Multi-resolution Models for Data Processing: an Experimental Sensitivity Analysis

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Abstract – Hierarchical Radial Basis Functions Networks (HRBF) have been recently introduced as a tool for adaptive multiscale image reconstruction from range data. They are based on local operation on the data and are able to give a sparse approximation. In this paper HRBF are reframed for the regular sampling case, and they are compared with Wavelet Decomposition. Results show that HRBF, thanks to their constructive approach to approximation, are much more tolerant on errors on the parameters when errors occurs in the configuration phase, while they are more sensitive to the errors which occurs since the network has been configured.

(HRBF) have been introduced, and it is shown that, although they do not perform a Wavelet Decomposition, they do enjoy the same asymptotic approximation properties. Moreover, HRBF are able to better achieve a given approximation error. This is useful, when measurement noise characterization is available.

Aim of this work is to reframe the HRBF configuration procedure for regular sampled data and compare the accuracy in the reconstruction of a given signal through HRBF and Wavelet Decomposition when error, due to numerical representation of their parameters, is introduced.

I. INTRODUCTION

Multi-resolution techniques are widely used in signal processing as they are able to analyze the signal properties and produce a local description both in temporal and in frequency domain. This feature is of fundamental importance when a continuos measurement field has to be recovered from spot measurements.

Wavelet Decomposition [1] is the most used tool for multi-resolution analysis thanks to the fast machinery adopted to compute its coefficients. An interesting alternative is offered by multi-scale approximation through Gaussian bases. In [2, 3] Hierarchical Radial Basis Functions Neural Networks

II. METHODOLOGICAL BACKGROUND

A. Multi-Resolution Analysis

A Multi-Resolution Analysis (MRA) is a sequence of function spaces $\{V_j\}_{j \in Z}$, where each space completely includes the previous ones. These spaces are completely characterized by a single function $\varphi(\cdot)$, called scaling function, as each space V_j is spanned by the set of functions $\{\varphi_{j,k}(\cdot) \mid k \in Z\}$ where $\varphi_{j,k}(x) = \sqrt{2^j}\varphi(2^j x - k)$. Thanks to the scaling factor 2^j, the scale of $\varphi(\cdot)$ doubles every layer. Approximations at a given scale are obtained as a linear combination of $\varphi_j(\cdot)$ and are contained in the corresponding V_j. As the union of $\{V_j\}$ is dense in L_2, every signal $f(\cdot)$ in L_2

(which are all the common ones) can be reconstructed with an arbitrary accuracy. Therefore, it is possible to define a set of spaces $\{W_j\}$ such that they complement $\{V_j\}$ in $\{V_{j+1}\}$: $V_{j+1} = V_j \oplus W_j$. The $\{W_j\}$ are the function spaces which contain the details: the portion of $f(\cdot)$ which is contained in V_{j+1} but not in V_j. A single function $\psi(\cdot)$, called *wavelet*, characterizes the spaces $\{W_j\}$: each space W_j is spanned by the base $\{\psi_{j,k}(\cdot) \mid k \in Z\}$, where $\psi_{j,k}(x) = \sqrt{2^j}\psi(2^j x - k)$. The details are therefore represented as a linear combination of (equally spaced) translated copies of $\psi(\cdot)$, whose scale also doubles every layer. The coefficients can be obtained by projecting the measured signal onto the wavelets and the scaling functions. When the signal is digitized, MRA theory allows to design a fast algorithm for Wavelet Decomposition, the *cascade algorithm*, which decomposes the signal by convolving it with two (suitable) mirror quadrature Finite Impulse Response (FIR) filters [4]. To obtain the multi-resolution description, the convolution is iterated on the coefficients which are obtained at each pass of the convolution.

More formally[1], given a sequence f obtained by regularly sampling the signal $f(\cdot)$, the approximation coefficients of the first layer are obtained as $a_1 = \downarrow 2(f * \tilde{g})$, and the detail coefficients as $d_1 = \downarrow 2(f * \tilde{h})$, where $\tilde{g}$ and $\tilde{h}$ are respectively the lowpass and the highpass decomposition FIR filters (called *dual* filters) correspondent to the considered MRA. $\downarrow 2(\cdot)$ is a subsampling operator: it discards one every two samples. The procedure is iterated in the higher layers using the coefficients computed in the previous layer: $a_j = \downarrow 2(a_{j-1} * \tilde{g})$ and $d_j = \downarrow 2(a_{j-1} * \tilde{h})$. After n iterations the signal f is represented by the following collection of coefficients $\{a_n, d_1, ..., d_n\}$. Given a_n and d_n, the approximation at a higher resolution level can be computed as $a_{n-1} = \uparrow 2(a_n)*g + \uparrow 2(d_n)*h$, where g and h are the lowpass and highpass reconstruction FIR filters, and $\uparrow 2(\cdot)$ is a supersampling operator which insert zeros in between the coefficients. The filters g and h are related by orthogonality or biorthogonality to $\tilde{g}$ and $\tilde{h}$.

Analitically, the original signal $f(x)$ can be represented as:

$$\tilde{f}(x) = a_n(x) + \sum_{j=1}^{n} d_j(x) \qquad (1)$$

where $a_n(\cdot)$ is the approximation at the n-th level of resolution, $a_n(x) = \sum_k a_{n,k} \varphi_{n,k}(x)$. Similarly, the details functions, $d_j(\cdot)$, are a linear combination of the wavelet

functions: $d_j(x) = \sum_k d_{j,k} \psi_{j,k}(x)$. Substituting the previous expressions in (1), it turns out:

$$\tilde{f}(x) = \sum_k a_{n,k} \varphi_n(x) + \sum_{j=1}^{n} \sum_k d_{j,k} \psi_{j,k}(x) \qquad (2)$$

B. Hierarchical Radial Basis Functions

In their original formulation, HRBF networks have been designed for range data (sparse data); their configuration algorithm is reframed here to the regular sampling case.

A HRBF network is composed of a hierarchical set of subnetworks, $\{a_j(\cdot)\}$, called *layers*. The j-th layer is composed of regularly spaced Gaussian units, with the same variance σ_j, and the output of the layer is constituted of a linear combination of them:

$$a_j(x) = \sum_k a_{j,k} g_{j,k}(x) =$$
$$= \sum_k a_{j,k} \exp\left(-(x - k\Delta x_j)^2 / \sigma_j^2\right)/(\sqrt{\pi}\sigma_j)^D \qquad (3)$$

where D is the input space dimension.

The ensemble of the Gaussian units of each layer, j, can therefore be seen as a function basis, which spans the input space at the scale j. The first layer, $a_1(\cdot)$ features the largest scale and it captures only the average behavior of the measurement field. The higher layers feature smaller scales and are devoted to reconstruct the details. To the scope, a residual function is computed at the output of each layer j, as the difference between the signal and the sum of the sum of the outputs of the first j layers:

$$r_j(x) = f(x) - \sum_{k=0}^{j} a_j(x) \qquad (4)$$

The configuration procedure is depicted in Fig 1c.

Equation (2) can also be regarded as a low-pass Gaussian filter. Under this perspective, it is possible to determine a relationship between the scale σ_j and the spacing between the units, Δx_j, by giving a maximum attenuation in the pass band and a maximum amplitude in the stop band [5].

Moreover, linear filtering theory can be used to design a non-iterative configuration algorithm. The simplest choice is to substitute the residual values $r_{j-1}(\cdot)$, sampled in the points $k \cdot \Delta x_j$, times Δx_j, to the coefficients $a_{j,k}$ in (2). This is a poor choice when samples are affected by noise. Exploiting the correlation between neighbor data, a better result can be obtained. We propose here a new schema to compute the coefficients $a_{j,k}$ which takes full advantage of the regular spacing of the data points. This is based on projecting the

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[1] In order to simplify the notation, vectors will be typed in italics bold, while function and scalars in italics plain.
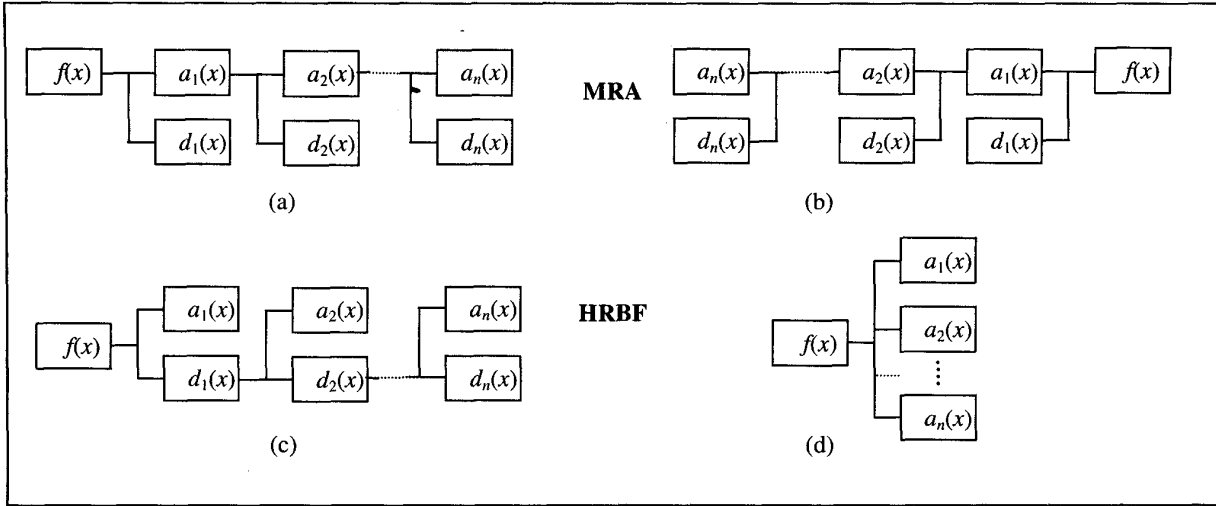
Fig. 1: Analysis and Synthesis with MRA and HRBF. (a) MRA determination of the coefficients; (b) MRA Reconstruction; (c) HRBF determination of the coefficients; (d) HRBF Reconstruction.

sampled data on the same Gaussian basis j and then down-sampling the result. The $a_{j,k}$ are therefore computed as:

$$a_j = \downarrow 2^{M-j}(r_{j-1}*g_j) \tag{5}$$

where M is the maximum number of layers. the function samples f are considered as residuals, r_0. for the first layer ($r_0 = f$). All the data points are considered in computing the weights $a_{j,k}$. We explicitly notice that (3) is equivalent to the analysis through filtering in the MRA.

The filter $g_j(\cdot)$ is obtained from $g_{j-1}(\cdot)$ by contracting it by a factor of two, similarly to the MRA scaling function:

$$g_{j+1,k}(x) = 2g_{j,k}(2x). \tag{6}$$

Downsampling (5) and contraction of the Gaussian basis (6) by a factor of two is not mandatory in the HRBF as it was in the MRA framework. An arbitrary integer value can be adopted and even a different integer value for the various layers.

The configuration procedure is iterated until a stop criterion is met, e.g. the predetermined number of level is reached or until a uniform error over the entire input domain is achieved. At the end, the original signal f is represented by the collection of HRBF coefficients $\{a_1, ..., a_n\}$:

$$\tilde{f}(x) = \sum_{j=1}^{n} a_j(x) = \sum_{j=1}^{n} \sum_{k} a_{j,k} g_{j,k}(x). \tag{7}$$

Once the network has been configured, it offers in a fast way a multi-scale approximation of the signal.

III. COMPARISON

A. Theoretical comparison

Although HRBF and MRA both offer a multi-resolution approximation, they work in a different way. MRA decomposes the signal decreasing the level of detail layer by layer (Fig 1a), where the first layer is constituted by the signal measured samples. HRBF, instead, works the other way round: the least detailed approximation is obtained first (Gaussian basis with large variance) and details are progressively added as the number of layers increases and the variance decreases (Fig. 1c).

MRA filters are generally shorter than a digital FIR implementation of the Gaussian filter, which, due to its large transition band, spans at least eight samples [3]. Moreover, the number of coefficients in a MRA is equal to the number of data points, N, while in HRBF it is equal to $2N - 1$. However, in practical applications, after zeroing the smallest coefficients the number of coefficients left is usually the same. HRBF, on the contrary, is much simpler as it needs only one filter while MRA requires two pairs (g, h, $\tilde{g}$, $\tilde{h}$), one for analysis and one for synthesis. As we will see in the experimental results, the configuration algorithm is also more error tolerant as it is based on the computation of the residual which can easily incorporate errors in the previous computations. Moreover, many basis functions used in MRA do not have an analytical expressions and are able to give the measurement field values only in the sampled points. The spacing in between them has to be interpolated while with

HRBF a continuos measurement field is directly output. Moreover, MRA is cast for a digital implementation, while HRBFs are suitable for both a digital and a hybrid implementation where the coefficients are stored digitally and the Guassians can be either digitalized (in a FIR filter) or computed analogically.

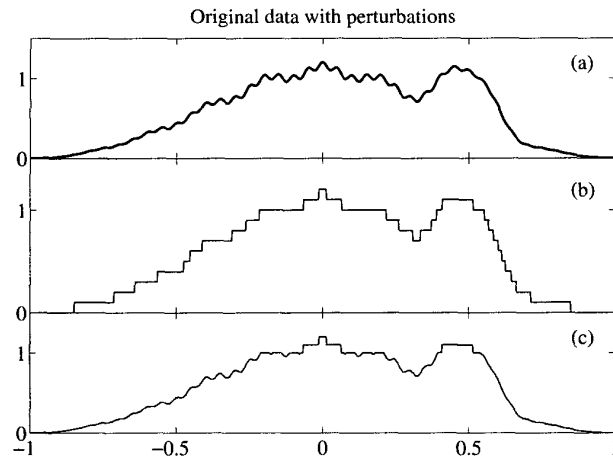## B. Experimental comparison

Original data with perturbations

Fig. 2: Original data (a), and the same signal affected by perturbation – fixed (b) and floating (c) point-like –.

The quality of the reconstruction of a signal depends on the accuracy in the parameters determined in the configuration phase.

We simulated the effect of quantization which affects the computation both in fixed and floating point implementation for MRA and HRBF algorithms. Fixed point notation is extensively used in hardware implementation since it allows circuitry simplification. The use of this notation involves – given the fixed number of bits available – a compromise between the range of the possible numbers and the resolution (i.e. the gap between two consecutive possible numbers). To simulate the fixed point implementation, the parameters value are constrained to assume a finite number of (equally spaced) values by rounding them. The reconstruction accuracy is evaluated for quantization steps 1/10000, 1/1000, 1/100, 1/50 and 1/10 of the maximum absolute value of the elements of the parameters vector considered.

Floating point notation allows to represent the data up to a given relative accuracy. In the floating point representation, the parameters are rounded according to the given number of bits dedicated to the mantissa and exponent. In our simulation, we limit the bits restrictions to the mantissa. This is equivalent to give a relative precision (namely, we used from two to six decimal digits) of the actual parameters value

(truncation). Computation precision is considered infinite for the scope of the experiment.

The parameter sets considered in the experiments are: the input data, f, the filters coefficients, g for HRBF and $\{g, h, \tilde{g}, \tilde{h}\}$ for the MRA, and the basis functions coefficients, $\{a_1, ..., a_n\}$ for HRBF and $\{a_n, d_1, ..., d_n\}$ for the MRA. Each session is characterized by both the noise type and the parameters set which is corrupted. In each session, HRBF and MRA receive the input data f and calculate the approximation coefficients. Error in the reconstruction is assessed through a different data set f_V. f and f_V are obtained by regularly sampling the function $f(\cdot)$ reported in Fig. 2 in 1,001 and 32,000 points respectively.

In each of the six plots of Figs 3 and 4, the results obtained perturbing the three sets of parameters in the two modalities (fixed and floating point) are reported. The reconstruction accuracy is measured as the standard deviation of the (signed) difference between the original function and the synthesized one. Biorthogonal wavelets 3.7 have been used in the MRA.

IV. RESULTS AND CONCLUSIONS

As Figs 3 and 4 show, HRBF does not loose accuracy when errors occur in the configuration procedure. This is due to two main reasons. The first reason is that MRA can loose the biorthogonality properties of its basis functions. The second and main reason is that the residual computed in the HRBF configuration procedure for the layer j (4) incorporates the errors introduced at the output of the previous layer as errors in the filter or approximation coefficients or in the computation per se. This allows error correction in the higher layers. Only the error introduced in the last layer cannot be corrected, but, as the residual decreases with the number of layers, it is of small amplitude its impact on the reconstruction is very small. This is not the case in MRA where the error in the coefficients propagates through the cascade algorithm.

Overall, the results suggest that HRBF networks can be a much more robust tool for hardware implemented multiresolution reconstruction of measurement fields.

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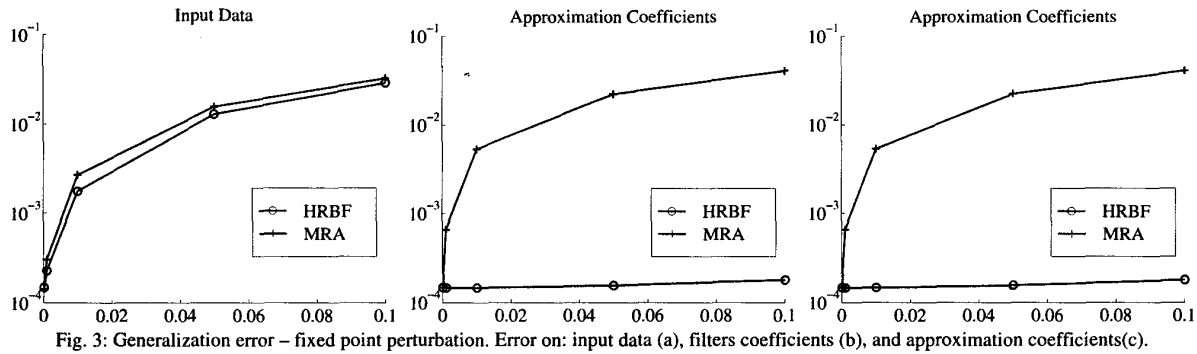
Fig. 3: Generalization error – fixed point perturbation. Error on: input data (a), filters coefficients (b), and approximation coefficients(c).
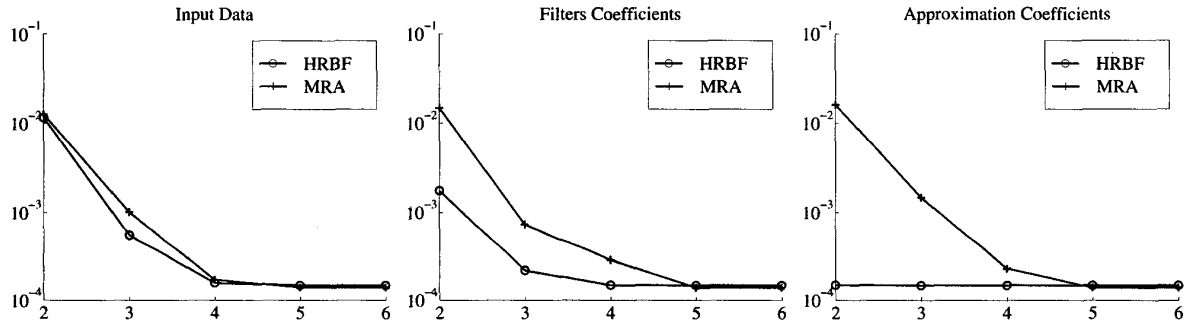

Fig. 4: Generalization error – fixed point perturbation. Error on: input data (a), filters coefficients (b), and approximation coefficients(c).