.

# Fourier transform of images

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**Elaborazione delle immagini**
**(Image processing I)**

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## Extension to bidimensional domain

- The concepts introduced for the monodimensional domain can be extended for the multidimensional case:
  - Impulse, $\delta$
  - Convolution
  - Fourier transform
  - Sampling theorem
- In particular, we are interested to the bidimensional domain.

## Impulse

The Dirac delta function, $\delta$, or impulse, is defined as:

$$\delta(t, z) = \begin{cases} \infty, & t = z = 0 \\ 0, & t \neq 0,\ z \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z)\, \mathrm{d}t\, \mathrm{d}z = 1$$

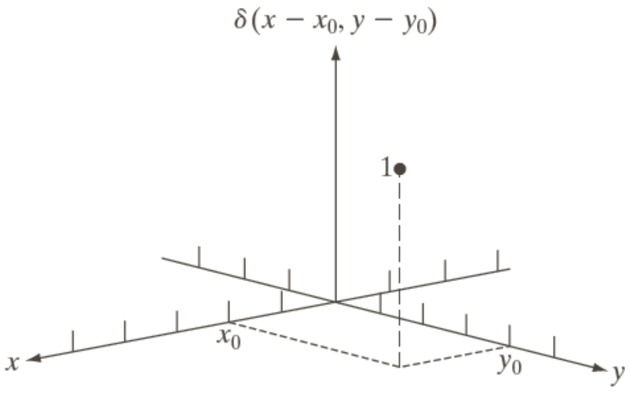
The *sifting property* holds also in this case:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z)\, \delta(t - t_0, z - z_0)\, \mathrm{d}t\, \mathrm{d}z = f(t_0, z_0)$$

## Impulse (2)

The discrete version of $\delta$ for the bidimensional case:

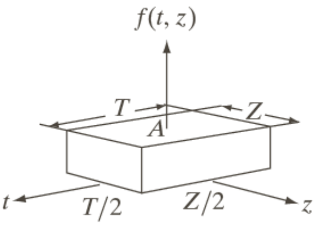
$$\delta(x, y) = \begin{cases} 1, & x = y = 0 \\ 0, & \text{otherwise} \end{cases}$$

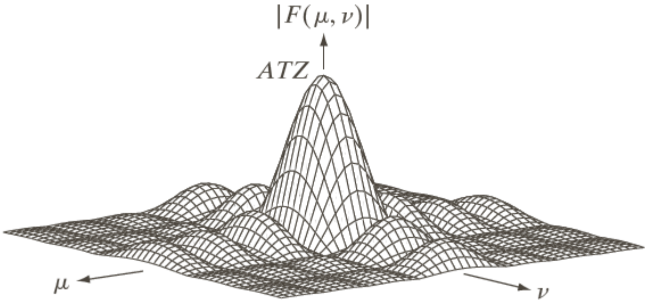


## 2D continuous Fourier transform pair

$$F(\nu,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\, e^{-\iota 2\pi(\nu t+\mu z)}\, \mathrm{d}t\, \mathrm{d}z$$

$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\nu,\mu)\, e^{\iota 2\pi(\nu t+\mu z)}\, \mathrm{d}\nu\, \mathrm{d}\mu$$

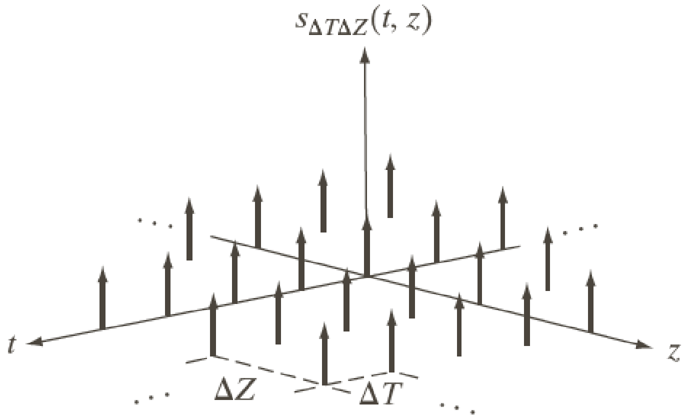


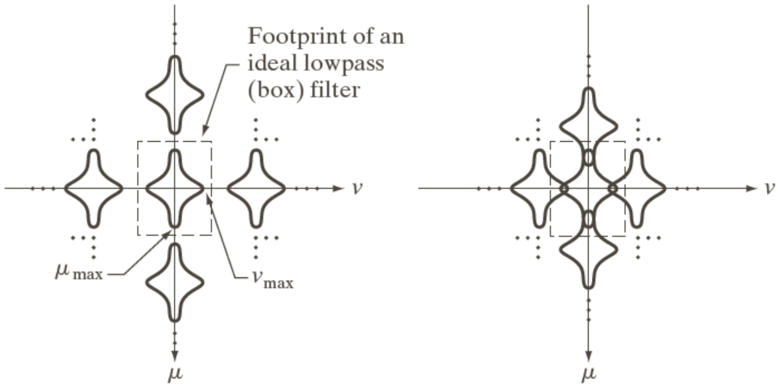

## 2D sampling theorem

$$\tilde{f}(t,z) = f(t,z)\, s_{\Delta T \Delta Z}(t,z) = \sum_{m,\, n=-\infty}^{\infty} f(t)\, \delta(t - n\Delta T,\; z - m\Delta Z)$$

$$\frac{1}{\Delta T} > 2\,\nu_{\max} \qquad \text{and} \qquad \frac{1}{\Delta Z} > 2\,\mu_{\max}$$

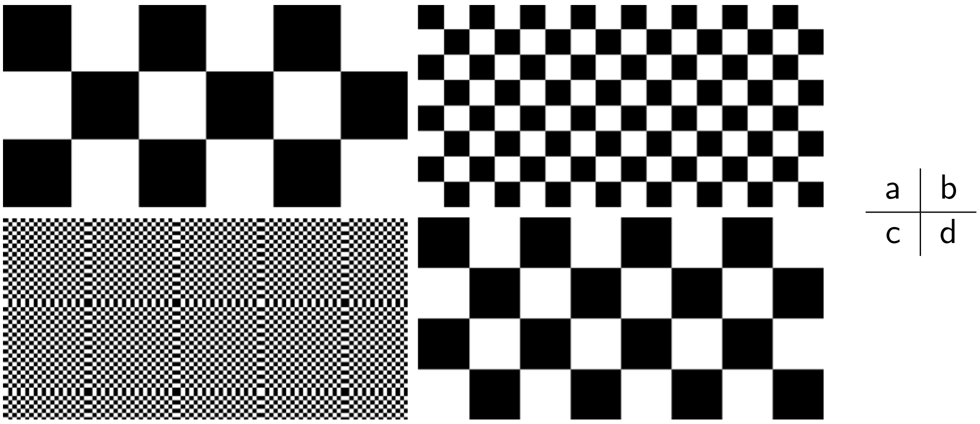


## Aliasing in images




- Aliasing effects: false borders and jagged edges.
- Reduction:
  - smoothing before sampling
  - oversampling and averaging
- Reduction (post)
  - smoothing

## Aliasing example: sampling a checkerboard




Sampling a checkerboard pattern where the sides of the squares are 96 units long.

(a) $\Delta T = \Delta Z = 6$
(b) $\Delta T = \Delta Z = 16$
(c) $\Delta T = \Delta Z = 105$
(d) $\Delta T = \Delta Z = 200$

## Resampling and interpolation

a | b | c

(a) Original image
(b) Resampled image
(c) Applying smoothing before resampling

Note: Resampling has been operated through rows and columns deletion.

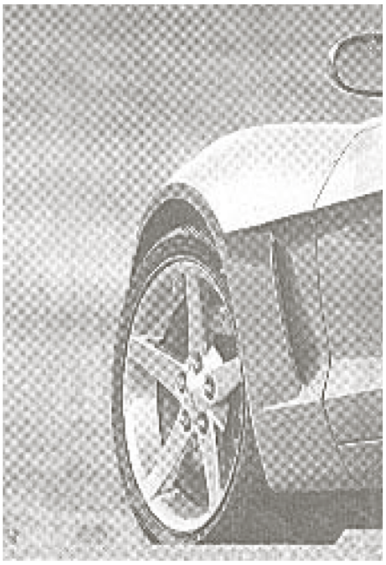
## Resampling and interpolation (2)

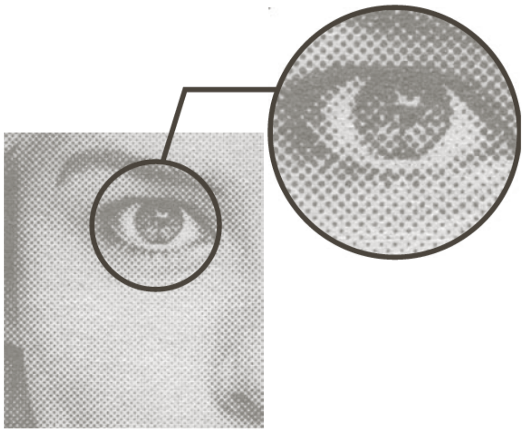
a | b | c

(a) Zooming by pixel replication
(b) Zooming by pixel bicubic interpolation
(b) Zooming by pixel sinc interpolation

## Resampling and interpolation (3)

The moirè effect is caused by the superimposition of two periodical patterns.





## Bidimensional DFT pair

$$F(u,\, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,\, y) e^{-\iota 2\pi (ux/M + vy/N)}$$

$$f(x,\, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,\, v) e^{\iota 2\pi (ux/M + vy/N)}$$

where

$$u = 0,\, \ldots,\, M-1 \qquad v = 0,\, \ldots,\, N-1$$

$$x = 0,\, \ldots,\, M-1, \qquad y = 0,\, \ldots,\, N-1$$

## DFT properties

- Traslation

$$\mathcal{F}\{f(x,y)\, e^{\iota 2\pi(u_0 x/M + v_0 y/N)}\} = F(u-u_0,\; v-_0)$$

$$\mathcal{F}\{f(x-x_0, y-y_0)\} = F(u,\; v)\, e^{-\iota 2\pi(x_0 u/M + y_0 v/N)}$$

  - Multiplying $f$ by an exponential produces a shift in the DTF.
  - Translating $f$ has the effect of multiplying its DFT.
- Rotation
  - Rotating $f$ produces an identical rotation in its DFT.

## DFT properties (2)

- Periodicity

$$F(u,\; v) = F(u + k_1 M,\; v + k_2 N)$$

$$f(x,\; y) = f(x + k_1 M,\; y + k_2 N)$$

where $k_1,\; k_2 \in \mathbb{Z}$

$$\mathcal{F}\{f(x,\; y)(-1)^{x+y}\} = F(u - M/2,\; v - N/2)$$

.

## DFT properties (3)

- Simmetry
  - Even (symmetric) functions

$$f(x, y) = f(-x, -y)$$

  - Odd (antisymmetric) functions

$$f(x, y) = -f(-x, -y)$$

Symmetry properties in $f$ involve corresponding properties in $F$ that are useful in processing.
E.g.: If $f$ is real and even, also $F$ is real and even.

## Fourier spectrum and phase angle

- The DFT can be expressed in polar form:

$$F(u, v) = |F(u, v)|\, e^{\iota\phi(u, v)}$$

where $|F(u, v)|$, called *Fourier spectrum*:

$$|F(u, v)| = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$$

and $\phi(u, v)$, called *phase angle*:

$$\phi(u, v) = \arctan\left(\frac{I(u, v)}{R(u, v)}\right)$$

- The *power spectrum*, $P(u, v)$, is defined as:

$$P(u, v) = |F(u, v)|^2 = R^2(u, v) + I^2(u, v)$$

.

## Fourier spectrum and phase angle (2)

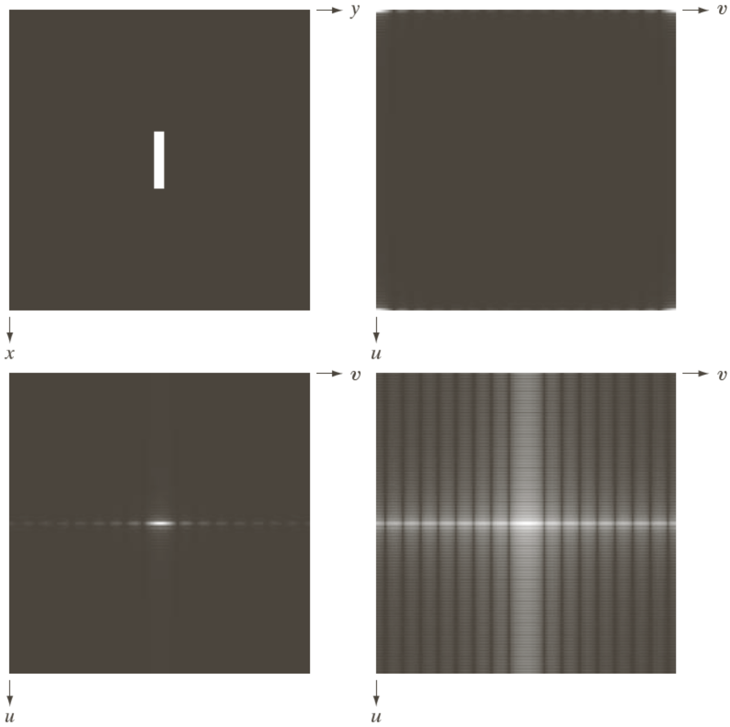
- It can be shown that:

$$|F(0, 0)| = MN|\bar{f}(x, y)|$$

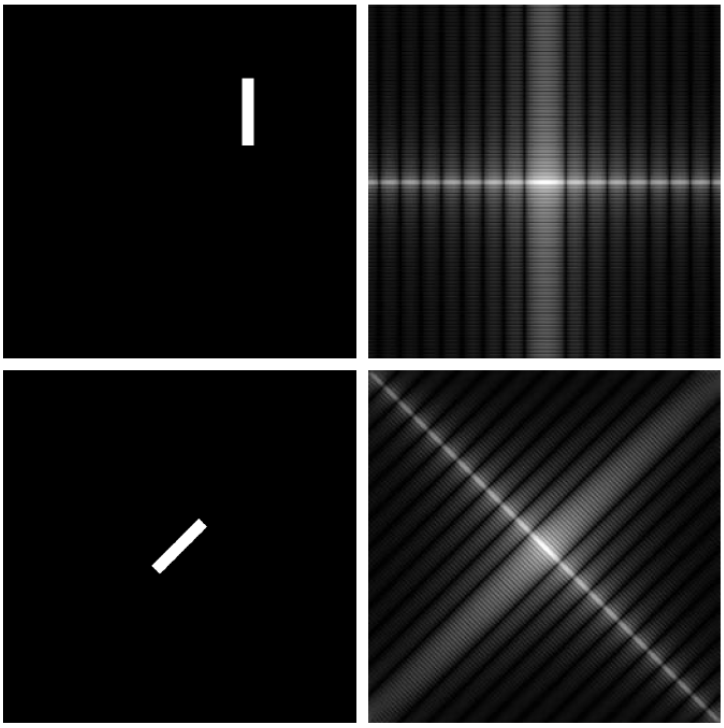
where $\bar{f}$ is the $f$ average value.

$F(0, 0)$ is generally much larger than the other terms of $F$;
  - logarithmic transform for displaying it.

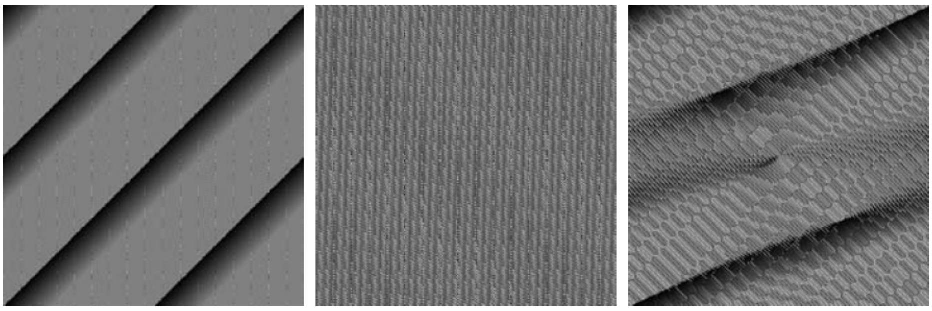
## Fourier spectrum and phase angle (3)



## Fourier spectrum and phase angle (4)



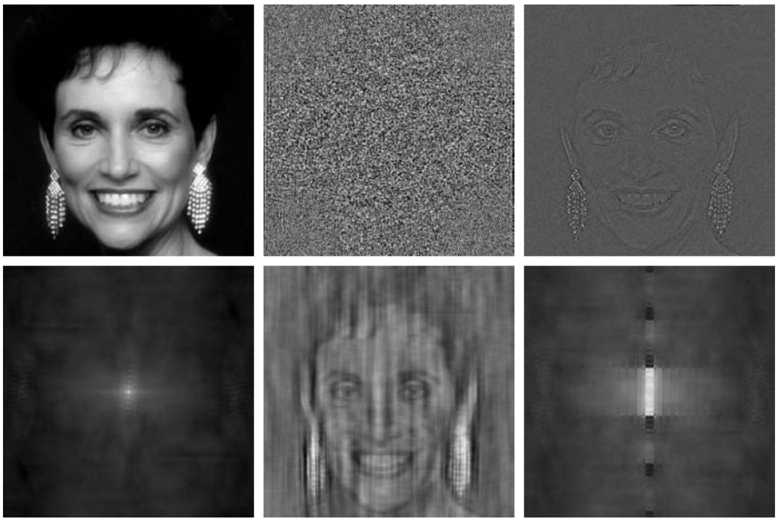
## Fourier spectrum and phase angle (5)



a | b | c

(a) phase angle of the centered rectangle image;
(b) phase angle of the shifted rectangle image;
(c) phase angle of the rotated rectangle image;

## Fourier spectrum and phase angle (6)



| | | |
|---|---|---|
| a | b | c |
| d | e | f |

- (b): phase angle of (a);
- (c) and (d): IDFT(phase angle of (a)) and IDFT(spectrum of (a));
- (e): IDFT(phase angle of the woman + spectrum of the rectangle);
- (f): IDFT(spectrum of the woman + phase angle of the rectangle).

## 2D convolution theorem

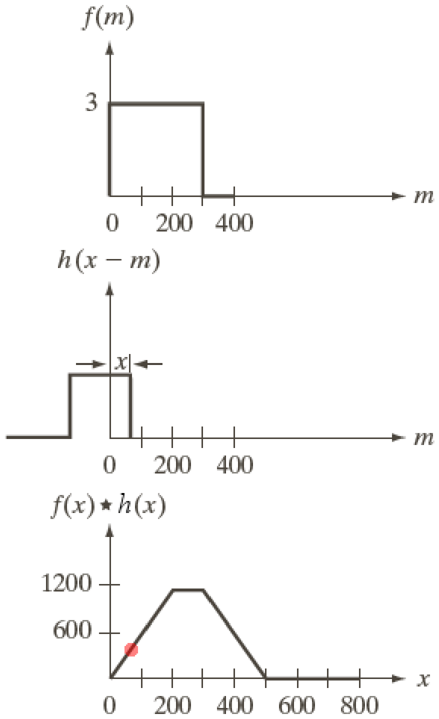
- The convolution theorem can be formulated for the 2D DFT:

$$\mathcal{F}\{f(x, y) * h(x, y)\} = F(u, v)\, H(u, v)$$

$$\mathcal{F}\{f(x, y)\, h(x, y)\} = F(u, v) * H(u, v)$$

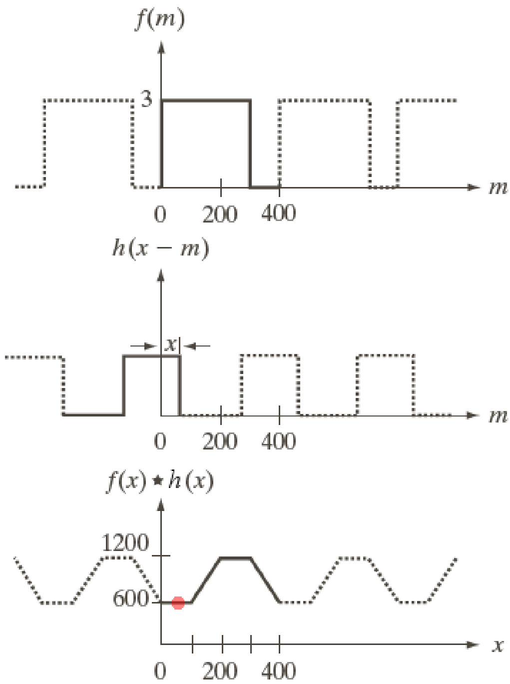
- The circular convolution has to be used.

## Convolution




$$f(x) * h(x) = \sum_{m=0}^{M-1} f(x)h(x-m)$$

## Circular convolution




$$f(x) * h(x) =$$

$$= \sum_{m=0}^{M-1} f(x)h(x-m)$$

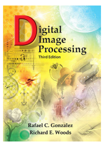
## Warparound error

- The (circular) convolution of two periodic function can cause the so called *warparound error*.
- It can be resolved using the *zero padding*.
  - Giving two sequences of respectively $A$ and $B$ samples, append zeros to them such that both will have $P$ elements:

$$P = A + B - 1$$

- If a function is not zero at the end of the interval, the zero padding introduces artifacts:
  - High frequency components in the transform.
- Attenuation with the windowing technique:
  - e.g., multiplying by a Gaussian.

## Homeworks and suggested readings




DIP, Sections 4.5–4.6
- pp. 225–254

GIMP
- Image
  - Scale Image
    - Cubic
    - Sinc