.

Fourier transform of images

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Methods for Image Processing

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Extension to bidimensional domain

- The concepts introduced for the monodimensional domain can be extended for the multidimensional case:
  - Impulse, δ
  - Convolution
  - Fourier transform
  - Sampling theorem
- In particular, we are interested to the bidimensional domain.

Impulse

The Dirac delta function, δ, or impulse, is defined as:

$$\delta(t,z) = \begin{cases} \infty, & t = z = 0 \\ 0, & t \neq 0,\ z \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)\,\mathrm{d}t\,\mathrm{d}z = 1$$

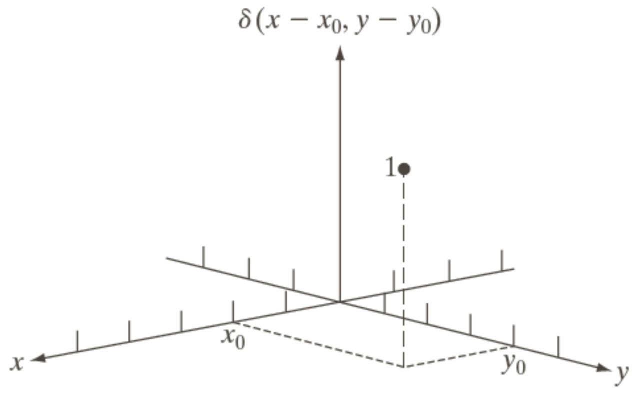
The *sifting property* holds also in this case:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\,\delta(t-t_0, z-z_0)\,\mathrm{d}t\,\mathrm{d}z = f(t_0,z_0)$$

Impulse (2)

The discrete version of δ for the bidimensional case:

$$\delta(x,y) = \begin{cases} 1, & x = y = 0 \\ 0, & \text{otherwise} \end{cases}$$



2D continuous Fourier transform pair

$$F(\nu, \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z)\, e^{-\iota 2\pi(\nu t + \mu z)}\, \mathrm{d}t\, \mathrm{d}z$$

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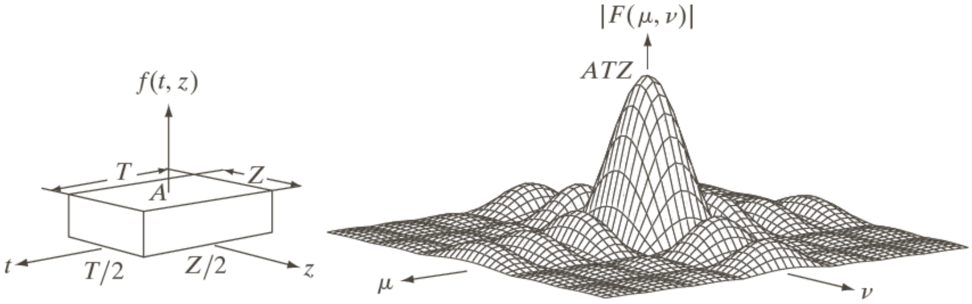
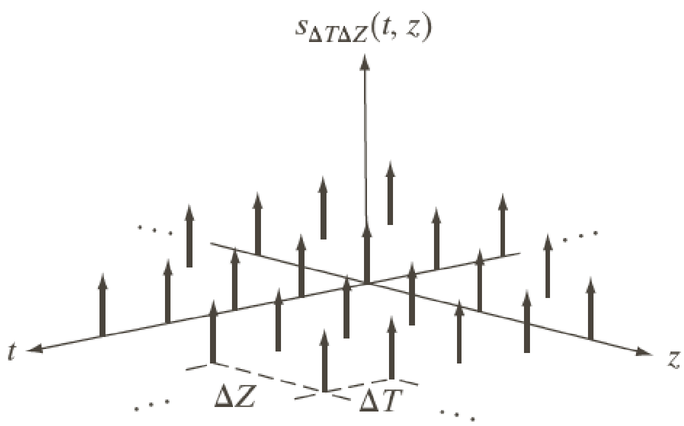


2D sampling theorem

$$\tilde{f}(t, z) = f(t, z)\, s_{\Delta T \Delta Z}(t, z) = \sum_{m,\, n=-\infty}^{\infty} f(t)\, \delta(t - n\Delta T,\ z - m\Delta Z)$$

$$\frac{1}{\Delta T} > 2\, \nu_{\max} \qquad \text{and} \qquad \frac{1}{\Delta Z} > 2\, \mu_{\max}$$



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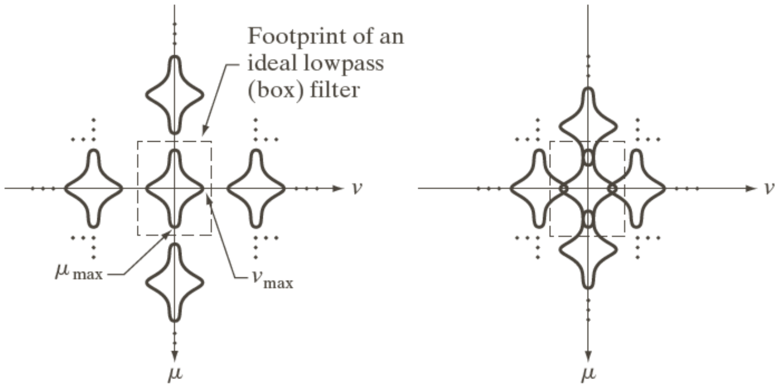
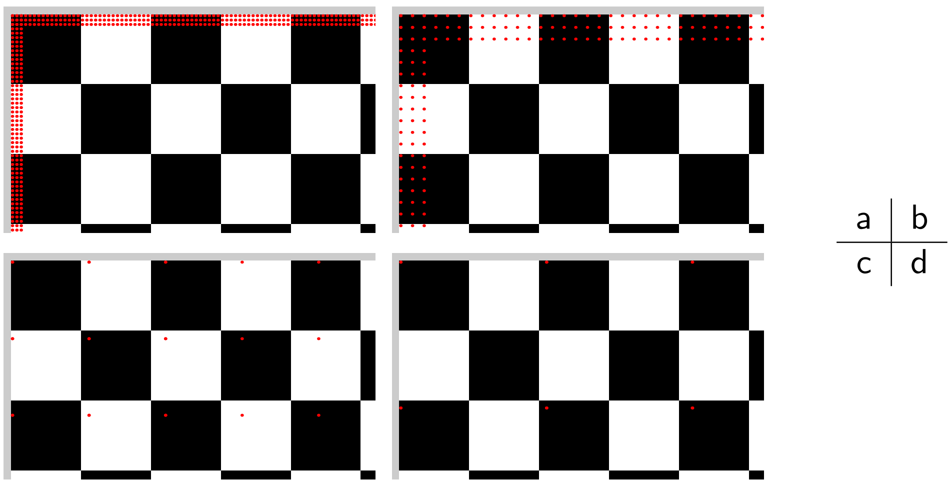
Aliasing in images



- Aliasing effects: false borders and jagged edges.
- Reduction:
  - smoothing before sampling
  - oversampling and averaging
- Reduction (post)
  - smoothing

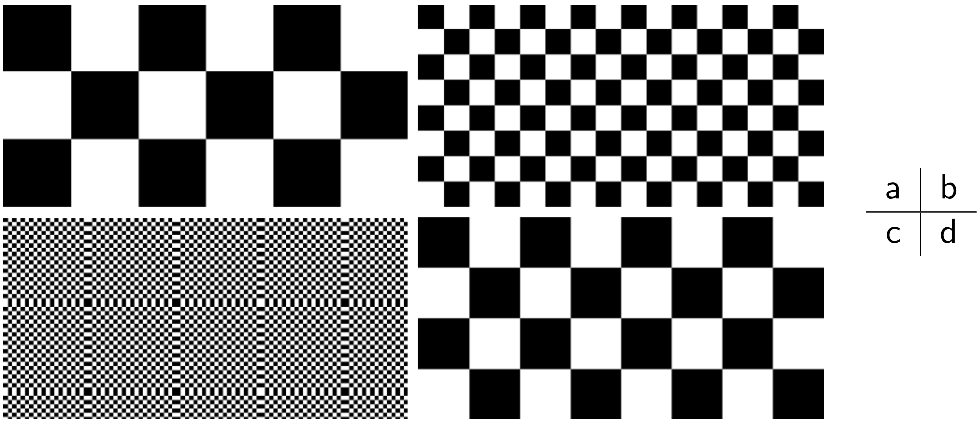
Aliasing example: sampling a checkerboard



Sampling a checkerboard pattern where the sides of the squares are 96 units long.

(a) $\Delta T = \Delta Z = 6$
(b) $\Delta T = \Delta Z = 16$
(c) $\Delta T = \Delta Z = 105$
(d) $\Delta T = \Delta Z = 200$

Aliasing example: sampling a checkerboard (2)



Sampling a checkerboard pattern where the sides of the squares are 96 units long.

(a) $\Delta T = \Delta Z = 6$

(b) $\Delta T = \Delta Z = 16$

(c) $\Delta T = \Delta Z = 105$

(d) $\Delta T = \Delta Z = 200$

Resampling and interpolation

a | b | c

(a) Original image

(b) Resampled image

(c) Applying smoothing before resampling

Note: Resampling has been operated through rows and columns deletion.

Resampling and interpolation (2)

a | b | c

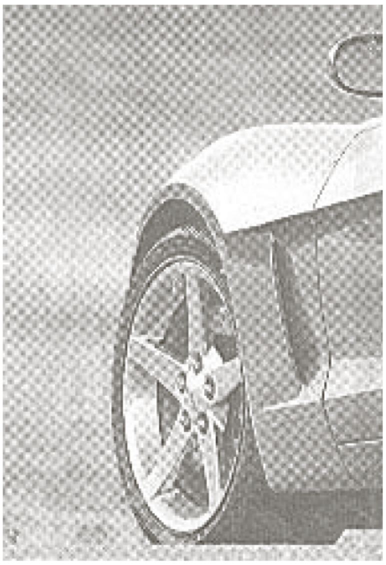
(a) Zooming by pixel replication

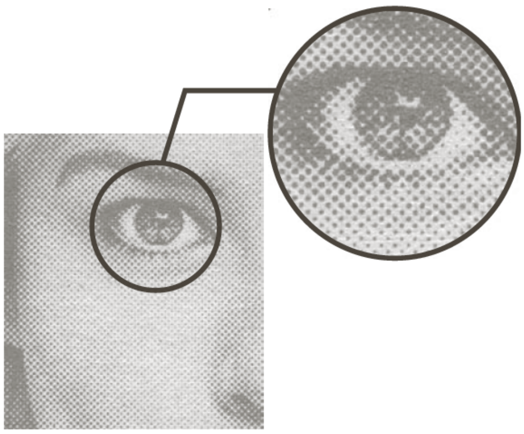
(b) Zooming by pixel bicubic interpolation

(b) Zooming by pixel sinc interpolation

Resampling and interpolation (3)

The moirè effect is caused by the superimposition of two periodical patterns.





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Bidimensional DFT pair

$$F(u,\, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,\, y) e^{-\imath 2\pi (ux/M + vy/N)}$$

$$f(x,\, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,\, v) e^{\imath 2\pi (ux/M + vy/N)}$$

where

$$u = 0,\, \ldots,\, M-1 \qquad v = 0,\, \ldots,\, N-1$$

$$x = 0,\, \ldots,\, M-1, \qquad y = 0,\, \ldots,\, N-1$$

DFT properties

- Traslation

$$\mathcal{F}\{f(x,y)\, e^{\imath 2\pi (u_0 x/M + v_0 y/N)}\} = F(u-u_0,\, v-v_0)$$

$$\mathcal{F}\{f(x-x_0, y-y_0)\} = F(u,\, v)\, e^{-\imath 2\pi (x_0 u/M + y_0 v/N)}$$

  - Multiplying f by an exponential produces a shift in the DTF.
  - Translating f has the effect of multiplying its DFT.
- Rotation
  - Rotating f produces an identical rotation in its DFT.

DFT properties (2)

- Periodicity

$$F(u,\, v) = F(u + k_1 M,\, v + k_2 N)$$

$$f(x,\, y) = f(x + k_1 M,\, y + k_2 N)$$

where $k_1,\, k_2 \in \mathbb{Z}$

$$\mathcal{F}\{f(x,\, y)(-1)^{x+y}\} = F(u - M/2,\, v - N/2)$$

DFT properties (3)

- Simmetry
  - Even (symmetric) functions

$$f(x,\, y) = f(-x,\, -y)$$

  - Odd (antisymmetric) functions

$$f(x,\, y) = -f(-x,\, -y)$$

Symmetry properties in f involve corresponding properties in F that are useful in processing.
E.g.: If f is real and even, also F is real and even.

Fourier spectrum and phase angle

- The DFT can be expressed in polar form:

$$F(u, v) = |F(u, v)|\, e^{\imath\phi(u, v)}$$

where $|F(u, v)|$, called *Fourier spectrum*:

$$|F(u, v)| = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$$

and $\phi(u, v)$, called *phase angle*:

$$\phi(u, v) = \arctan\left(\frac{I(u, v)}{R(u, v)}\right)$$

- The *power spectrum*, $P(u, v)$, is defined as:

$$P(u, v) = |F(u, v)|^2 = R^2(u, v) + I^2(u, v)$$

Fourier spectrum and phase angle (2)

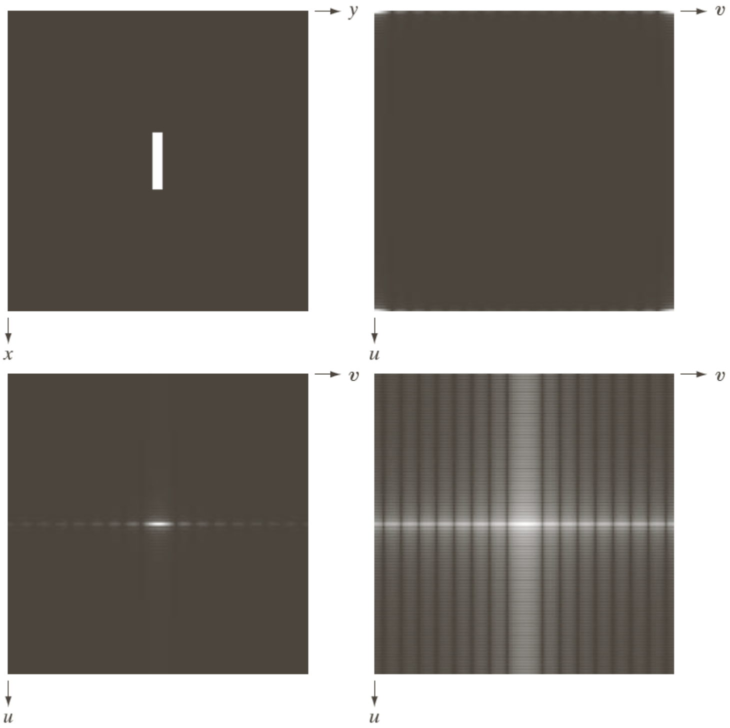
- It can be shown that:

$$|F(0, 0)| = MN|\bar{f}(x, y)|$$

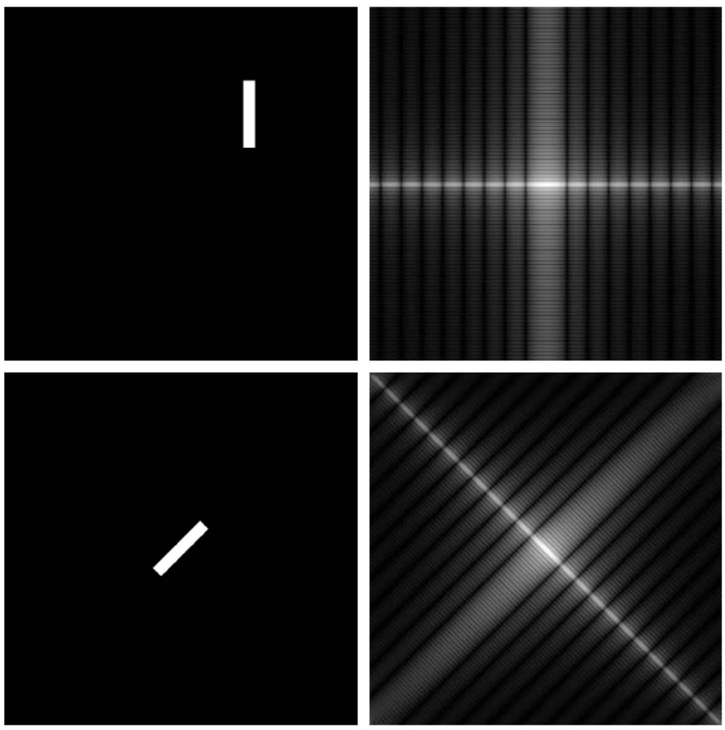
where $\bar{f}$ is the f average value.

$F(0, 0)$ is generally much larger than the other terms of F;
  - logarithmic transform for displaying it.

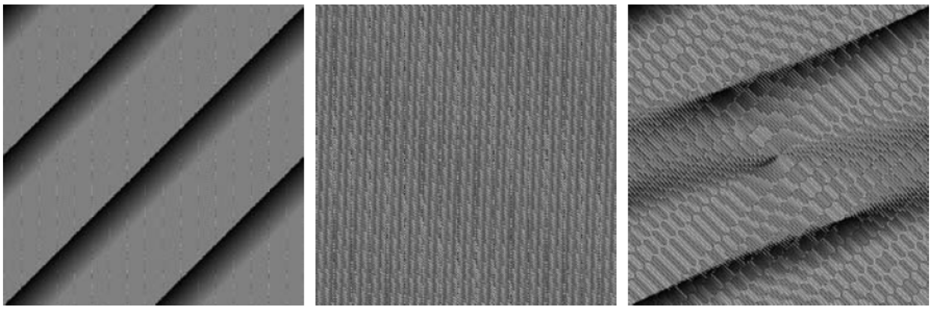
Fourier spectrum and phase angle (3)



Fourier spectrum and phase angle (4)



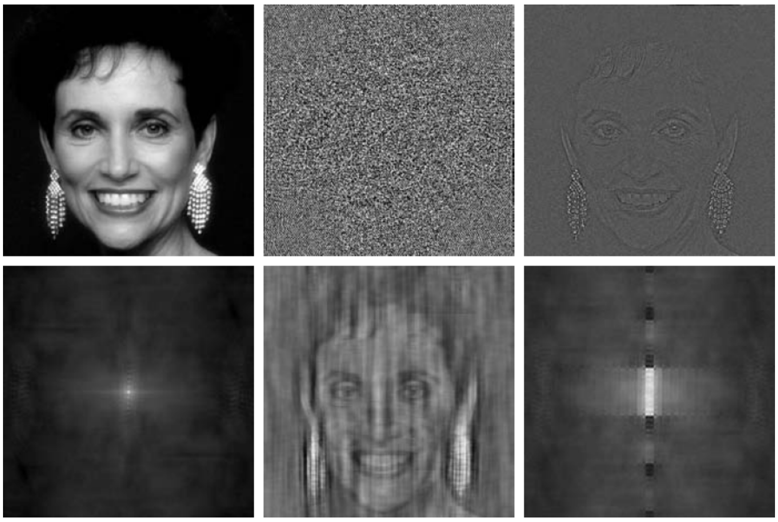
Fourier spectrum and phase angle (5)



a | b | c

(a) phase angle of the centered rectangle image;
(b) phase angle of the shifted rectangle image;
(c) phase angle of the rotated rectangle image;

Fourier spectrum and phase angle (6)



a	b	c
d	e	f

- (b): phase angle of (a);
- (c) and (d): IDFT(phase angle of (a)) and IDFT(spectrum of (a));
- (e): IDFT(phase angle of the woman + spectrum of the rectangle);
- (f): IDFT(spectrum of the woman + phase angle of the rectangle).

2D convolution theorem

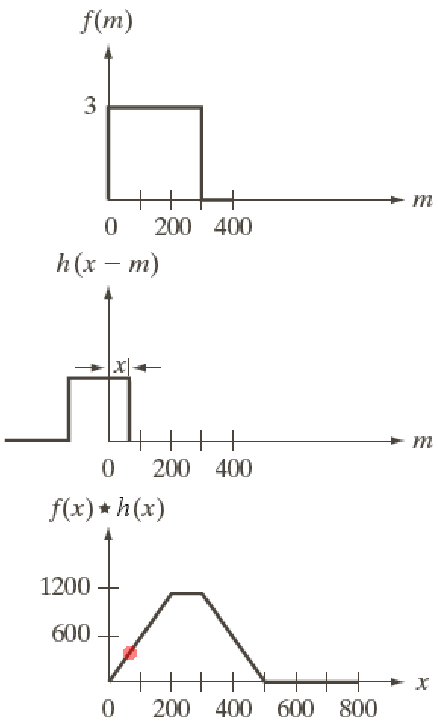
- The convolution theorem can be formulated for the 2D DFT:

$$\mathcal{F}\{f(x, y) * h(x, y)\} = F(u, v)\, H(u, v)$$

$$\mathcal{F}\{f(x, y)\, h(x, y)\} = F(u, v) * H(u, v)$$

- The circular convolution has to be used.

Convolution



$$f(x)*h(x) = \sum_{m=0}^{M-1} f(x)h(x-m)$$

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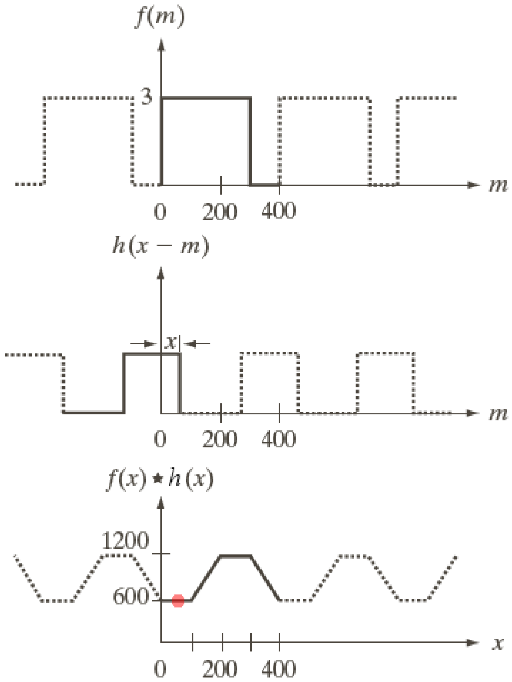
Circular convolution



$$f(x) * h(x) = \\ = \sum_{m=0}^{M-1} f(x)h(x-m)$$

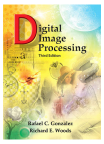
Wraparound error

- The (circular) convolution of two periodic function can cause the so called *wraparound error*.
- It can be resolved using the *zero padding*.
  - Giving two sequences of respectively A and B samples, append zeros to them such that both will have P elements:

$$P = A + B - 1$$

- If a function is not zero at the end of the interval, the zero padding introduces artifacts:
  - High frequency components in the transform.
- Attenuation with the windowing technique:
  - e.g., multiplying by a Gaussian.

Homeworks and suggested readings



DIP, Sections 4.5–4.6

- pp. 225–254

GIMP

- Image
  - Scale Image
    - Cubic
    - Sinc