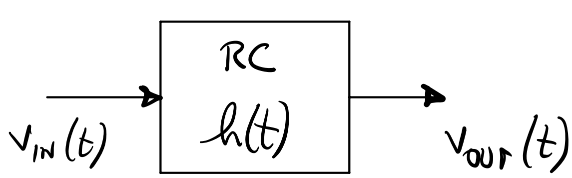


→ la TRANSIZIONE è GRADUALE!

Esempio di PASSA-BASSO REALE:

CIRCUITO R-C





$$v_{IN}(t) = \delta(t) \longrightarrow V_{OUT}(t) = \underbrace{\frac{1}{RC} u(t) e^{-t/RC} = h(t)}$$

$$h(t) = \frac{1}{RC} u(t) e^{-t/RC}$$

FOURIER PAIR:

$$e^{-at} u(t) \longleftrightarrow \frac{1}{a + j2\pi f}$$

$$\left[a = \frac{1}{RC} \right] \longrightarrow H(f) = \frac{1}{RC} \frac{1}{\frac{1}{RC} + j2\pi f} = \frac{1}{1 + j2\pi RC f}$$

Definisco $\frac{1}{2\pi RC} = f_T$: $\longrightarrow$ $\boxed{H(f) = \frac{1}{1 + j f/f_T}}$

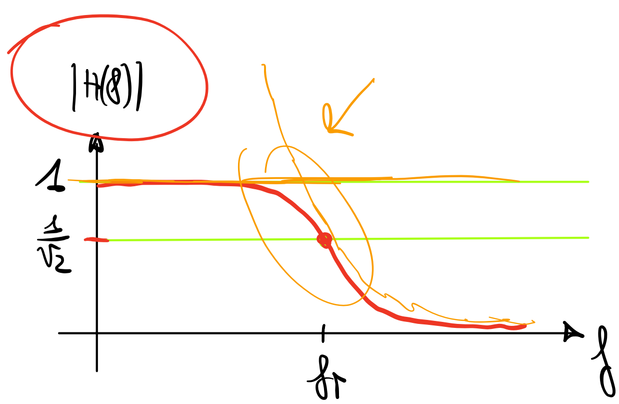
Risposta in AMPIEZZA e FASE:

$$|H(f)| = \frac{|1|}{|1 + j f/f_T|} = \frac{1}{\sqrt{1^2 + (f/f_T)^2}} = \begin{cases} f \ll f_T \longrightarrow |H(f)| = 1 \\ f = f_T \longrightarrow |H(f)| = \frac{1}{\sqrt{2}} \\ f \gg f_T \longrightarrow |H(f)| = \frac{f_T}{f} \end{cases}$$

$$\measuredangle H(f) = \measuredangle 1 - \measuredangle \left(1 + j\frac{f}{f_T}\right) =$$

$\measuredangle 1 \to 0$

$$= -\operatorname{atan2}\left(\frac{f}{f_T}\right) = \begin{cases} f \ll f_T \longrightarrow \approx 0 \\ f = f_T \longrightarrow -\pi/4 \\ f \gg f_T \longrightarrow -\pi/2 \end{cases}$$



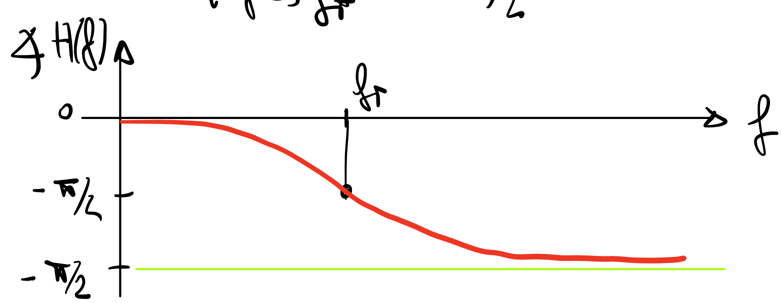


DIAGRAMMA di BODE: GRAFICO "LOG-LOG" di $|H(f)|$

$\log_{10} |H(f)|$ vs $\log_{10} f$

FILTRO del 1° ORDINE

FAMIGLIE di FILTRI REALI:

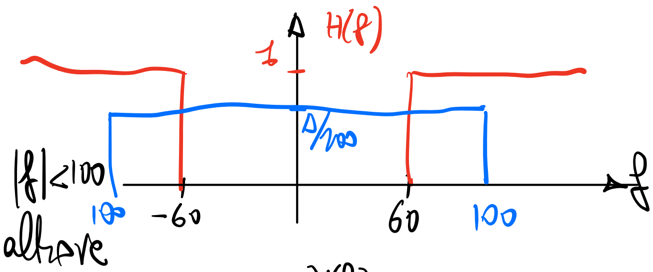
- Filtri di BUTTERWORTH
- Filtri di CHEBYSHEV
- Filtri di BESSEL

ESERCIZIO: Dato un filtro PASSA-ALTO IDEALE con $f_T = 60$ Hz in cui entra il segnale: $s(t) = \underbrace{A\,\mathrm{sinc}(200t)}$, calcolare il segnale in uscita dal filtro

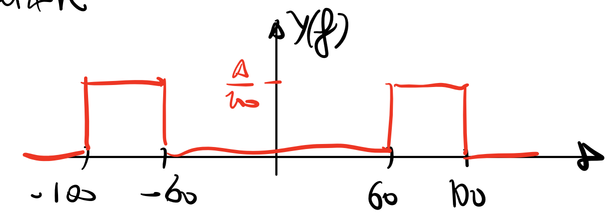
$$H_{HP}(f) = \begin{cases} 1 & |f| > 60 \\ 0 & \text{altrove} \end{cases} = 1 - \mathrm{rect}\left(\frac{f}{120}\right)$$

Calcolo S(f):

$$S(f) = A\frac{1}{200}\,\mathrm{rect}\left(\frac{f}{200}\right) = \begin{cases} \frac{A}{200} & |f| < 100 \\ 0 & \text{altrove} \end{cases}$$



$$\rightarrow \quad Y(f) = S(f) \cdot H(f) = \frac{A}{200}\left[\mathrm{rect}\left(\frac{f-80}{40}\right) + \mathrm{rect}\left(\frac{f+80}{40}\right)\right]$$



Antitrasformo Y(f):

$$\mathrm{rect}(f) \longrightarrow \mathrm{sinc}(t)$$

$$\frac{1}{40} \qquad \mathrm{rect}\left(\frac{f}{40}\right) \longrightarrow 40\,\mathrm{sinc}(40t)$$

$$\mathrm{rect}\left(\frac{f \pm 80}{40}\right) \longrightarrow 40\,\mathrm{sinc}(40)\,e^{\pm j160\pi t}$$

$$\rightarrow \quad y(t) = \frac{A}{\cancel{200}_{5}}\left[\cancel{40}\,\mathrm{sinc}(40t)\,e^{-j160\pi t} + \cancel{40}\,\mathrm{sinc}(40t)\,e^{+j160\pi t}\right] = \frac{A}{5}\,\mathrm{sinc}(40t)\left[\underbrace{e^{-j160\pi t} + e^{+j160\pi t}}_{2\cos(160\pi t)}\right]$$

$$= \frac{2}{5}A\,\mathrm{sinc}(40t)\cos(160\pi t)$$