

Fourier transform of images

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Extension to bidimensional domain

- ▶ The concepts introduced for the monodimensional domain can be extended for the multidimensional case:
 - ▶ Impulse, δ
 - ▶ Convolution
 - ▶ Fourier transform
 - ▶ Sampling theorem
- ▶ In particular, we are interested to the bidimensional domain.

Impulse

The Dirac delta function, δ , or impulse, is defined as:

$$\delta(t, z) = \begin{cases} \infty, & t = z = 0 \\ 0, & t \neq 0, z \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z) dt dz = 1$$

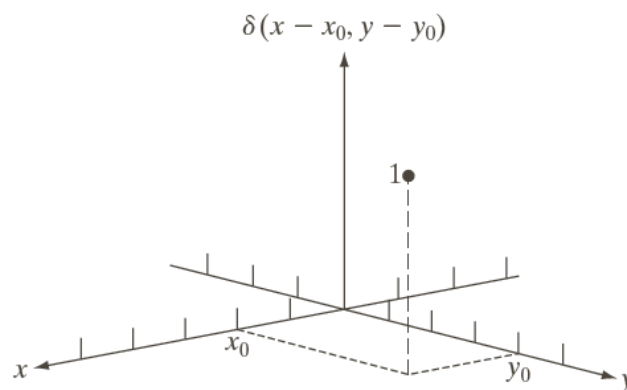
The *sifting property* holds also in this case:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) \delta(t - t_0, z - z_0) dt dz = f(t_0, z_0)$$

Impulse (2)

The discrete version of δ for the bidimensional case:

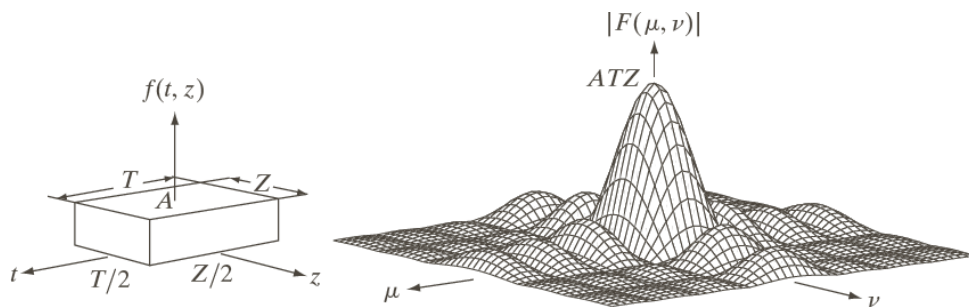
$$\delta(x, y) = \begin{cases} 1, & x = y = 0 \\ 0, & \text{otherwise} \end{cases}$$



2D continuous Fourier transform pair

$$F(\nu, \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-i2\pi(\nu t + \mu z)} dt dz$$

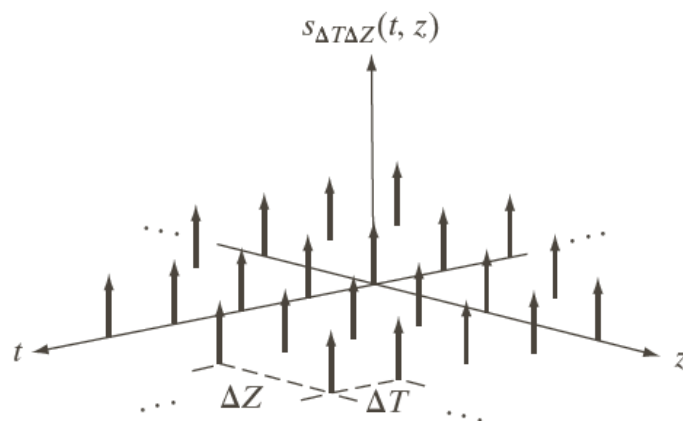
$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\nu, \mu) e^{i2\pi(\nu t + \mu z)} d\nu d\mu$$



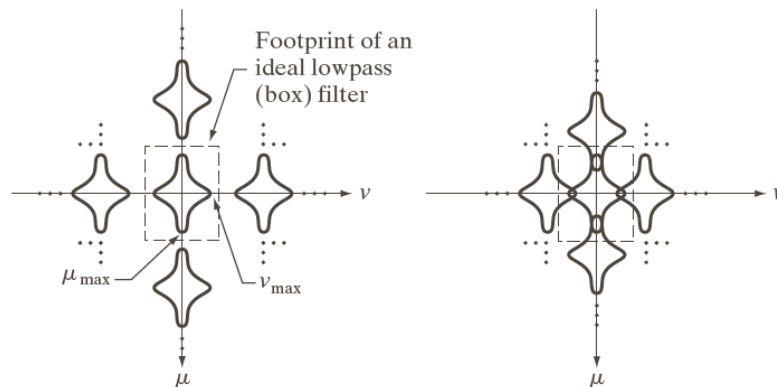
2D sampling theorem

$$\tilde{f}(t, z) = f(t, z) s_{\Delta T \Delta Z}(t, z) = \sum_{m, n=-\infty}^{\infty} f(t) \delta(t - n\Delta T, z - m\Delta Z)$$

$$\frac{1}{\Delta T} > 2\nu_{\max} \quad \text{and} \quad \frac{1}{\Delta Z} > 2\mu_{\max}$$

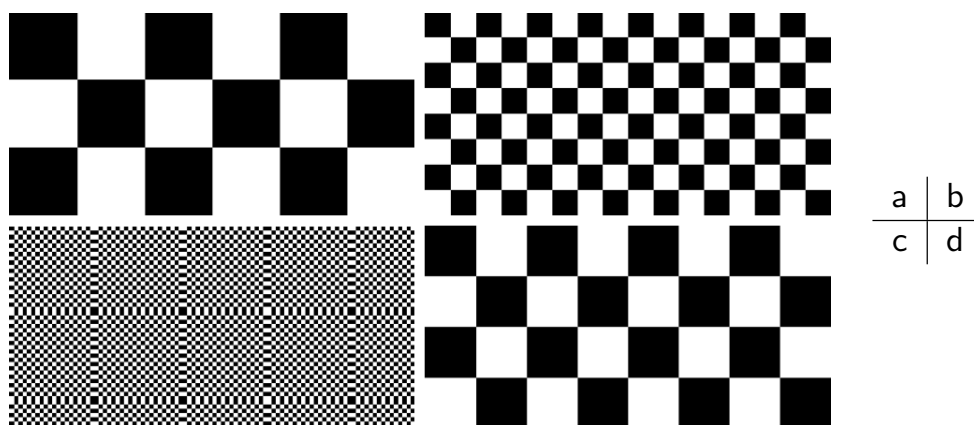


Aliasing in images



- ▶ Aliasing effects: false borders and jagged edges.
- ▶ Reduction:
 - ▶ smoothing before sampling
 - ▶ oversampling and averaging
- ▶ Reduction (post)
 - ▶ smoothing

Aliasing example: sampling a checkerboard



Sampling a checkerboard pattern where the sides of the squares are 96 units long.

(a) $\Delta T = \Delta Z = 6$

(b) $\Delta T = \Delta Z = 16$

(c) $\Delta T = \Delta Z = 105$

(d) $\Delta T = \Delta Z = 200$

Resampling and interpolation

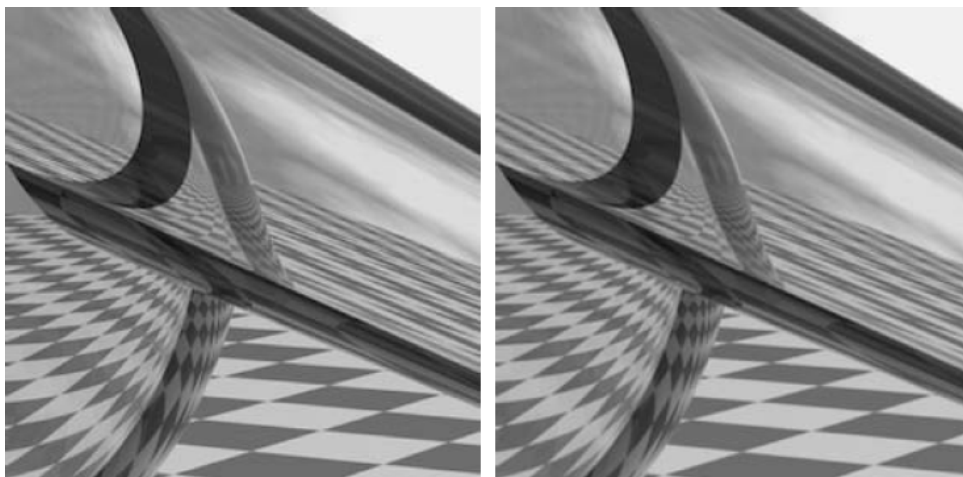


a | b | c

- (a) Original image
- (b) Resampled image
- (c) Applying smoothing before resampling

Note: Resampling has been operated through rows and columns deletion.

Resampling and interpolation (2)

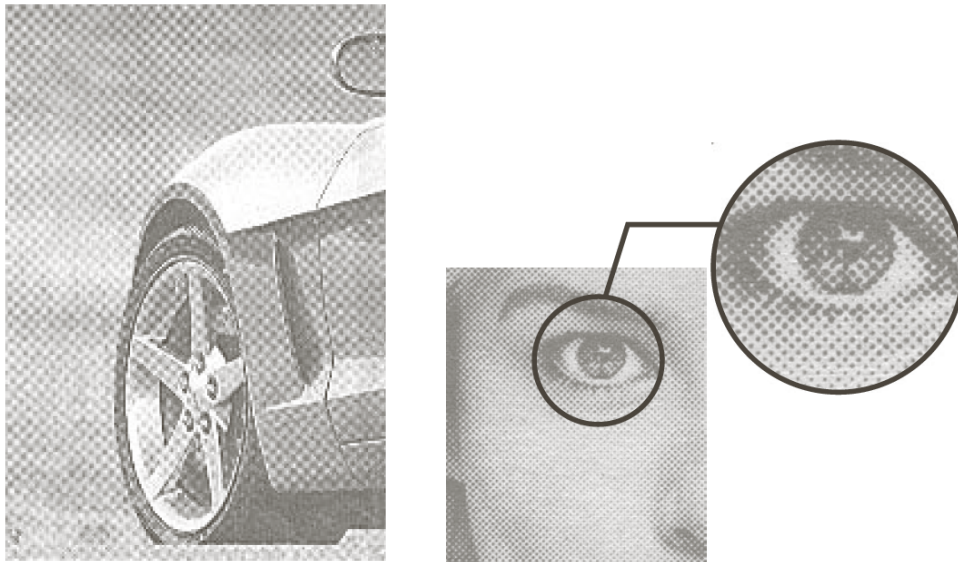


a | b

- (a) Zooming by pixel replication
- (b) Zooming by pixel interpolation

Resampling and interpolation (3)

The moirè effect is caused by the superimposition of two periodical patterns.



Bidimensional DFT pair

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

where

$$u = 0, \dots, M-1 \quad v = 0, \dots, N-1$$

$$x = 0, \dots, M-1, \quad y = 0, \dots, N-1$$

DFT properties

- ▶ Traslation

$$\mathcal{F}\{f(x, y) e^{i2\pi(u_0 x/M + v_0 y/N)}\} = F(u - u_0, v - v_0)$$

$$\mathcal{F}\{f(x - x_0, y - y_0)\} = F(u, v) e^{-i2\pi(x_0 u/M + y_0 v/N)}$$

- ▶ Multiplying f by an exponential produces a shift in the DTF.
- ▶ Translating f has the effect of multiplying its DFT.

- ▶ Rotation

- ▶ Rotating f produces an identical rotation in its DFT.

DFT properties (2)

- ▶ Periodicity

$$F(u, v) = F(u + k_1 M, v + k_2 N)$$

$$f(x, y) = f(x + k_1 M, y + k_2 N)$$

where $k_1, k_2 \in \mathbb{Z}$

$$\mathcal{F}\{f(x, y)(-1)^{x+y}\} = F(u - M/2, v - N/2)$$

DFT properties (3)

- ▶ Symmetry

- ▶ Even (symmetric) functions

$$f(x, y) = f(-x, -y)$$

- ▶ Odd (antisymmetric) functions

$$f(x, y) = -f(-x, -y)$$

Symmetry properties in f involve corresponding properties in F that are useful in processing.

E.g.: If f is real and even, also F is real and even.

Fourier spectrum and phase angle

- ▶ The DFT can be expressed in polar form:

$$F(u, v) = |F(u, v)| e^{i\phi(u, v)}$$

where $|F(u, v)|$, called *Fourier spectrum*:

$$|F(u, v)| = [R^2(u, v) + I^2(u, v)]^{1/2}$$

and $\phi(u, v)$, called *phase angle*:

$$\phi(u, v) = \arctan \left(\frac{I(u, v)}{R(u, v)} \right)$$

- ▶ The *power spectrum*, $P(u, v)$, is defined as:

$$P(u, v) = |F(u, v)|^2 = R^2(u, v) + I^2(u, v)$$

Fourier spectrum and phase angle (2)

- It can be shown that:

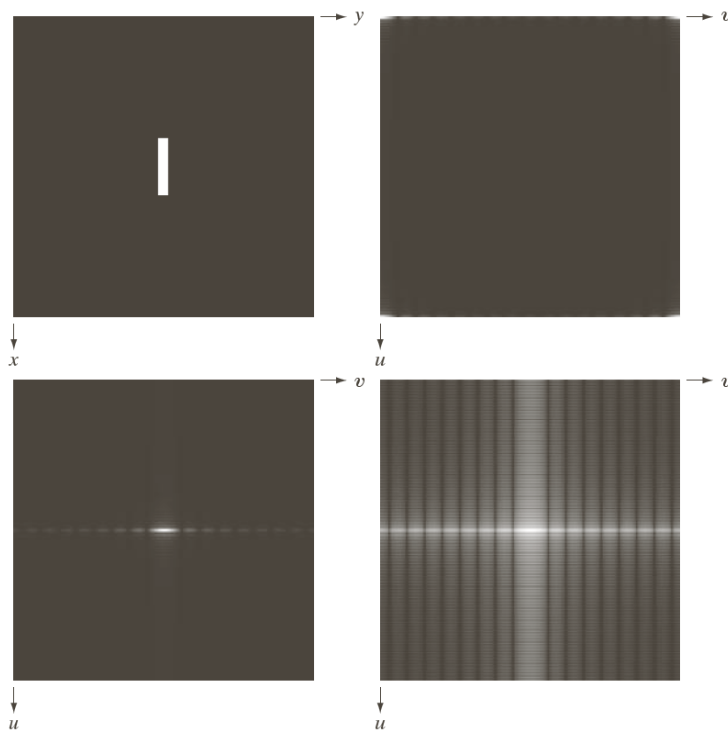
$$|F(0, 0)| = MN|\bar{f}(x, y)|$$

where $\bar{f}$ is the f average value.

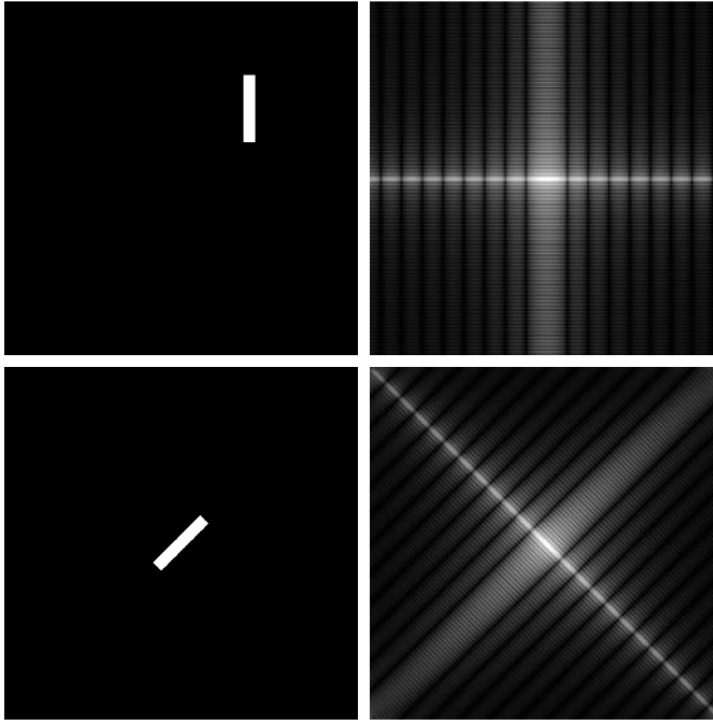
$F(0, 0)$ is generally much more larger than the other terms of F ;

- logarithmic transform for displaying it.

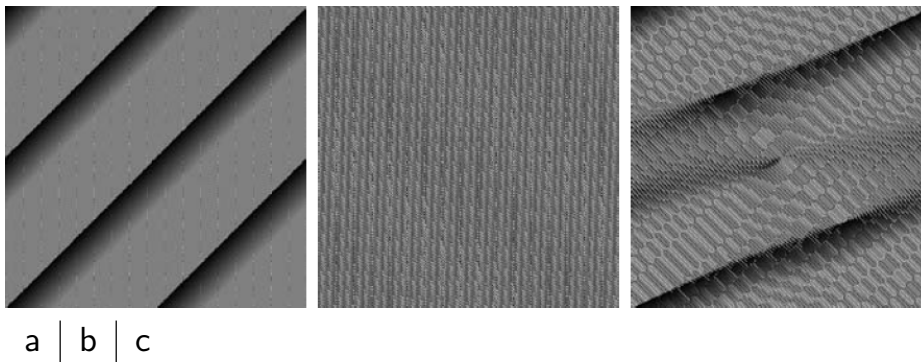
Fourier spectrum and phase angle (3)



Fourier spectrum and phase angle (4)

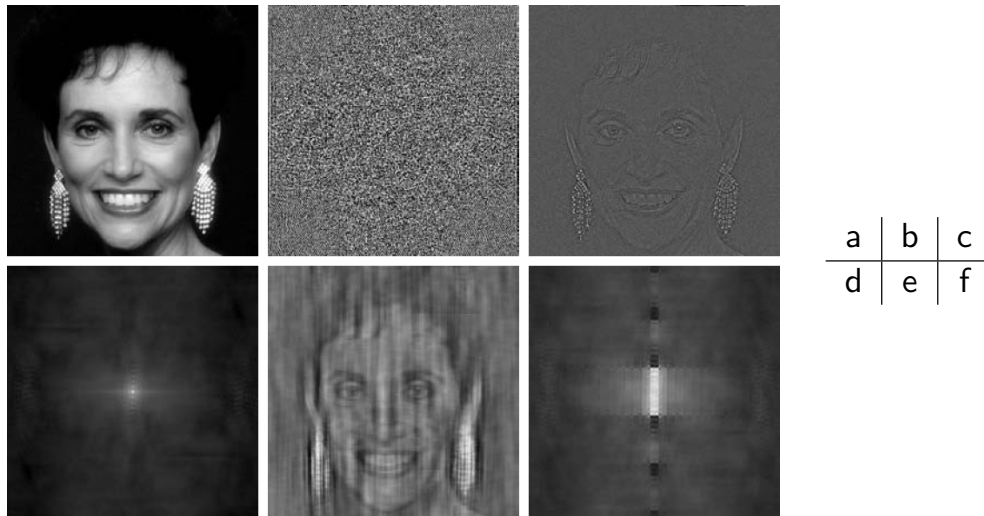


Fourier spectrum and phase angle (5)



- (a) phase angle of the centered rectangle image;
- (b) phase angle of the shifted rectangle image;
- (c) phase angle of the rotated rectangle image;

Fourier spectrum and phase angle (6)



- ▶ (b): phase angle of (a);
- ▶ (c) and (d): IDFT(phase angle of (a)) and IDFT(spectrum of (a));
- ▶ (e): IDFT(phase angle of the woman + spectrum of the rectangle);
- ▶ (f): IDFT(spectrum of the woman + phase angle of the rectangle).

2D convolution theorem

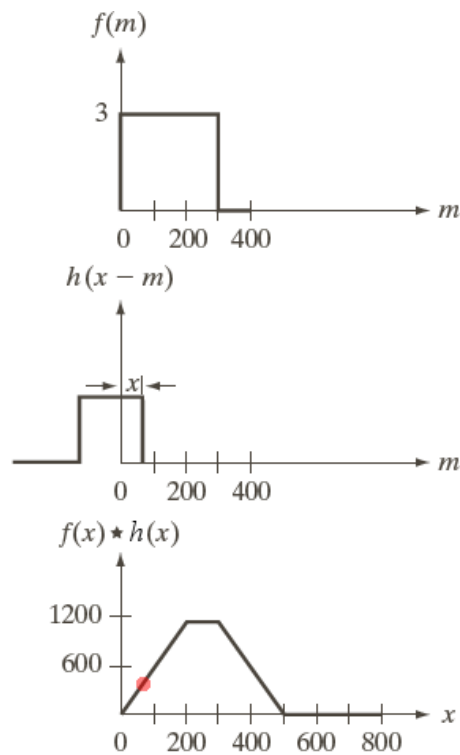
- ▶ The convolution theorem can be formulated for the 2D DFT:

$$\mathcal{F}\{f(x, y) * h(x, y)\} = F(u, v) H(u, v)$$

$$\mathcal{F}\{f(x, y) h(x, y)\} = F(u, v) * H(u, v)$$

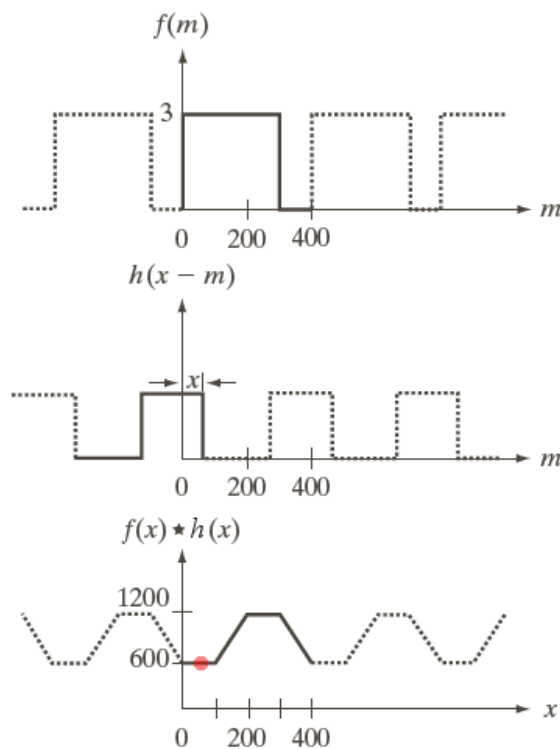
- ▶ The circular convolution has to be used.

Convolution



$$f(x) * h(x) = \sum_{m=0}^{M-1} f(x)h(x-m)$$

Circular convolution



$$f(x) * h(x) = \sum_{m=0}^{M-1} f(x)h(x-m)$$

Warparound error

- ▶ The (circular) convolution of two periodic function can cause the so called *warparound error*.
- ▶ It can be resolved using the *zero padding*.
 - ▶ Giving two sequences of respectively A and B samples, append zeros to them such that both will have P elements:

$$P = A + B - 1$$

- ▶ If a function is not zero at the end of the interval, the zero padding introduces an artificial
 - ▶ High frequency components in the transform.
- ▶ Attenuation with the windowing technique:
 - ▶ e.g., multiplying by a Gaussian.