Local Wavelet decomposition and its application to face reconstruction

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Abstract Wavelets are a powerful tool for multi-resolution analysis as they combine spatial and frequency locality. In this paper an efficient procedure to compute the Wavelet coefficients, called lifting schema, is illustrated. Results on face reconstruction operated at different resolution are reported and discussed.

1 Introduction

Wavelet theory provides a unifying framework for different techniques, developed independently. In particular it provides an efficient tool for a Multi-Resolution reconstruction of surfaces whose frequency content is non-stationary [1, 2]. The Wavelet Transform (WT) decomposes the surface onto a set of basis functions, called *Wavelets,* which are obtained from a single prototype, the *mother Wavelet,* through dilation, contraction and shift. This decomposition is achieved by projecting the surface onto the base set. An efficient implementation requires that the surface is projected iteratively on a base of decreasing scale. The analytical key operation is the internal product of the surface with the Wavelet base, which is theoretically extended over the entire R^2 plane. To cope with real data, which have a finite support, some approximations have to be made: the contribution of a Wavelet basis is supposed to vanish over a certain distance from its centre, or windowing of the input surface is performed [3]. A simpler analytical solution has been recently proposed under the name of lifting schema [4]. It implements only local operations and implicitly performs Wavelet windowing. Results on the application of this schema to surface reconstruction are reported and discussed.

2 The Multi-Resolution Analysis and the Wavelet Transform

The Wavelet Transform is based on the Multi-Resolution Analysis (MRA). This is obtained projecting $f(x) \in L^2(R)$ over a suitable set of bases which form a Riesz base for $L^2(R)$. These projections give an approximation of $f(x)$: $P_j[f(x)]$, at a resolution j. The bases are obtained by scaling a given function, called *Scaling Function,* $\varphi(x)$, such that each base is orthogonal to the previous one. The MRA produces a sequence of approximations of $f(x)$ such that its limit coincides with $f(x)$ itself:

$$\lim_{j\to\infty} P_j[f(x)] = f(x) \quad (1)$$

If we move into the discrete (sampled functions and discrete resolutions), the concept of filter bank tree substitutes the concept of Multi-Resolution. In particular, if we define:

$$R_j = P_j[f(x)] - P_{j-1}[f(x)] \quad (2)$$

it can be shown that it is possible to define a base, the *Wavelet*, such that projecting $f(x)$ over this basis, R_j is obtained. If we call $Q_j[f(x)]$ this projection, the following relationship holds:

$$Q_{j-1}[f(x)] = P_j[f(x)] - P_{j-1}[f(x)] \quad (3)$$

From Eq. (2), it follows that the series of the $Q_j(.)$ gives the function itself:

$$\sum_j Q_j[f(x)] = f(x) \quad (4)$$

and that $f(x)$ can be approximated ad libitum, by a finite number of the $Q_j[f(x)]$. The *Wavelet Transform* is defined as the set of the coefficients, $\{\gamma_{j,k}\}$, which produce $Q_j[f(x)]$:

$$f(x) = \sum_j Q_j[f(x)] = \sum_{j,k} \gamma_{j,k} \psi_{j,k}(x) \quad (5)$$

The bases $\psi_{j,k}(.)$ are orthogonal and are constructed by dilation, from a single Wavelet function, called *mother Wavelet*. The set of the $\psi_{j,k}(.)$ constitutes a Riesz base for $L^2(R)$. The lifting schema, reported in Section 4, gives a simple and local method to set the values of the $\{\gamma_{j,k}\}$. Similarly, the approximation at the resolution j, can be written as:

$$P_j[f(x)] = \sum_{j,k} \lambda_{j,k} \varphi_{j,k}(x) \quad (6)$$

It can be shown [5] that the coefficients $\{\gamma_{j,k}\}$ and $\{\lambda_{j,k}\}$ can be computed recursively from the set of coefficients at the higher resolution, $j+1$. This decomposition is exemplified in Fig. 1 and takes the name of pyramid algorithm or Mallat algorithm.

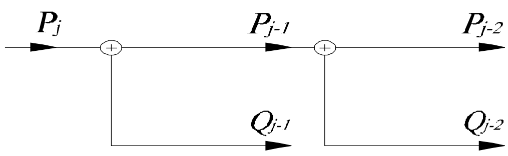


Fig. 1. The decomposition of a function $f(x)$: at each resolution or scale j, a pair of Approximation $P_j(.)$ and Residual $Q_j(.)$ is obtained. $Q_j(.)$ can be obtained by projecting the function, $f(x)$, over the Wavelet of scale j, $\psi_j(.)$. $P_0[f(x)]$ is usually assumed equal to $f(x)$.

3 Computing the Wavelet Transform

Haar is the simplest orthogonal Wavelet and it will be used to exemplify the construction machinery. Haar Base is a piecewise constant function. If we consider a finite support of length 1, the basis- function, $\psi_{j,0}(.)$, is $2^{j/2}$ on as subinterval of length $0.5 * 2^{-j}$ and $-2^{j/2}$ on the next subinterval. There are therefore 2^j wavelets at level j. Its graphical representation is a square wave. The coefficients of the Haar Wavelet Transform are computed projecting the function $f(x)$ on this square wave as:

$$\gamma_{j,k} = \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{+\infty} f(x) 2^{j/2} \psi(2^j t - k) dt \tag{7a}$$

The Scaling function, which has to be orthogonal to the Wavelet, is a box function which is constant and equal to $2^{j/2}$ in the interval of length 2^{-j}, starting at $t = k2^{-j}$. The coefficients of the scaling function at the resolution j, can be computed by projection as:

$$\lambda_{j,k} = \langle f, \varphi_{j,k} \rangle = \int_{-\infty}^{+\infty} f(x) 2^{j/2} \varphi(2^j t - k) dt = \sum_{k=0}^{k=2j} 2^{j/2} f(x_k) \tag{7b}$$

As can be seen, the Haar Scaling function, $\varphi_{j,k}$ computes the local mean value of $f(x)$ at the resolution j. This can be recast into a recursive schema where $\lambda_{j,k}$ is computed as:

$$\lambda_{j,k} = \frac{\sqrt{2}}{2} \left(\lambda_{j+1,2k} + \lambda_{j+1,2k+1} \right) \tag{8a}$$

which is the mean value between two consecutive samples at the higher resolution. It is usually assumed, as for the other Scaling functions: $\varphi_{0,k} = f(x_k)$. From the definition of Haar Wavelet, the following equation for the coefficients $\{\gamma_{j,k}\}$ holds:

$$\gamma_{j,k} = \frac{\sqrt{2}}{2} \left(\lambda_{j+1,2k} - \lambda_{+1j,2k+1} \right) \tag{8b}$$

Eq. (8b) is used to compute the coefficients of the Haar Wavelet Transform. Given the coefficients $\{\lambda_{j,k}\}$ and $\{\gamma_{j,k}\}$, the higher resolution coefficients $\{\lambda_{j+1,k}\}$ can be recovered by:

$$\lambda_{j+1,2k} = \frac{\sqrt{2}}{2} \left(\lambda_{j,k} + \gamma_{j,k} \right) \quad \text{and} \quad \lambda_{j+1,2k+1} = \frac{\sqrt{2}}{2} \left(\lambda_{j,k} - \gamma_{j,k} \right) \tag{8c}$$

The (8c) formalize the Inverse Wavelet Transform (IWT). The algorithm described above can be generalized to perform a general WT. Every wavelet can be characterized by a set of four finite filters, $\{h, g, \tilde{h}, \tilde{g}\}$, which can be used to move between different resolutions. The general form of Eqs. (8a-c) is:

$$\lambda_{j,l} = \sqrt{2} \sum_k \tilde{h}_{k-2l} \lambda_{j+1,k} \tag{9a}$$

$$\gamma_{j,l} = \sqrt{2} \sum_k \tilde{g}_{k-2l} \lambda_{j+1,k} \; . \tag{9b}$$

$$\lambda_{j+1,l} = \sqrt{2}\sum_{l} h_{k-2l}\lambda_{j,k} + \sqrt{2}\sum_{l} g_{k-2l}\gamma_{j,k} \tag{9c}$$

In the following, simple filters used for some of the most common Wavelets are reported [3]: Haar Wavelet: h = $\tilde{h}$ = [1/2, 1/2], g = $\tilde{g}$ = [1/2, -1/2]; Hat (linear) Wavelet: h = [1/4 1/2 1/4], $\tilde{h}$ = [-1/8, 1/4, 3/4, 1/4, -1/8], g = [-1/8, -1/4, 3/4, -1/4, -1/8], $\tilde{g}$ = [-1/4, 1/2, -1/4]; Cubic Wavelet: h = [-1/32 0 9/32 1/2 9/32 0 -1/32], $\tilde{h}$ = [1/64 0 -1/8 1/4 23/32 1/4 -1/8 0 1/64], g = [1/64 0 -1/8 -1/4 23/32 -1/4 -1/8 0 1/64], $\tilde{g}$ = [1/32 0 -9/32 1/2 -9/32 0 1/32].

4 The Lifting schema

The lifting schema, recently proposed by [4], is a very efficient way to compute the Wavelet Transform. We will illustrate here how does it work when the Hat (linear) Wavelet is used. Let us start with a sampled function: $\{f(x_k)\}$, $k \in Z$. Subsampling by a factor two we obtain the even and the odd sequence. These can be seen as the approximation of $f(x_k)$, the even samples, and the residual, $r(x_k)$, the odd samples. This schema produces a Lazy Wavelet where:

$$\lambda_{-1,k} = \lambda_{0,2k} = f(x_{0,2k}) \tag{10a}$$

are the coefficients of the Scaling functions $\varphi_{-1,k}(.)$, and:

$$\gamma_{-1,k} = \lambda_{0,2k+1} = f(x_{0,2k+1}) \tag{10b}$$

are the coefficients of the Wavelet, $\psi_{-1,k}(.)$. A more efficient coding of $f(x_k)$ is achieved by considering that the correlation between neighbour samples is usually high, at least for smooth functions. The odd samples can therefore be computed as the deviation from the average of the value of pairs of neighbour even samples. The Wavelet coefficients are thus:

$$\gamma_{1,k} = \lambda_{0,2k+1} - 1/2(\lambda_{1,k} + \lambda_{1,k+1}) \tag{11a}$$

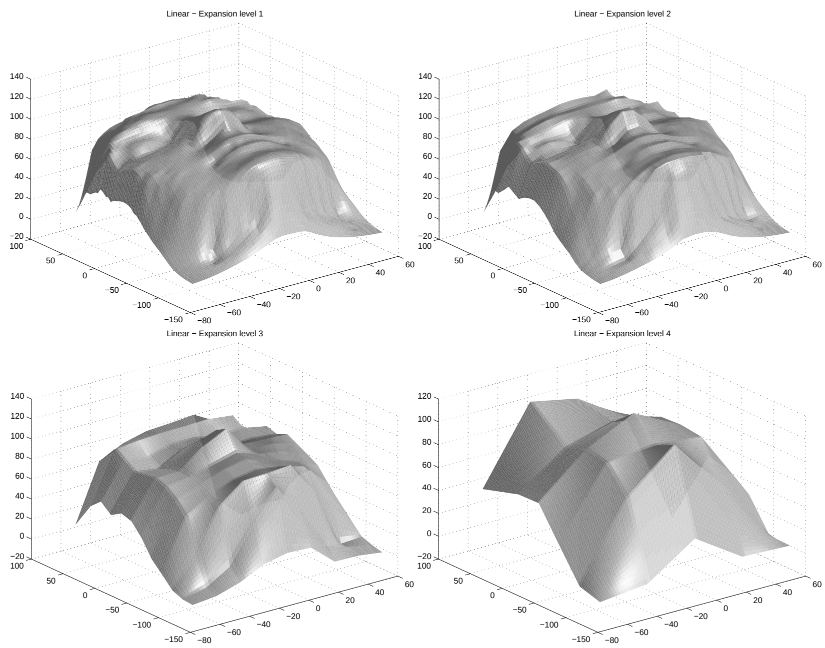
As these coefficients encode the deviation from linearity of the function $f(x_k)$, they will usually be small. A good reconstruction of $f(x_k)$ can be obtained even if the small coefficients are discarded. We can now improve the coding of the coefficients of the Scaling function, $\{\lambda_{j,k}\}$. In fact, if the given function samples are composed by 1's interleaved with 0's (i.e. $\{\lambda_{0,k}\}$ = [1 0 1 0 1 0 1 0 1 0 1 0 1]), the coding reported above will suffer from aliasing and poor approximation, as the approximation at the first level (j=1) will result in a sequence of 1's. A better schema to compute the $\{\lambda_{j,k}\}$ is the following:

$$\lambda_{j,k} = \lambda_{j+1,2k} + 1/4(\gamma_{j,k-1} + \gamma_{j,k}) \tag{11b}$$

which guarantees that the average of the $\lambda_{j,k}$ is at the same at every level of resolution j. Eqs. (10) and (11) describe the lifting schema for the Hat Wavelet. This schema can be generalised to other WTs and it allows to cut the computational time to half with respect to the classical computation which performs a convolution. It does not require additional memory, as the calculation can be carried out in place. Moreover, the algorithm is well suited for SIMD parallelisation into SIMD machines. In literature, the WT was heavily linked to Fourier transform, and its use supposes regularly spaced samples. Since the lifting schema does not depend on frequency analysis, it allows to generalise the WT to deal directly with range data.

5 Reconstruction of a human face

The WT can be applied to MRA and reconstruction of 3D surfaces ($z = f(P)$, with $P \in \mathbb{R}^2$ and $z \in \mathbb{R}^1$). Provided that the sampled data are organised into a matrix, it could be shown that the WT can be realised transforming first all the rows and, successively, all the columns of the matrix, i.e. considering alternatively the rows and the columns as a battery of monodimensional function. This schema has been applied to the reconstruction of human faces at different resolutions. The results are obtained applying the Hat wavelet, but the results can be generalised easily to quadratic and cubic Bases. The face in Figure 2 has been reconstructed starting from 12,641 data points sampled over the face using the Autoscan system [6]. A Hierarchical RBF network [7][8] has been used to resample the data points with a spacing of 1,5mm. The Hat Wavelet has been applied to this set of data.



6 Conclusion

Wavelets perform a local analysis in the combined frequency/space domain. In this respect they are similar to the Short-Time-Fourier-Transform with the difference that WT uses a short window at high frequency and a large window at low frequency while the STFT uses a fixed window. It is similar also to other MRA schemas like HRBF with the difference that in HRBF the number of coefficients increases one level after the other while in the Wavelet transform the number of coefficients decreses. On the other side, the WT does not have the redundant representation in the frequency domain which the HRBF has. Wavelets are made even more powerful by the lifting schema which gives to the MRA more freedom than the classical approach as different metrix can be used in computing the coefficients, windowing can be implemented locally, saving in computational time is achieved and only local operations are carried out.

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