Denoising in digital radiography:
A total variation approach

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Images are corrupted by noise...

i) When measurement of some physical parameter is performed, noise corruption cannot be avoided.

ii) Each pixel of a digital image measures a number of photons.

Therefore, from i) and ii)... ...Images are corrupted by noise!
Gaussian noise
(not so useful for digital radiographs, but a good model for learning...)

- Measurement noise is often modeled as Gaussian noise...
- Let \( x \) be the measured physical parameter, let \( \mu \) be the noise free parameter and let \( \sigma^2 \) be the variance of the measured parameter (noise power); the probability density function for \( x \) is given by:

\[
p(x | \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right)
\]

Gaussian noise and likelihood

- Images are composed by a set of pixels, \( x \) (\( x \) is a vector!)
- How can we quantify the probability to measure the image \( x \), given the probability density function for each pixel?
- Let us assume that the variance is equal for each pixel;
- Let \( x_i \) and \( \mu_i \) be the measured and noiseless values for the i-th pixel;
- Likelihood function, \( L(x | \mu) \):

\[
L(x | \mu) = \prod_{i=1}^{N} p(x_i | \mu_i) = \prod_{i=1}^{N} \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{x_i - \mu_i}{\sigma} \right)^2 \right)
\]
- \( L(x | \mu) \) describes the probability to measure the image \( x \), given the noise free value for each pixel, \( \mu \).
What about denoising??

- **What is denoising then?**
  
  Denoising = estimate $\mu$ from $x$.

- **How can we estimate $\mu$?**
- Maximize $p(\mu|x)$ -> this usually leads to an hard, inverse problem.
- It is easier to maximize $p(x|\mu)$, that is $\Rightarrow$ maximize the likelihood function (a “simple”, direct problem).
- But... Is maximization of $p(\mu|x)$ different from that of $p(x|\mu)$?

Bayes and likelihood

- **Bayes theorem:**
  
  $$p(\mu | x)p(x) = p(x | \mu)p(\mu) \Rightarrow$$

  $$\Rightarrow p(\mu | x) = \frac{p(x | \mu)p(\mu)}{p(x)}$$

  - **Likelihood**
  - **A priori hypothesis on the estimated parameters $\mu$. For the moment, let us suppose $p(\mu) = \text{Cost.}$**

  - **Probability density function for the data $x$. Just a normalization factor!!!**

- **In this case, maximizing $p(\mu|x)$ or $p(x|\mu)$ is the same!**
So, let us maximize the likelihood...

- Instead of maximizing $L(\mathbf{x} | \mu)$, it is easier to minimize $-\log[L(\mathbf{x} | \mu)]$.

- When the noise is Gaussian, we get:

$$L(\mathbf{x} | \mu) = \prod_{i=1}^{N} p(x_i | \mu_i) = \prod_{i=1}^{N} \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x_i - \mu_i}{\sigma} \right)^2 \right]$$

$$f(\mathbf{x} | \mu) = -\ln[L(\mathbf{x} | \mu)] = -\sum_{i=1}^{N} \ln \left( \frac{1}{\sigma \sqrt{2\pi}} \right) + \frac{1}{2\sigma^2} \sum_{i=1}^{N} (x_i - \mu_i)^2$$

- Maximize $L \Rightarrow$ Least squares problem!

However, what about noise in digital radiography?

- Noise in digital radiography is Poisson (photon counting noise)!

- Let $p_{n,i}$ be the noisy (measured) number of photons associated to pixel $i$, and $p_i$ the unnoisy number of photons. Then:

$$p(p_{n,i} | p_i) = \frac{p_i^{p_{n,i}} e^{-p_i}}{p_{n,i}!}$$
Gaussian noise: example

Constant variance

Poisson noise: example

Lower variance for low signal
**Likelihood for Poisson noise**

- Let us write the negative log likelihood for the Poisson case:
  
  \[
  L(p_n | p) = \prod_{i=1}^{N} p(p_{n,i} | p) = \prod_{i=1}^{N} \frac{p_{n,i} \cdot e^{-p}}{p_{n,i}!}
  \]

  \[
  f(p_n | p) = -\ln[L(x | μ)] = -\sum_{i=1}^{N} [p_{n,i} \cdot \ln(p_i)] + \sum_{i=1}^{N} p_i + \sum_{i=1}^{N} \ln(p_{n,i}) = \sum_{i=1}^{N} [p_i - p_{n,i} \cdot \ln(p_i)]
  \]

- \( L(p_n | p) \) is also known as Kullback-Leibler divergence (apart from a constant term, which does not affect the minimization process), \( KL(p_n | p) \).

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**Maximize \( L \)**

\( L \) is maximized \( \Leftrightarrow \) \( f \) is minimized;

- Optimization (Gaussian noise) can be performed posing:
  
  \[
  \frac{\partial f(x | μ)}{\partial μ} = 0 \Leftrightarrow \frac{\partial f(x | μ)}{\partial μ} = 0, \ \forall i \Rightarrow \frac{\partial}{\partial μ} \sum_{j=1}^{N} (x_j - μ_j)^2 = 0, \ \forall i \Rightarrow \exists 2(x_i - μ_i) = 0, \ \forall i \Rightarrow x_i = μ_i, \ \forall i
  \]

- The noisy image gives the highest likelihood!!!
- This solution is not so interesting... The likelihood approach suffers from a severe overfitting problem.
Maximize $L$!

$L$ is maximized $\iff$ $f$ is minimized;

- Optimization (Poisson noise) can be performed posing:
  \[
  \frac{\partial f(p_n | p)}{\partial p} = 0 \iff \frac{\partial f(p_n | p)}{\partial p} = 0, \quad \forall i \Rightarrow \frac{\partial}{\partial p_i} \sum_{i=1}^{N} [p_i - p_{w,i} \cdot \ln(p_i)] = 0, \quad \forall i \Rightarrow \\
  1 - \frac{p_{w,i}}{p_i} = 0, \quad \forall i \Rightarrow p_i = p_{w,i}, \quad \forall i
  \]

- The noisy image gives the highest likelihood!!!
- This solution is not so interesting… The likelihood approach suffers from a severe overfitting problem.

Back to Bayes

- Bayes theorem:
  \[
  \Rightarrow p(p | p_n) = \frac{p(p_n | p)p(p)}{p(p_n)}
  \]

- If we introduce a-priori knowledge about the solution $\mu$, we get a Maximum A Posteriori (MAP) solution – $p(p | p_n)$ is maximized!
What do we have to minimize now?

- We want to maximize \( p(\mathbf{p}|\mathbf{p}_n) \sim p(\mathbf{p}_n|\mathbf{p}) \ p(\mathbf{p}) \), that is:

\[
- \ln[p(\mathbf{p}|\mathbf{p}_n)] = - \ln[p(\mathbf{p}_n|\mathbf{p})p(\mathbf{p})] = - \ln \prod_{i=1}^{N} [p(\mathbf{p}_n,\mathbf{p})] = \\
- \sum_{i=1}^{N} \ln[p(\mathbf{p}_n,\mathbf{p})] = - \sum_{i=1}^{N} \ln[p(\mathbf{p}_n,\mathbf{p})] - \sum_{i=1}^{N} \ln[p(\mathbf{p})] = \\
= - \ln[L(\mathbf{p}_n|\mathbf{p})] - \sum_{i=1}^{N} \ln[p(\mathbf{p})]
\]

Negative log likelihood

Regularization term (a priori information)

A priori term

- Let us call \( p_x \) and \( p_y \) the two components of the gradient of the image.
- These are easily computed, for instance as:
  - \( p_x = p(i,j) - p(i-1,j) \);
  - \( p_y = p(i,j) - p(i,j-1) \);
- The gradient (a vector!) will be indicated as \( \nabla \mathbf{p} \);
- \( || \nabla \mathbf{p} || \) indicates the norm of the gradient.
A priori term – image gradients (no noise)

\[ p_x = p(i,j) - p(i-1,j) \]
\[ p_y = p(i,j) - p(i,j-1) \]

A priori term – image gradients (noise)

\[ p_x = p(i,j) - p(i-1,j) \]
\[ p_y = p(i,j) - p(i,j-1) \]
A priori term – norm of image gradient

No noise

In the real image, most of the areas are characterized by an (almost) null gradient norm;

*We can for instance suppose that \( \| \nabla p \| \) is a random variable with Gaussian distribution, zero mean and variance equal to \( \beta^2 \).

(Note that, in the noisy image, the norm of the gradient assume higher values \( \| \nabla p \| \) means low noise!)

MAP and regularization theory

• Poisson noise, normal distribution for the norm of the gradient:

\[
\begin{align*}
    f(p_a | p) &= -\ln[L(p_a | p)] - \sum_{i=1}^{N} \ln p(\nabla p_i) = \\
    &= \sum_{i=1}^{N} [p_i - p_{a,i} \cdot \ln(p_i)] - \sum_{i=1}^{N} \ln \left( \frac{1}{\sqrt{2\pi} \beta} \exp \left( -\frac{1}{2} \frac{\| \nabla p_i \|}{\beta^2} \right) \right) = \\
    &= \sum_{i=1}^{N} [p_i - p_{a,i} \cdot \ln(p_i)] + N \ln(\sqrt{2\pi}) + \frac{1}{2\beta^2} \sum_{i=1}^{N} \| \nabla p_i \| ^2
\end{align*}
\]

[Note that, \( \sum_{i=1}^{N} \| \nabla p_i \| \) means low noise!]

Negative log likelihood

Regularization term (a priori information)
MAP and regularization theory

- We look for the minimum of $f$...
- ... The likelihood is maximized (data fitting term)...
- ... At the same time, the squared norm of the gradient is minimized (regularization term)...
- ... The regularization parameter $(1/2\beta^2)$ balances between a perfect data fitting and very regular image...

$$f(p_n | p) = \sum_{i=1}^{N} \left[ p_i - p_{n,i} \cdot \ln(p_i) \right] + \frac{1}{2\beta^2} \sum_{i=1}^{N} \| \nabla p_i \|^2$$

For $(1/2\beta^2) = 0$ we get the maximum likelihood solution; increasing $(1/2\beta^2)$ we get a more regular (less noisy) solution.; For $(1/2\beta^2) \rightarrow \infty$, a completely smooth image is achieved.

Noise reduction.

Noise and edge reduction.
Fix the ideas

- A statistical based denoising filter is achieved minimizing:
  \[ f = -\ln[L(p_n|p)] - \lambda \ln[p(p)] \]

- The data fitting term is derived from the noise statistical distribution (likelihood of the data); generally, the choice for this term is unquestionable.

- The regularization term is derived from a-priori knowledge regarding some properties of the solution; this term is generally user defined.

- Depending on the regularization parameter \( \lambda \), the first or the second term assume more or less importance. For \( \lambda \to 0 \), the maximum likelihood solution is obtained.

Gibbs prior

- Up to now, we assumed a normal distribution for the norm of the gradient, \( \nabla \) Tikhonov regularization (quadratic penalization).

- A more general framework is obtained considering:
  \[ p(p) = \exp[-R(p)] \quad \text{(Gibb's prior)} \]

- \( R(p) \) \( \to \) Energy function ~ regularization term (note that \( -\ln \exp[-R(p)] = R(p)! \))

- Tikhonov assumes \( R(p) = \frac{1}{2} (\|\nabla p\|^2 / \beta) \)
**Edge preserving denoising?**

- Tikhonov term penalizes the image edges (high gradient) more than the noise gradients.
- It is well known that Tikhonov regularization does not preserve edges.
- An edge preserving algorithm is obtained considering $R(p) = ||\nabla p||$ (Total Variation, TV).

**Tikhonov vs. TV (preview)**

- Tikhonov =>
- Original image
- TV =>
- Filtered image
- Difference
**TV in digital radiography: starting point and problems**

- $p_n$, noisy image affected by Poisson noise (likelihood $\Rightarrow$ KL);
- $p$, noise free image (unknown);
- $R(p) = ||\nabla p||$ (Total Variation);
- Minimize $f(p|p_n) = KL(p_n,p) + \lambda \cdot \Sigma_{i=2..N} ||\nabla p_i||$.

- How to compute $||\nabla p_i||$? $\Rightarrow$ A compromise between computational efficiency and accuracy has to be achieved.
- How to minimize $f(p|p_n)$? $\Rightarrow$ An iterative optimization technique is required.

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**How to compute $||\nabla p_i||$?**

- $p_x = p(u,v) - p(u-1,v)$
- $p_y = p(u,v) - p(u,v-1)$
- $||p_i||_1 = |p_x| + |p_y|$ L1 norm
- $||p_i||_2 = (p_x^2 + p_y^2)^{1/2}$ L2 norm
- Computational cost increases with the number of neighbours considered for computing the gradient;
- The computational cost is higher for L2 norm with respect to L1 norm;
- What about accuracy? $\Rightarrow$ See experimental results!
How to minimize $f(p|p_n)$?

- $f(p|p_n)$ is strongly non-linear; solving $\frac{df(p|p_n)}{dp}=0$ directly is not possible => iterative optimization methods.

1) Steepest descent + line search (SD+LS)
2) Expectation – Maximization (damped with line search - EM)
3) Scaled gradient (SG)

Steepest descent + line search (SD+LS)

- $p^{k+2}=p^k-\alpha \cdot \frac{df(p|p_n)}{dp} \Rightarrow p^{k+2}=p^k-\alpha \cdot df(p|p_n)/dp$
- The damping parameter $\alpha$ is estimated at each iteration to assure convergence ($f^{k+1}<f^k$);

+: easy implementation;
-: slow convergence, the method has been damped (line search) to improve convergence ($\alpha>1$).
**EM + line search (EM)**

- Consider the pixel $i$, then:
  
  \[
  \frac{df(p|p_i)}{dp} = 0 \Rightarrow \frac{dK(p|p_i)}{dp} + \frac{dR}{dp} = 0 \Rightarrow \frac{\beta \cdot dR}{dp} + p_i - p_n,i = 0 \Rightarrow p_i = p_n,i / (\beta \cdot dR/dpi + 1) \]  
  [Fixed point iteration]

- Damped formula: $p_i = p_i \cdot (1 - \alpha) + \alpha \cdot p_n,i / (\beta \cdot dR/dpi + 1)$

- The damping parameter $\alpha$ is estimated at each iteration to assure convergence ($f^{k+1} < f^k$);
  
  $+$: easy implementation, fast convergence;  
  $-$: the method has been damped to assure convergence ($\alpha < 1$, what happens when $\beta \cdot dR/dpi + 1 \rightarrow 0$??).

**Scaled gradient (SG)**

- Consider the gradient method formula;
- Each component of the gradient is scaled to improve convergence ($S$ is a diagonal matrix containing the scaling parameters):
  
  \[
  p^{k+1} = p^k - \alpha \cdot S \cdot \frac{df(p|p_i)}{dp}
  \]

- The matrix $S$ is computed from an opportune gradient decomposition and KKT conditions;
  
  $+$: easy implementation, fastest convergence; it can also be demonstrated that, for positive initial values, the estimated solution remains positive at each iteration!  
  $-$: ???
Problems with \( \text{dR/dp}_i \)

- Independently from the optimization method, the term \( \text{dR/dp}_i \) has to be computed at each iteration for any \( i \);
- We have:

\[
\frac{\text{dR}}{\text{dp}_i} = \frac{\text{d} \left[ \sum_{i=1..N} (\text{ll} \nabla \text{p}_i) \right]}{\text{dp}_i}
\]

\[
\frac{\text{dR}}{\text{dp}_i} = \frac{\text{d} \left[ \sum_{i=1..N} (\text{ll} \nabla \text{p}_i) \right]}{\text{dp}_i}
\]

Problems with \( \text{dR/dp}_i \)

- Let us compute it for \( \text{ll}_{l_2} \) \( (\text{R/dp}_i = \frac{\text{d} \left[ \sum_{i=1..N} (\text{ll} \nabla \text{p}_i) \right]}{\text{dp}_i}) \)

\[
\frac{\text{dR}}{\text{dp}_i} = \sum_{j=1}^{N} \sqrt{p_{x,j}^2 + p_{y,j}^2} \frac{\text{d} \left[ \sqrt{[p(u,v) - p(u-1,v)]^2 + [p(u,v) - p(u,v-1)]^2} \right]}{\text{dp}_i} + \ldots =
\]

\[
\frac{\text{dR}}{\text{dp}_i} = \frac{2[p(u,v) - p(u-1,v)] + 2[p(u,v) - p(u,v-1)]}{\sqrt{[p(u,v) - p(u-1,v)]^2 + [p(u,v) - p(u,v-1)]^2}} + \ldots = \frac{2 p_{x,i} + p_{y,i}}{\sqrt{p(u,v)}} + \ldots
\]

- To avoid division by zero:

\[
\frac{\text{dR}}{\text{dp}_i} = \frac{2 p_{x,i} + p_{y,i}}{\sqrt{p(u,v)}} + \ldots \rightarrow 2 \frac{p_{x,i} + p_{y,i}}{\sqrt{[p(u,v) - p(u-1,v)]^2 + [p(u,v) - p(u,v-1)]^2}} \delta + \ldots
\]
Problems with $\frac{dR}{dp_i}$

- Let us compute it for $\| \nabla p \|_2$ $(\frac{dR}{dp} = \frac{\sum_{i=1..N}(\| \nabla p_i \|_2)}{dp})$

$$\frac{dR}{dp} = \frac{d\sum \| p_{i,j} \|_2}{dp} = \left[ \sum \frac{d}{dp} \sqrt{p(u,v)^2 - p(u-1,v)^2} + \sqrt{p(u,v)^2 - p(u,v-1)^2} \right] + \ldots = \sum_{i=1}^N \left[ \text{sign}(p_{i,j}) + \text{sign}(p_{i,j}) \right] + \ldots$$

- Here divisions by zero are automatically avoided – only “sign” is required -> computationally efficient!

Questions

- How many neighbor pixels do we have to consider to achieve a satisfying accuracy at low computational cost?

- Best norm, $\| \nabla p \|_1$ vs $\| \nabla p \|_2$?

- Best optimization method (SD+LS, EM, SG)?
Results (answers)

- 75 simulated radiographs with different frequency content, corrupted by Poisson noise (max 15,000 photons).

- For any filtered image, measure:

  \[
  \text{MAE} = \frac{1}{N} \sum_{i=1}^{N} |p_i^{\text{noisefree}} - p_i^{\text{filtered}}| \\
  \text{RMSE} = \left( \frac{1}{N} \sum_{i=1}^{N} (p_i^{\text{noisefree}} - p_i^{\text{filtered}})^2 \right)^{1/2} \\
  \text{KL} = \sum_{i=1}^{N} [p_i^{\text{noisefree}} \ln(p_i^{\text{noisefree}} / p_i^{\text{filtered}}) + p_i^{\text{noisefree}} - p_i^{\text{filtered}}]
  \]
2 neighbors (01010000) vs. 4 neighbors (11110000)

II._II_2 vs. II._II_2
EM vs. SD+LS

EM vs. SG
Convergence and iterations

Filter effect

Original  Filtered
Filter effect: before filtering

Filter effect: after filtering
Conclusion

- Effective edge preserving filter;
- 2 neighbors, $l_2$, $l_1$ and EM achieve the best compromise between accuracy and computational cost;
- SD achieves results better than EM when the regularization parameter is not correctly selected.

- Adaptive regularization parameter;
- GPU (CUDA) implementation;
- Expanding the likelihood model
  - Mixture of Poisson, Gaussian and Impulsive noise;
  - Include the sensor point spread function.

http://aislab.dsi.unimi.it