Fourier transform

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Methods for Image Processing

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Function transforms

- Sometimes, operating on a class of functions is easier if the functions are expressed in another domain.
- Hence a given function is transformed in another function on a different domain (the codomain of the transform).
- Time (or space) functions can be transformed in functions defined on another domain.
- The operations applied on the original function domain correspond to simpler operations when applied on the transformed function.
- Once processed, the transformed function can be back-transformed to the original domain by way of a suitable inverse transform.

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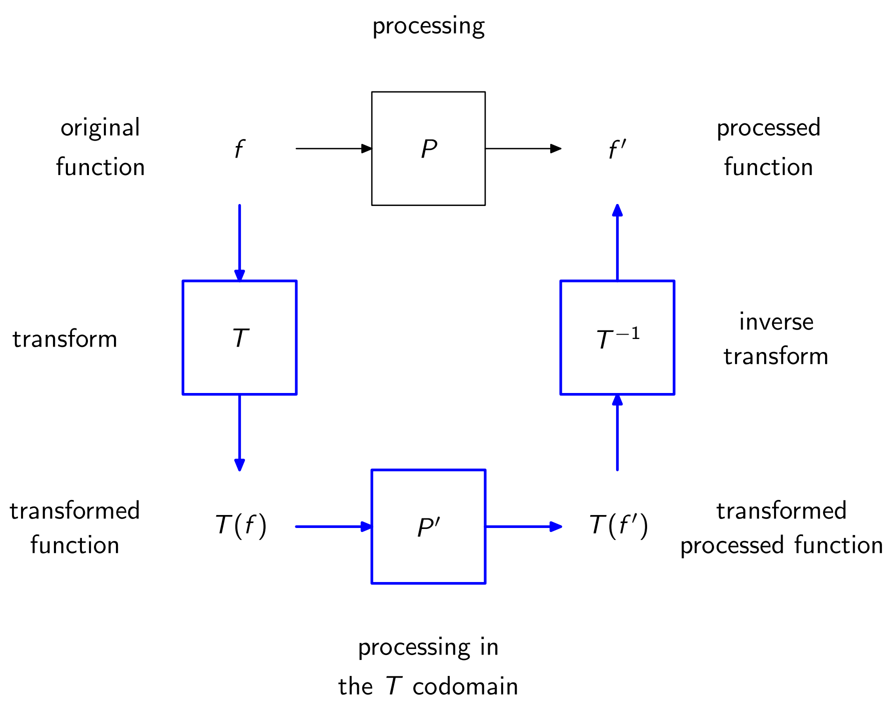
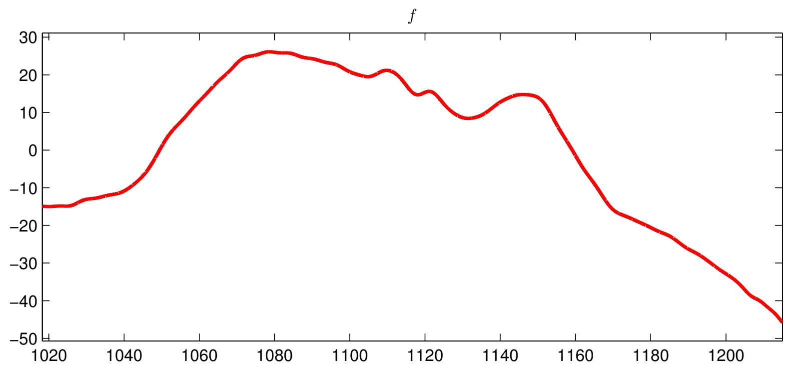
Function transforms (2)

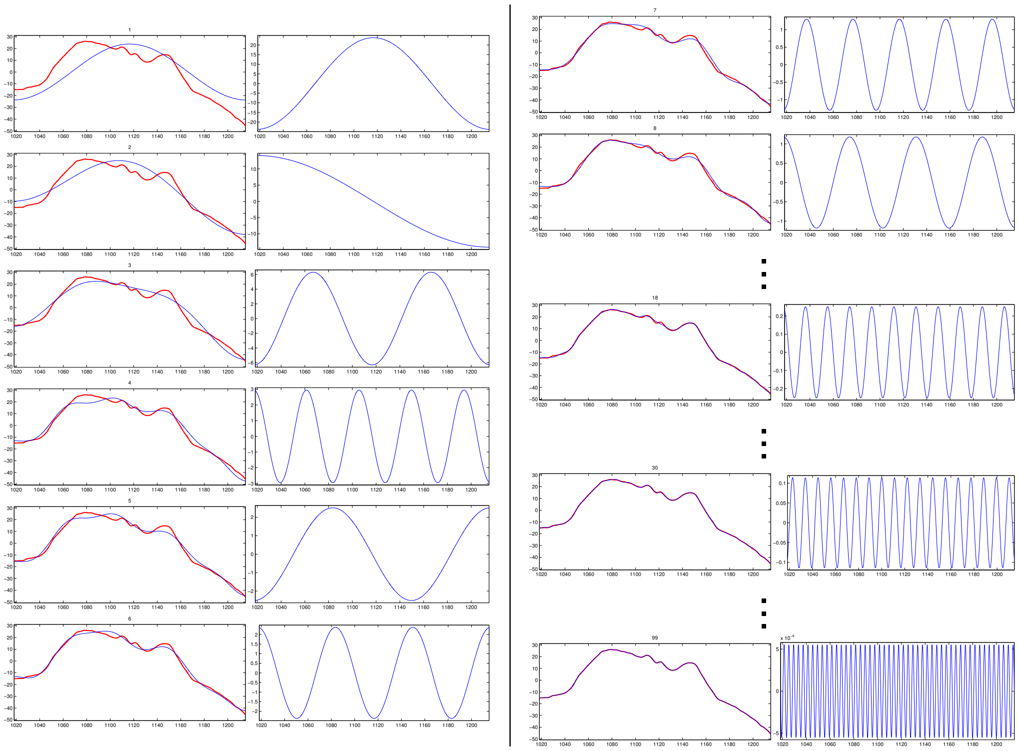


Example: Discrete Cosine Transform



Through the Discrete Cosine Transform, the function f can be expressed in terms of sum of weighted (scaled) cosinusoidal functions.

Example: Discrete Cosine Transform (2)



Fourier series

Every periodic function, f, with period T, can be represented as a linear combination of sines and cosines:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\iota \frac{2\pi n}{T} t}$$

where

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-\iota \frac{2\pi n}{T} t} \mathrm{d}t$$

Note: the base is composed of an infinite set of sines and cosines.

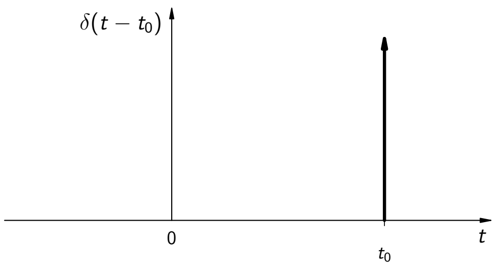
Impulse

The Dirac delta function, δ, or impulse, is defined as:

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\, \mathrm{d}t = 1$$



Impulse (2)

The impulse has the interesting property, called the *sifting property*:

$$\int_{-\infty}^{\infty} f(t)\, \delta(t)\, \mathrm{d}t = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, \mathrm{d}t = f(t_0)$$

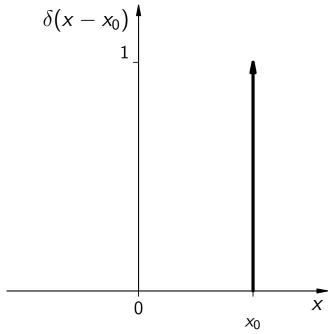
Hence, it can be used for sampling a function, f, by convolution.

Discrete impulse

For working with functions defined only in a discrete set of points (e.g., $f : \mathbb{Z} \rightarrow \mathbb{R}$), the discrete impulse can be defined:

$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$



The sifting property is satisfied also in this case:

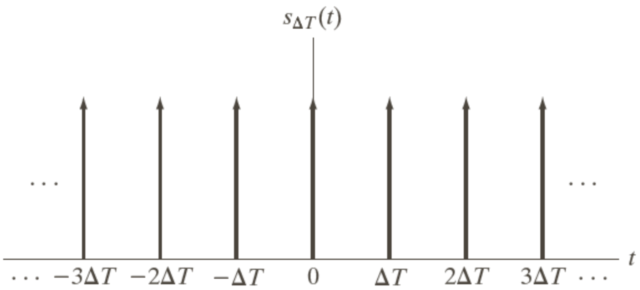
$$\sum_{x=-\infty}^{\infty} f(x)\,\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\,\delta(x - x_0) = f(x_0)$$

Impulse train

The impulse train, $s_{\Delta T}$, will be important:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



$s_{\Delta T}$ is a periodic function with period ΔT.

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Continuous Fourier transform

Under mild conditions, for every function $f(t)$ the *continuous Fourier transform* can be computed as:

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-\iota 2\pi\nu t}\, \mathrm{d}t = F(\nu)$$

Since t is integrated, the Fourier transform of $f(t)$ is a function of the variable ν.
Hence it is usually indicated as $\mathcal{F}\{f(t)\} = F(\nu)$.

The Fourier transform describes $f(t)$ as a linear (complex) combination of sines and cosines:

$$F(\nu) = \int_{-\infty}^{\infty} f(t)\ [\cos(2\pi\nu t) - \iota \sin(2\pi\nu t)]\ \mathrm{d}t$$

since:

$$e^{\iota\theta} = \cos\theta + \iota\sin\theta$$

Continuous Fourier transform (2)

A complementary transform, called *inverse Fourier transform* can be defined:

$$f(t) = \mathcal{F}^{-1}\{F(\nu)\} = \int_{-\infty}^{\infty} F(\nu)\, e^{\iota 2\pi\nu t}\, \mathrm{d}\nu$$

It allows to obtain $f(t)$ from $F(\nu)$.
The $\mathcal{F}\{f(t)\}$ and $\mathcal{F}^{-1}\{F(\nu)\}$ transforms are called the *Fourier transforms pair*.

For some operations, the spectrum of the transform carries valuable information:

$$\|F(\nu)\|$$

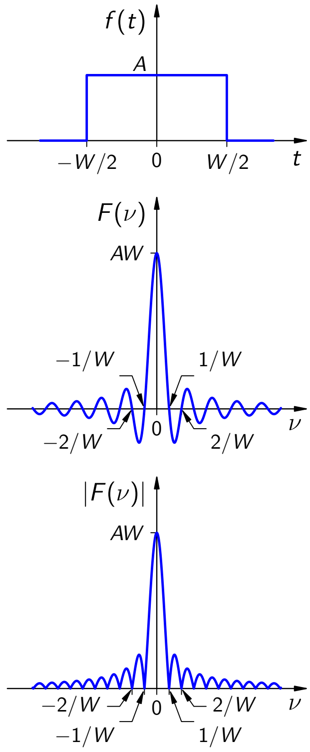
- the coefficient of the basis function gives the relative importance of the basis function in the representation.

Example: FT of the box function

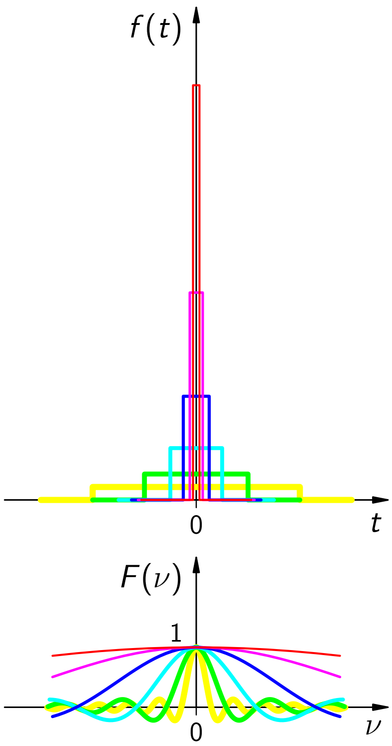
$$f(t) = \begin{cases} A, & -W/2 \le t \le W/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} F(\nu) &= \int_{-\infty}^{\infty} f(t)\, e^{-\iota 2\pi\nu t}\, \mathrm{d}t \\ &= \int_{-W/2}^{W/2} A\, e^{-\iota 2\pi\nu t}\, \mathrm{d}t \\ &= AW \frac{\sin(\pi\nu W)}{\pi\nu W} \\ &= AW \operatorname{sinc}(\nu W) \end{aligned}$$

$$\operatorname{sinc}(t) = \begin{cases} 1, & t = 0 \\ \frac{\sin \pi t}{\pi t}, & \text{otherwise} \end{cases}$$



Example: FT of the box function (2)



- The box function can be exploited for computing the Fourier transform of δ.
- When considering only boxes function where $AW = 1$, the limit for $W \to 0$ is δ:
  - $A \to \infty$,
  - the integral is 1,
  - outside $[-W/2, W/2]$ the function is 0.
- Since the main lobe of $F(\nu)$ extends over $[-1/W, 1/W]$, for $W \to 0$ it tends to occupy the whole real line, $(-\infty, \infty)$.
- Hence, the limit of the Fourier transform of these sequence is the constant function 1.

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Example: FT of the impulse

Using the sifting property of δ, its Fourier transform can be computed directly:

$$\mathcal{F}(\delta(t)) = F(\nu) = \int_{-\infty}^{\infty} \delta(t)\, e^{-\iota 2\pi\nu t}\, \mathrm{d}t = e^{-\iota 2\pi\nu 0} = 1$$

It can be also shown that:

$$\mathcal{F}\{s_{\Delta T}(t)\} = S(\nu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\nu - \frac{n}{\Delta T}\right)$$

Note: $S(\nu)$ is a periodic function with period $\frac{1}{\Delta T}$.

Convolution and Fourier transform

The convolution between two continuous functions, f and h, is defined as:

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\mathrm{d}\tau$$

It can be shown that:

$$\mathcal{F}\{f(t) * h(t)\} = F(\nu)\, H(\nu)$$

Also the opposite holds:

$$\mathcal{F}\{f(t)\, h(t)\} = F(\nu) * H(\nu)$$

This is called the *convolution theorem*.

Sampling

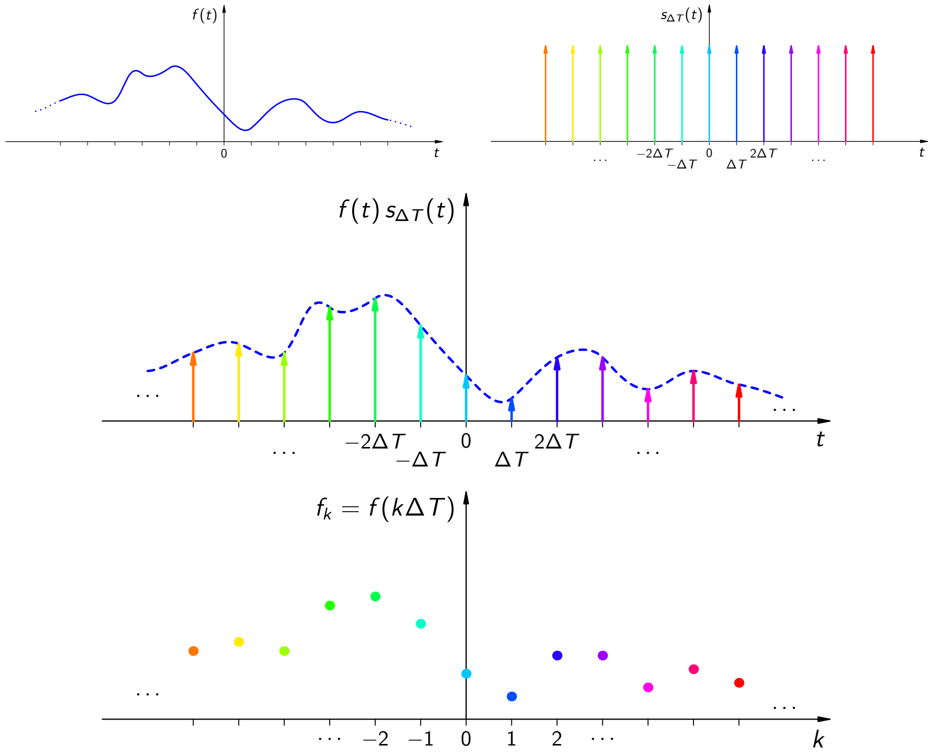
- Sampling is a processing step often required for operating on a function with a digital computer.
- This operation can be modeled as the multiplication of the considered function, f, with the impulse train of a suitable period, ΔT, called *sampling step*:

$$\tilde{f}(t) = f(t)\, s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\, \delta(t - n\Delta T)$$

- The value of each sample results:

$$f_k = \int_{-\infty}^{\infty} f(t)\, \delta(t - k\Delta T)\, \mathrm{d}t = f(k\Delta T), \qquad k \in \mathbb{Z}$$

Sampling (2)



Fourier transform of a sampled function

Using the convolution theorem, the Fourier transform of a sampled function can be computed as:

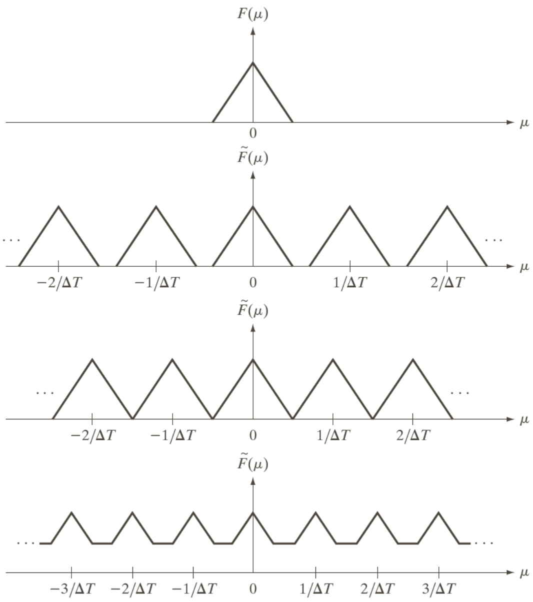
$$\tilde{F}(\nu) = \mathcal{F}\{\tilde{f}(t)\} = \mathcal{F}\{f(t)\, s_{\Delta T}(t)\} = F(\nu) * S(\nu)$$

From the sifting property of δ:

$$\tilde{F}(\nu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\nu - \frac{n}{\Delta T}\right)$$

Hence, the Fourier transform of the sampled function, $\tilde{F}$ is the infinite summation of scaled copies (for a factor of $\frac{1}{\Delta T}$) of the Fourier transform of the original function, F, shifted by $\frac{1}{\Delta T}$ (i.e., $\tilde{F}$ is a $\frac{1}{\Delta T}$ periodic function).

Fourier transform of a sampled function (2)



- Although $\tilde{f}$ is a sampled function, $\tilde{F}$ is continuous.
  - It is the summation of scaled copies of a continuous function, F.
- The separation between the copies is ruled by $\frac{1}{\Delta T}$:
  - oversampling
  - critical sampling
  - undersampling

Sampling theorem

We are interested in recovering the original function, f, from its samples, $\tilde{f}$:

- Is it always possible?
- What are the necessary conditions?

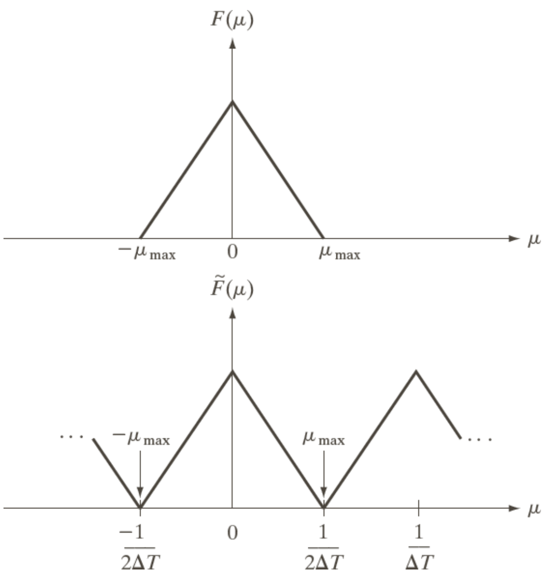
The Fourier transform of a sampled function, $\tilde{F}$, provides infinite copies of F:

- if a single copy of F can be isolated,
- f can be obtained using the inverse Fourier transform.

Since $\tilde{F}$ is composed of shifted copies of F:

- F have to be different from zero only in a finite interval $[-\nu_{\text{max}}, \nu_{\text{max}}]$
  - f is a *band limited function*
- the separation of $\frac{1}{\Delta T}$ must be sufficient to avoid overlapping of copies.

Sampling theorem (2)



- The previous conditions can be summarized in the *sampling theorem*:

$$\frac{1}{\Delta T} > 2\,\nu_{\text{max}}$$

  - The sampling rate must be at least twice the higher frequency component of f.
- The critical sampling rate is called the *Nyquist rate*.

Reconstruction of a sampled function

- Multiplying $\tilde{F}$ by a box function H:

$$H(\nu) = \begin{cases} \Delta T, & -\nu_{\max} \leq \nu \leq \nu_{\max} \\ 0, & \text{otherwise} \end{cases}$$

  a single copy of F can be recovered.
- Then, the inverse Fourier transform can be applied to obtain f.

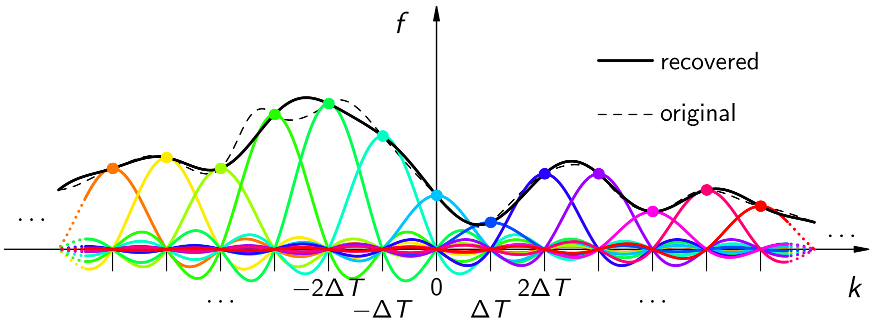
$$f(t) = \mathcal{F}^{-1}\{F(\nu)\} = \mathcal{F}^{-1}\{H(\nu)\,\tilde{F}(\nu)\} = h(t) * \tilde{f}(t)$$

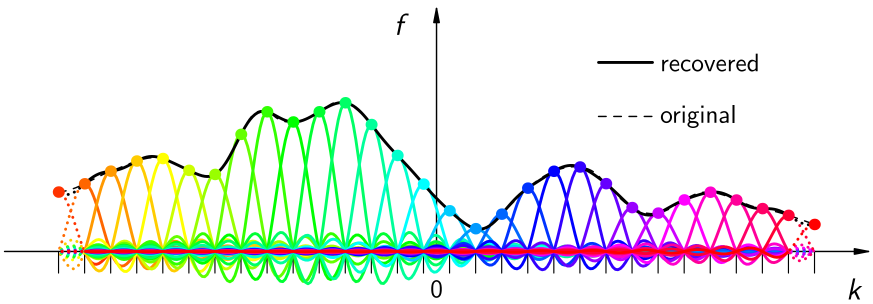
- It can be shown that:

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\,\Delta T)\,\text{sinc}\left(\frac{t - n\,\Delta T}{\Delta T}\right)$$

  - $f(t)$ is equal to f_k in $t = k\,\Delta T$;
  - elsewhere it is obtained using a shifted sinc functions basis.

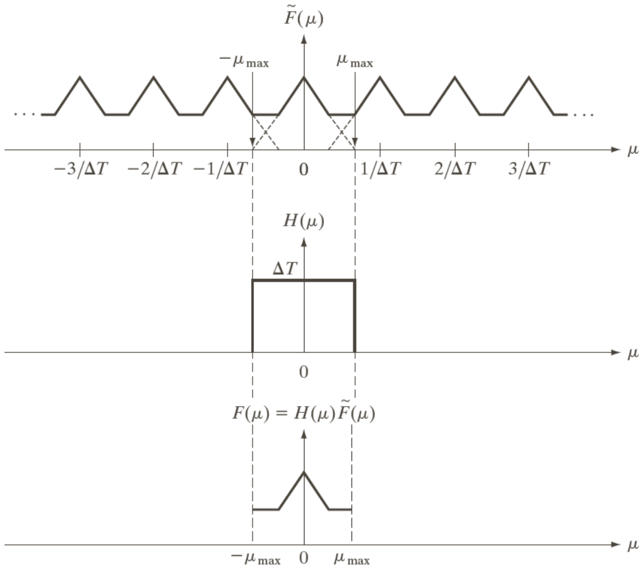
Reconstruction of a sampled function (2)





Aliasing

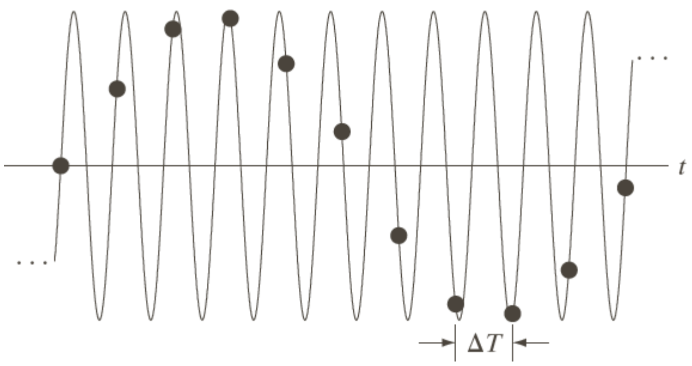
- If the function is sampled under the Nyquist rate ($\frac{1}{\Delta T} < 2\,\nu_{\text{max}}$), the periods overlap.
  - The components between $\frac{1}{2\,\Delta T}$ and ν_{max} are mixed with other components.
- This effect is called *aliasing*.



Aliasing (2)

- For practical cases, aliasing is almost unavoidable.
- Since only a finite interval of the signal can be considered, the processed function can be modelled multiplying the real (band limited) function by a box function.
- For the convolution theorem, the Fourier transform of the processed function is equal to the Fourier transform of the original function convolved with the sinc function (that has infinite support).
  - No function that has finite duration can be band limited.
  - Every band limited function must have infinite support.
- Aliasing effects can be reduced by attenuating the high frequency components (i.e., smoothing) *before* sampling.
  - Or *windowing*: use of a different function instead of the box, such that its spectrum vanishes more rapidly than sinc.

Aliasing: an example



- The sampled function looks like a sinusoidal function having a frequency much smaller than the original function.
- Since the only component of the Fourier transform of the original function is higher than the sampling frequency, its copy in the overlapping period is positioned in a lower frequency.

Discrete Fourier Transform (DFT)

- $\tilde{F}$ can be expressed in terms of $\tilde{f}$:

$$\tilde{F}(\nu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-\iota 2\pi\nu t}\, dt$$

- From which can be shown:

$$\tilde{F}(\nu) = \sum_{n=-\infty}^{\infty} f_n e^{-\iota 2\pi\nu n \Delta T}$$

- Since $\tilde{F}$ is periodic, all the information carried out by $\tilde{F}$ is contained in a single period.

Discrete Fourier Transform (DFT) (2)

- If M samples of $\tilde{F}$ are considered in one period, ΔT, the following frequencies are inspected:

$$\nu_m = \frac{m}{M \Delta T}, \qquad m = 0, \ldots, M-1$$

  and the samples are:

$$F_m = \sum_{n=0}^{M-1} f_n e^{-\iota 2\pi mn/M}$$

- The M samples $\{F_m\}$ are computed using only M samples of f.
- This transform is called the *Discrete Fourier Transform*.

Inverse Discrete Fourier Transform (IDFT)

- The M samples $\{f_n\}$ can be reconstructed from $\{F_m\}$ using the following transformation:

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{\iota 2\pi mn/M}, \qquad n = 0, \ldots, M-1$$

- This transform is called the *Inverse Discrete Fourier Transform*.

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Discrete Fourier Transform pair

- The forward and inverse Fourier transform are usually represented as:

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-\iota 2\pi ux/M}, \qquad u = 0, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{\iota 2\pi ux/M}, \qquad x = 0, \ldots, M-1$$

- No direct reference to time and frequency is present.
  - $\{f_n\}$ and $\{F_m\}$ are just sequences.
- It can be shown that $F(u)$ and $f(x)$ are periodic:

$$F(u) = F(u + kM) \quad \text{and} \quad f(x) = f(x + kM), \quad k \in \mathbb{Z}$$

  - Although if (the original) f is not.

Circular convolution

- The convolution of finite sequence of M elements can be defined through:

$$g(x) = f(x) * h(x) = \sum_{m=0}^{M-1} f(m)\, h(x - m)$$

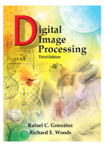
- The periodicity of g derives from the periodicity of f and h.
- This operation is called *circular convolution*.
- Through this operation, the convolution theorem (for continuous FT) can be extended to the DFT.
  - The circularity causes the *wraparound* problem, that will be discussed later.

Space and frequency resolution

- f is composed of M samples taken ΔT apart.
- The sequence covers an interval that is $T = M \Delta T$ long.
- In the frequency domain, the samples of F are $\Delta u = \frac{1}{M \Delta T} = \frac{1}{T}$ apart.
- Hence, the DFT is defined over a frequency interval $\Omega = M \Delta u = \frac{1}{\Delta T}$ long.

- The frequency resolution depends on the length of the sampled interval in the space domain, T.
- The range of the frequencies covered by the DFT depends on the sampling step, ΔT.

Homeworks and suggested readings



DIP, Sections 4.1–4.4

- pp. 199–224