MATHEURISTICS FOR COMBINATORIAL OPTIMIZATION PROBLEMS

Module 1- Lesson 4

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OUTLINE - Decomposition based matheur.

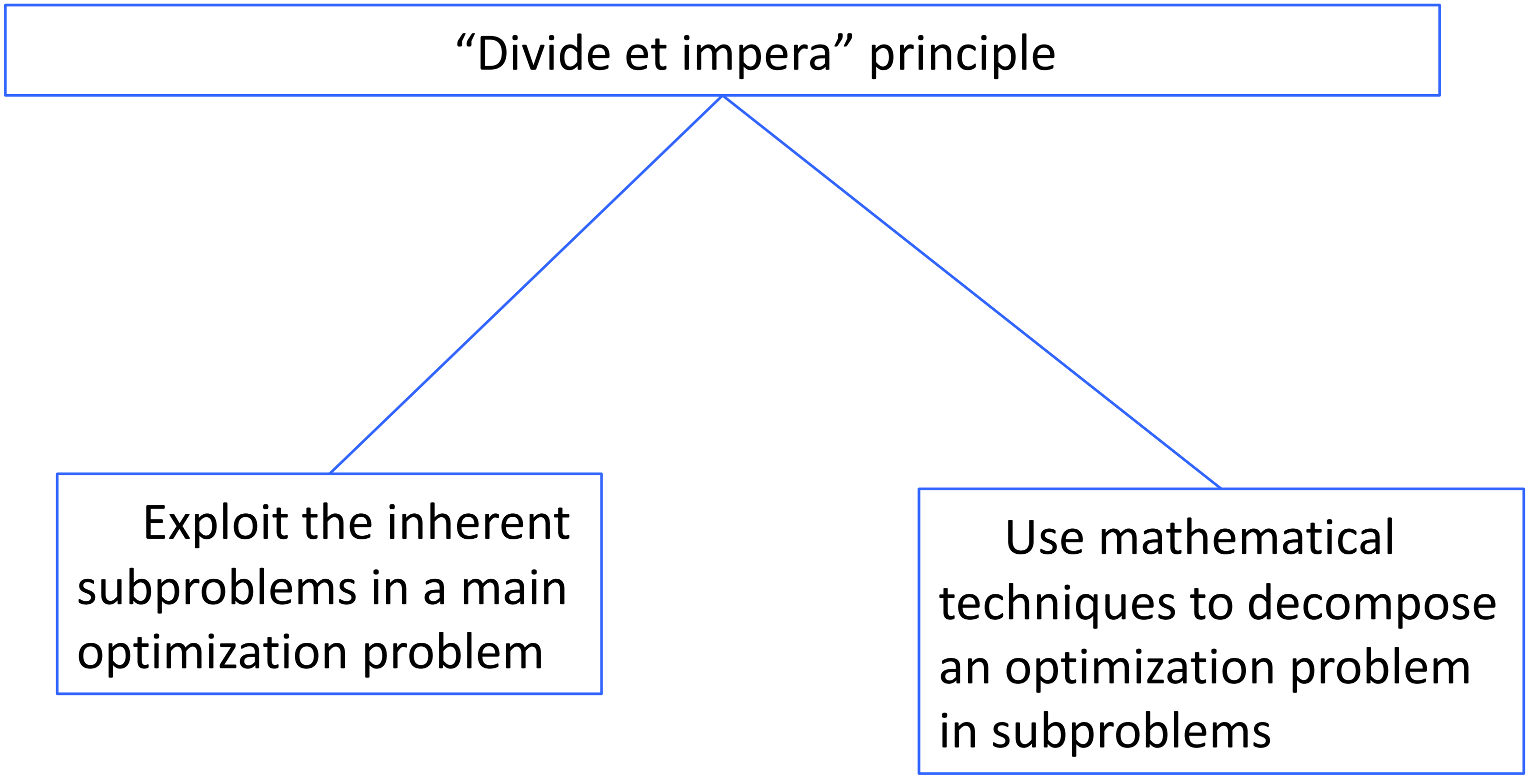
- Classification of decomposition based matheuristics (“natural” vs “artificial” decomposition)

- Decomposition based matheuristics for the VRP:
  - Generalized Assignment heuristic
  - Location heuristic
  - Route-first, cluster-second heuristic

- Lagrangean decomposition matheuristic
  - Application to hazmat transport

- Dantzig-Wolfe decomposition matheuristic

Classification of decomposition matheur.



"Natural" decomposition matheuristics

- Some optimization problems are naturally structured as a sequence of optimization subproblems.

- In the airline crew and fleet planning:

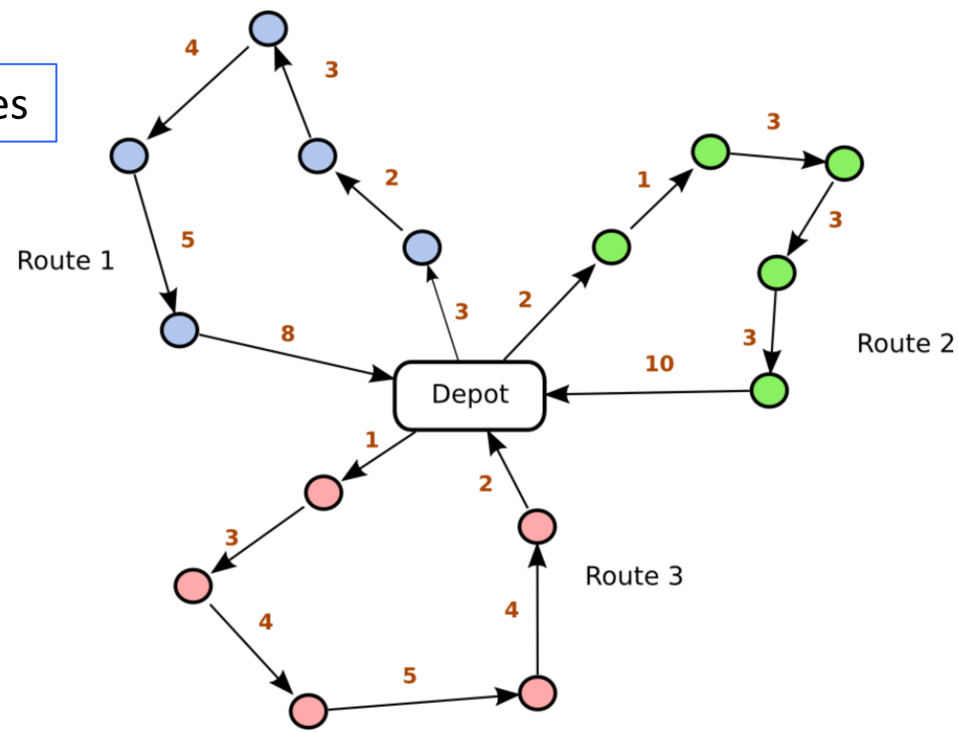
airline fleet planning → passenger schedule → crew pairing problem

- In the Vehicle Routing Problem (VRP):

assignment of customers to the vehicles

↓

routing of each vehicle (TSP)



Decomposition matheuristics for VRP

- Formulation for the VRP:

$$\text{Min: } \sum_j \hat{c}_{TSP}(S_j) \qquad (1)$$

$$\text{s.t.} \quad \sum_j x_{ij} = 1 \qquad \text{for all customers } i \qquad (2)$$

$$\text{s.t.} \quad \sum_i q_i x_{ij} \leq Q \qquad \text{for all vehicles } j \qquad (3)$$

$$S_j = \{i : x_{ij} = 1\} \quad \text{for all vehicles } j \qquad (4)$$

$$x_{ij} \in \{0, 1\} \qquad \text{for all } i \text{ and } j. \qquad (5)$$

$\hat{c}_{TSP}(S_j)$ = cost of an optimal TSP over node set $S_j \cup \{0\}$

- Constraints (2)-(5) define a Generalized Assignment Problem (GAP)
- Since it is challenging evaluating $\hat{c}_{TSP}(S_j)$ we replace it with a heuristic value (e.g., double tree algorithm) $\Leftrightarrow$ CLUSTER-FIRST ROUTE-SECOND

Generalized Assignment matheur. for VRP

- Introduced by Fisher and Jaikumar (1981):
  1. Choose seed nodes sj for j=1,…,m
  2. Solve the GA: $\min\left\{\sum_{ij} c_{is_j} x_{ij} : (2) - (5)\right\}$
  3. Solve (heuristically) a TSP over each cluster $S_j \cup \{0\}$ defined in step 2

- Drawback of this approach: its dependency on the seed selection step

- For this reason in the next heuristic, steps 1 and 2 are combined

Location matheuristic for VRP

- Introduced by Bramel and Simchi-Levi (1995):
  1. Choose a set of candidate seed nodes
  2. Solve a Concentrator Location Problem (CLP) to determine a seed node s_j and a cluster S_j for each vehicle j
  3. Solve (heuristically) a TSP over each cluster $S_j \cup \{0\}$ defined in step 2

CLP:

$$\min \sum_{ij} c_{ij} x_{ij} + \sum_{j} v_j y_j$$

$$\sum_{j} x_{ij} = 1 \quad \text{for all customers } i$$

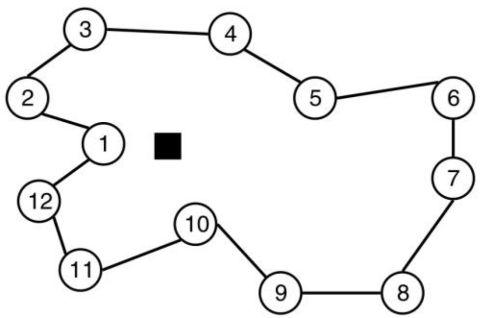
$$\sum_{i} q_i x_{ij} \leq Q \quad \text{for all vehicles } j$$

$$x_{ij} \leq y_j \quad \forall i, j$$

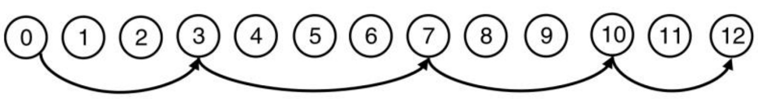
$$x_{ij}, y_j \in \{0,1\} \quad \forall i, j$$

Route-first cluster-second heuristic for VRP

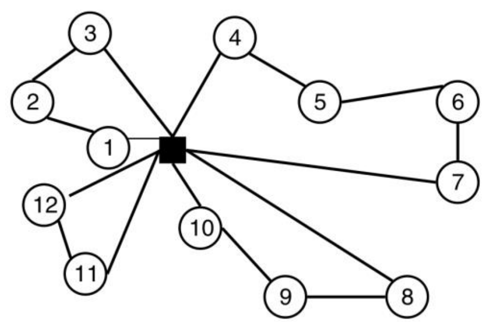
1. Solve the TSP on the whole set of customers.
   Let $\{i_1, i_2, \dots, i_n\}$ the customer sequence of TSP.
2. Choose a set of m "cut points": $k_1 < k_2 < \cdots < k_m$ and a cluster S_j , ∀ vehicle j
3. Let $S_1 = \{0, i_{k_1}, \dots, i_{k_2-1}, 0\}$, $S_2 = \{0, i_{k_2}, \dots, i_{k_3-1}, 0\}$,…,$S_m = \{0, i_{k_m}, \dots, i_{k_1-1}, 0\}$



- Step 2 can be solved optimally using a shortest path model.
- An arc from node i to node j, with i < j represents the VRP route starting at the depot, proceeding to node i + 1 then following the TSP node order to node j and returning to the depot.



- By varying the start node, the best partition over all start nodes can be obtained.



Lagrangean decomposition

- Suppose that in the ILP of a COP the constraints can be partitioned in two groups (1) and (2) such that the optimization over each single group is easy (or easier):

$$\min cx$$

$$Ax \leq b \quad (1)$$

$$Dx \leq e \quad (2)$$

$$x \in \{0,1\}^n$$

$$\Leftrightarrow \quad \min \alpha cx + \beta cy$$

$$Ax \leq b \quad (1)$$

$$Dy \leq e \quad (2)$$

$$x = y \quad (3)$$

$$x \in \{0,1\}^n, y \in \{0,1\}^n$$

with $\alpha + \beta = 1$

The Lagrangean relaxation of (3) provides:

$$\mathrm{Z}(D(P,\lambda)) = \min(\alpha c + \lambda)x + (\beta c - \lambda)y$$

$$Ax \leq b \quad (1)$$

$$Dy \leq e \quad (2)$$

$$x \in \{0,1\}^n, y \in \{0,1\}^n$$

Lagrangean decomposition

$$Z(D(P,\lambda)) = \min(\alpha c + \lambda)x + (\beta c - \lambda)y$$

$$Ax \leq b \quad (1)$$

$$Dy \leq e \quad (2)$$

$$x \in \{0,1\}^n, y \in \{0,1\}^n$$

- $Z(D(P,\lambda))$ can be decomposed in:

$$Z(D_1(P,\lambda)) = \min(\alpha c + \lambda)x$$

$$Ax \leq b \quad (1)$$

$$x \in \{0,1\}^n$$

$$Z(D_2(P,\lambda)) = \min(\beta c - \lambda)y$$

$$Dy \leq e \quad (2)$$

$$y \in \{0,1\}^n$$

- $Z(D(P,\lambda)) = Z(D_1(P,\lambda)) + Z(D_2(P,\lambda))$

Lagrangean decomposition dual problem

- $\mathrm{Z}\big(D(P,\lambda^*)\big) = \max_\lambda \mathrm{Z}(D(P,\lambda))$

- $\mathrm{Z}(D(P,\lambda^*)) \geq \max\{\, \mathrm{Z}\big(L_1(P,\mu^*)\big), \mathrm{Z}(L_2(P,\pi^*))\}$

  where $\mathrm{Z}\big(L_1(P,\mu^*)\big)$ is the optimal value of the lagrangean dual by relaxing (1) and $\mathrm{Z}(L_2(P,\pi^*))$ is the optimal value of the lagrangean dual by relaxing (2)

- With this kind of bound it is possible to develop effective matheuristics

Problem definition

- Given a road network $G=(N, A)$ and a set of s hazmat shipment requests: origin-destination (o_k,d_k), amount φ^k, hazmat category $h_k \in H$, $\forall\, k=1,\ldots,s$

- Two stakeholders: the **government (Gov)** and the **carriers (Crs)**

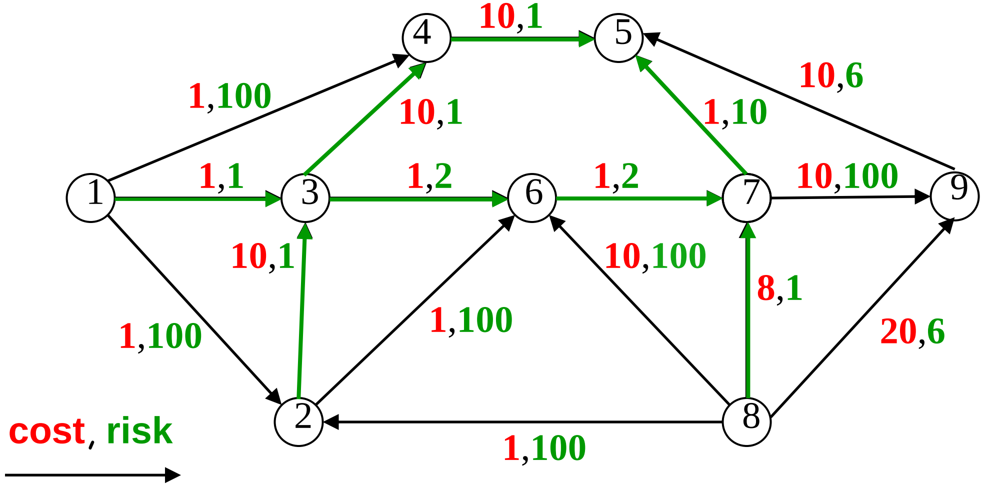
- **Gov** is interested in **minimizing the overall risk** of shipments while each **Cr** is interested in **minimizing the route cost**

Tunnel Interdiction Problem for Hazmat Transportation (TIPHT)
Given $G, \{(o_k,d_k), \varphi^k, h_k: k=1,\ldots,s\}$ and $T \subseteq A$ set of interdictable arcs (tunnels), decide which arcs of *T* to forbid to each hazmat category so as to minimize the total risk of shipments while ensuring that if a tunnel is interdicted to a hazmat of type h_2 it is also for any more dangerous hazmat (i.e., with $h_1 \prec h_2\ \forall h_1 \in H$)

Bilevel optimization problem

- Due to the hierarchy between the decision makers (**Gov** and **Crs**) the Hazmat Network Design Problem (HNDP) is a bilevel optimization problem:
  - **Gov** decision is the ***leader problem***
  - **Crs** decisions are the ***follower problem***

Example 1 (T=A,|H|=1)



4 commodities:

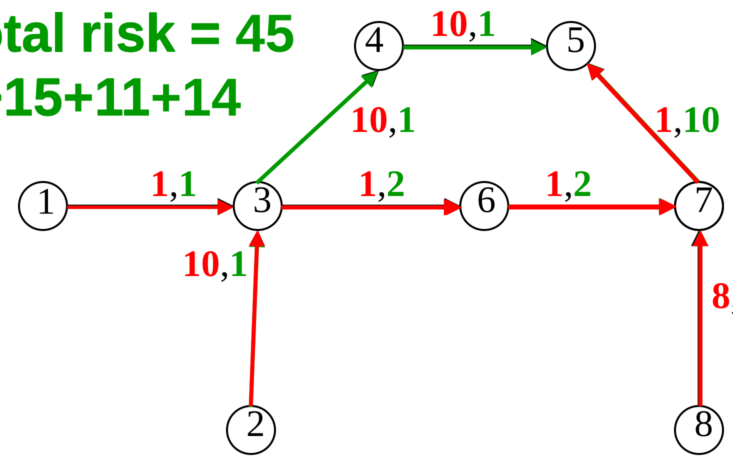
$(o_1,d_1)=(1,7)$

$(o_2,d_2)=(2,5)$

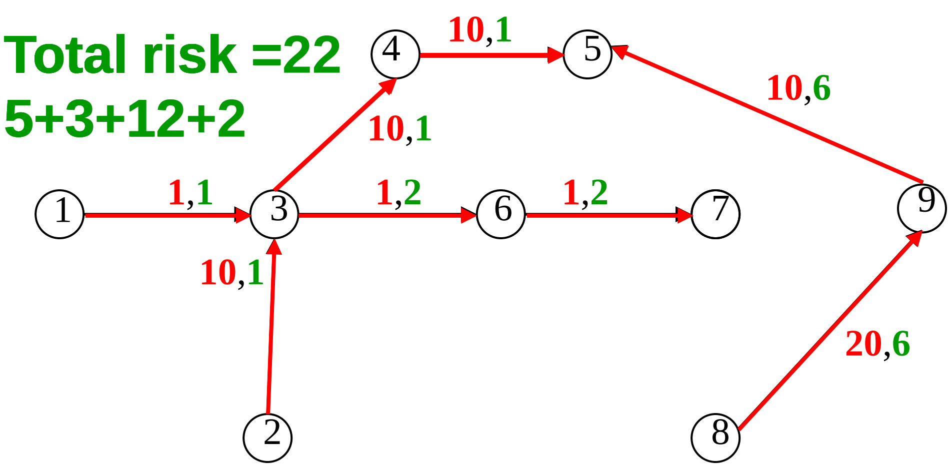
$(o_3,d_3)=(8,5)$

$(o_4,d_4)=(3,5)$

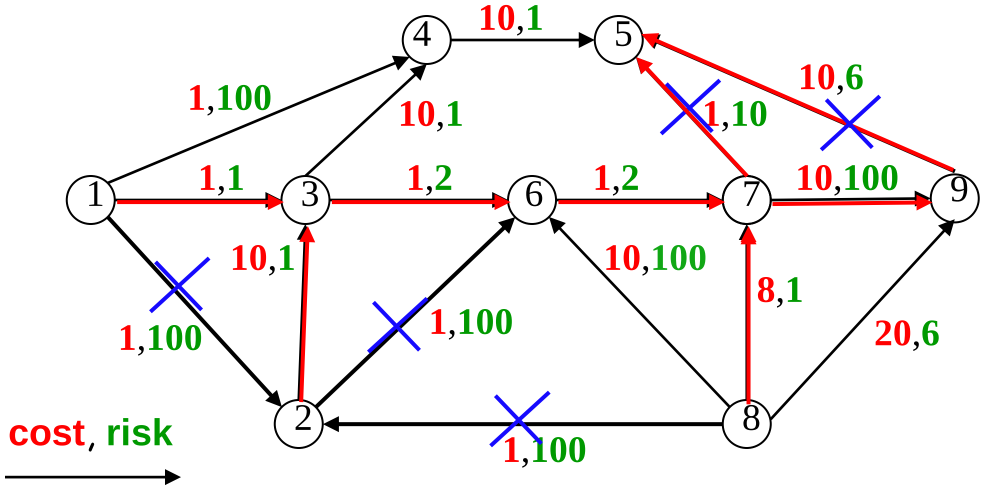
Total risk = 45
5+15+11+14



Total risk =22
5+3+12+2



Example 2 ($T \subset A$, $|H|=1$)



4 commodities:

$(o_1,d_1)=(1,7)$

$(o_2,d_2)=(2,5)$

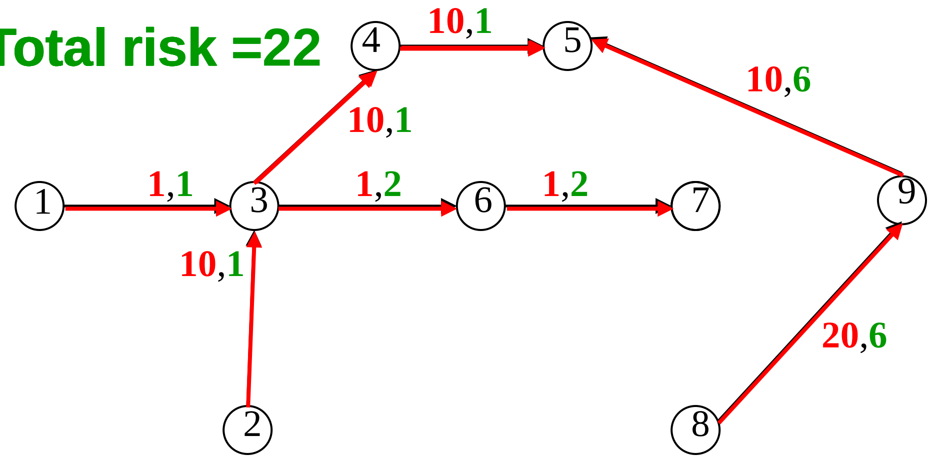
$(o_3,d_3)=(8,5)$

$(o_4,d_4)=(3,5)$

Total risk >100!

Total risk =11+5+15+14=45

Total risk =22



Literature

- Kara, Verter "Designing a road network for hazardous materials transportation", Transp.Science 04:

  first bilevel formulation (just with $T = A$) and a single level one obtainend linearizing the KKT conditions of the follower

- Erkut, Alp "Designing a road network for dangerous goods shipments", C.O.R. 09:

  restrict the network to a tree

- Erkut, Gzara "Solving the hazmat transport network design problem", C.O.R. 08:

  heuristics to find stable solutions (that consider the worst case risk when the follower has multiple optimal solutions)

Literature

- E.Amaldi, M.Bruglieri, B.Fortz, *On the hazmat transport network design problem,* INOC '11:

  - HNDP extension where a **subset** T of roads can be interdicted
  - proof of NP-hardness even for a single o-d pair
  - bilevel MILP formulation that guarantees stability
  - single-level MILP reformulation that can be solved more efficiently than Kara-Verter's one ($|T|$ binary variables rather than $s|A|+|T|$)

Current work

TIPHT problem differs from the HNDP of INOC'11 for the **tunnel interdiction hierarchical conditions** (required by ADR 2007)

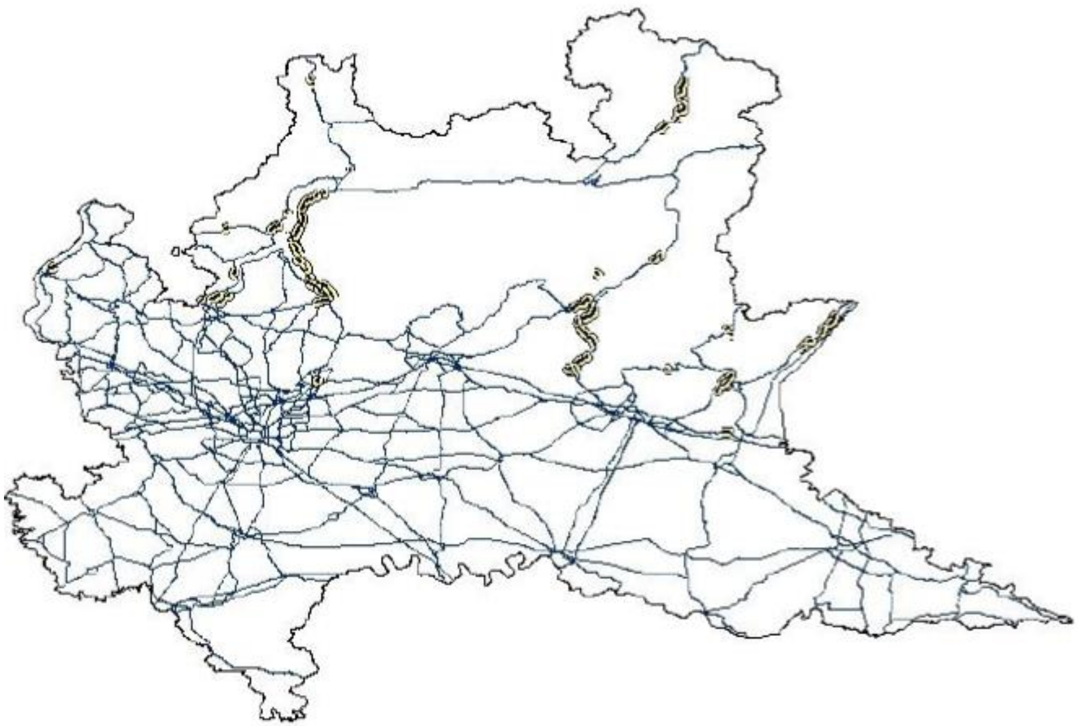
We need to solve large scale instances

Lombardia case study:

1560 o-d pairs

34899 road links

333 tunnels



MILP requires

$s\ |A| \approx 10^8$

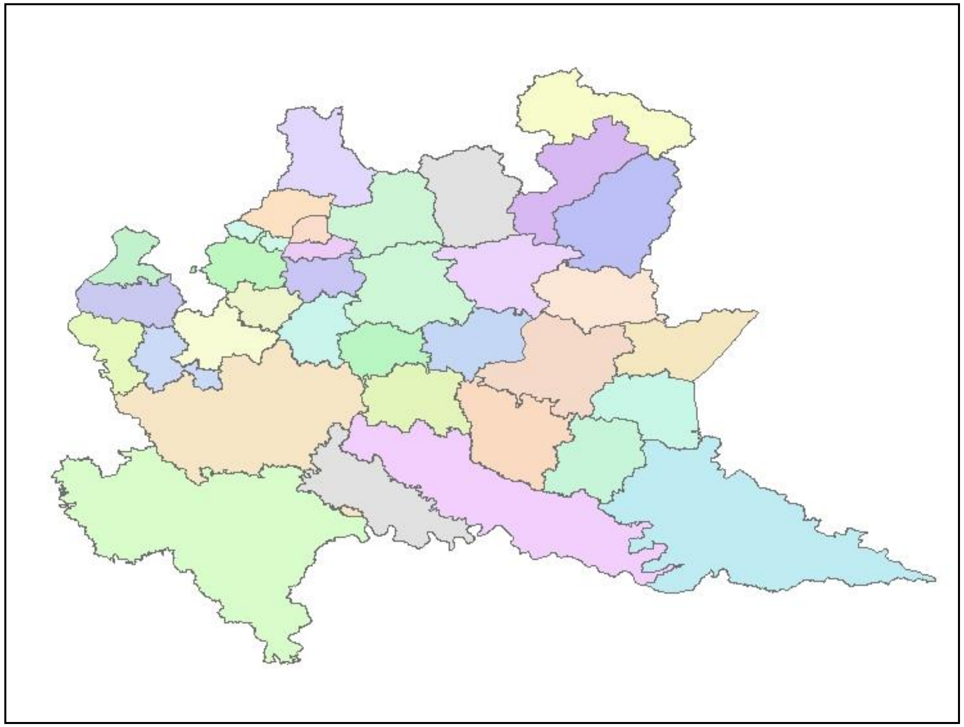
continuous var. !

Instance of Lombardia

Lombardia region: a large and interesting case (many tunnels and o-d pairs)

P.Gandini, MS Thesis (in Italian), PoliMI, 2009

- o-d pairs: 40x40 (partitioning each province in subareas)
- Hazmat shipment estimation: Conto Nazionale Trasporto (ISTAT 2004) + Gross Domestic Product
- Risk assessment in tunnels and in the open-topped roads for each hazmat category (population exposed, environment,…)



Input Data

- N = node set
- A = road arc set
- T = set of interdictable arcs ($T \subseteq A$)
- s = number of hazardous commodities
- (o_k, d_k) = origin-destination pair of commodity k
- h_k = hazmat category of commodity k
- φ^k = shipment request amount of commodity k
- c_{ij}^k = travel cost of arc (i,j) per unit of commodity k
- r_{ij}^k = risk to travel arc (i,j) per unit of commodity k

Decision variables

Gov variables

$$y_{ij}^{h} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is allowed} \\ & \text{to hazmat category } h \\ 0 & \text{otherwise} \end{cases} \qquad \forall (i,j) \in T, \forall h \in H$$

Crs variables

$$x_{ij}^{k} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is chosen for shipment } k \\ 0 & \text{otherwise} \end{cases} \qquad \forall (i,j) \in A, \forall k = 1,\ldots,s$$

Bilevel ILP formulation

$$\min \sum_{k=1}^{s} \sum_{(i,j)\in A} r_{ij}^{h_k} \varphi^k x_{ij}^k \qquad (1)$$

$$y_{ij}^{h_1} \le y_{ij}^{h_2} \qquad \forall (i,j)\in T, \forall\, h_1, h_2 \in H : h_1 \prec h_2 \qquad (2)$$

$$y_{ij}^{h} \in \{0,1\}, \quad \forall (i,j)\in T, \forall h \in H \qquad (3)$$

where variables x_{ij}^k are solution of:

$$\min \sum_{k=1}^{s} \sum_{(i,j)\in A} c_{ij}^k x_{ij}^k \qquad (4)$$

$$\sum_{i\in\delta^-(j)} x_{ij}^k - \sum_{l\in\delta^+(j)} x_{jl}^k = \begin{cases} 0 & \text{if } j \ne o_k, d_k \\ -1 & \text{if } j = o_k \\ 1 & \text{if } j = d_k \end{cases}, \quad \forall j \in N, \forall k = 1,\dots,s \qquad (5)$$

$$x_{ij}^k \le y_{ij}^{h_k} \qquad \forall (i,j)\in T, \forall k = 1,\dots,s \qquad (6)$$

$$0 \le x_{ij}^k \le 1 \qquad \forall (i,j)\in A, \forall k = 1,\dots,s \qquad (7)$$

Solving bilevel formulation

Applying strong LP duality, we substitute the inner problem with constraints:

Primal feasibility

$$\sum_{i\in\delta^-(j)} x_{ij}^{\ k} - \sum_{l\in\delta^+(j)} x_{jl}^{\ k} = \begin{cases} 0 & \text{if } j \neq o_k, d_k \\ -1 & \text{if } j = o_k \\ 1 & \text{if } j = d_k \end{cases}, \quad \forall j \in N, \forall k = 1,\dots,s \qquad (5)$$

$$x_{ij}^{\ k} \leq y_{ij}^{h_k} \qquad \forall (i,j) \in T, \forall k = 1,\dots,s \qquad (6)$$

$$x_{ij}^{\ k} \geq 0 \qquad \forall (i,j) \in A, \forall k = 1,\dots,s \qquad (8)$$

Dual feasibility

$$w_j^{\ k} - w_i^{\ k} \leq c_{ij}^{\ k} \qquad \forall (i,j) \in A \setminus T, \forall k = 1,\dots,s \qquad (9)$$

$$w_j^{\ k} - w_i^{\ k} \leq c_{ij}^{\ k} + M(1 - y_{ij}^{h_k}) \qquad \forall (i,j) \in T, \forall k = 1,\dots,s \qquad (10)$$

primal o.f. = dual o.f.

$$\sum_{(i,j)\in A} c_{ij}^{\ k} x_{ij}^{\ k} = w_{d_k}^{\ k} - w_{o_k}^{\ k} \qquad \forall k = 1,\dots,s \qquad (11)$$

Single level reformulation

$$\min \sum_{k=1}^{s} \sum_{(i,j)\in A} r_{ij}^{h_k} \varphi^k x_{ij}^{k} \qquad (1)$$

$$y_{ij}^{h_1} \le y_{ij}^{h_2} \qquad \forall (i,j) \in T, \forall\, h_1, h_2 \in H : h_1 \prec h_2 \qquad (2)$$

$$\sum_{i\in\delta^-(j)} x_{ij}^{k} - \sum_{l\in\delta^+(j)} x_{jl}^{k} = \begin{cases} 0 & \text{if } j \ne o_k, d_k \\ -1 & \text{if } j = o_k \\ 1 & \text{if } j = d_k \end{cases}, \quad \forall j \in N, \forall k = 1,\dots,s \qquad (5)$$

$$x_{ij}^{k} \le y_{ij}^{h_k} \qquad \forall (i,j) \in T,\ \forall k = 1,\dots,s \qquad (6)$$

$$w_j^{k} - w_i^{k} \le c_{ij}^{k} \qquad \forall (i,j) \in A \setminus T, \forall k = 1,\dots,s \qquad (7)$$

$$w_j^{k} - w_i^{k} \le c_{ij}^{k} + M(1 - y_{ij}^{h_k}) \qquad \forall (i,j) \in T, \forall k = 1,\dots,s \qquad (8)$$

$$\sum_{(i,j)\in A} c_{ij}^{k} x_{ij}^{k} = w_{d_k}^{k} - w_{o_k}^{k} \qquad \forall k = 1,\dots,s \qquad (9)$$

$$x_{ij}^{k} \ge 0 \qquad \forall (i,j) \in A, \forall k = 1,\dots,s$$

$$y_{ij}^{h} \in \{0,1\} \qquad \forall (i,j) \in T, \forall h \in H$$

Lagrangean relaxation

$$L(\lambda,\mu) = \min \sum_{k=1}^{s} \sum_{(i,j)\in A} r_{ij}^{h_k} \varphi^k x_{ij}^{k} + \sum_{k=1}^{s} \sum_{(i,j)\in T} \lambda_{ij}^{k} (x_{ij}^{k} - y_{ij}^{h_k}) + \sum_{k=1}^{s} \sum_{(i,j)\in T} \mu_{ij}^{k} [\ldots]$$

$$y_{ij}^{h_1} \le y_{ij}^{h_2} \qquad \forall (i,j)\in T, \forall\, h_1, h_2 \in H : h_1 \prec h_2 \qquad (2)$$

$$\sum_{i\in\delta^-(j)} x_{ij}^{k} - \sum_{l\in\delta^+(j)} x_{jl}^{k} = \begin{cases} 0 & \text{if } j \ne o_k, d_k \\ -1 & \text{if } j = o_k \\ 1 & \text{if } j = d_k \end{cases}, \qquad \forall j \in N, \forall k = 1,\ldots,s \qquad (5)$$

$$x_{ij}^{k} \le y_{ij}^{h_k} \qquad \forall (i,j)\in T, \forall k = 1,\ldots,s \qquad (6)$$

$$w_j^{k} - w_i^{k} \le c_{ij}^{k} \qquad \forall (i,j)\in A \setminus T, \forall k = 1,\ldots,s \qquad (7)$$

$$w_j^{k} - w_i^{k} \le c_{ij}^{k} + M(1 - y_{ij}^{h_k}) \qquad \forall (i,j)\in T, \forall k = 1,\ldots,s \qquad (8)$$

$$\sum_{(i,j)\in A} c_{ij}^{k} x_{ij}^{k} = w_{d_k}^{k} - w_{o_k}^{k} \qquad \forall k = 1,\ldots,s \qquad (9)$$

$$x_{ij}^{k} \ge 0 \qquad \forall (i,j)\in A, \forall k = 1,\ldots,s$$

$$y_{ij}^{h} \in \{0,1\} \qquad \forall (i,j)\in T, \forall h \in H$$

Lagrangean decomposition

$$L(\lambda,\mu) = \min L_1(y,\lambda,\mu) + L_2(x,w,\lambda,\mu)$$

S_2^k

$$y_{ij}^{h_1} \le y_{ij}^{h_2} \qquad \forall (i,j) \in T, \forall h_1, h_2 \in H : h_1 \prec h_2 \qquad (2)$$

S_1

$$\sum_{i \in \delta^-(j)} x_{ij}^k - \sum_{l \in \delta^+(j)} x_{jl}^k = \begin{cases} 0 & \text{if } j \ne o_k, d_k \\ -1 & \text{if } j = o_k \\ 1 & \text{if } j = d_k \end{cases}, \quad \forall j \in N, \forall k = 1,\dots,s \qquad (5)$$

$$w_j^k - w_i^k \le c_{ij}^k \qquad \forall (i,j) \in A \setminus T, \forall k = 1,\dots,s \qquad (7)$$

$$\sum_{(i,j) \in A} c_{ij}^k x_{ij}^k = w_{d_k}^k - w_{o_k}^k \qquad \forall k = 1,\dots,s \qquad (9)$$

$$x_{ij}^k \ge 0 \qquad \forall (i,j) \in A, \forall k = 1,\dots,s$$

$$y_{ij}^h \in \{0,1\} \qquad \forall (i,j) \in T, \forall h \in H$$

Lagrangean dual

We solve the Lagrangean dual via the **subgradient method**:

$$\lambda_{ij}^{k+1} = \max\left\{0, \lambda_{ij}^{k} + \alpha \frac{\partial L(\lambda,\mu)}{\partial \lambda_{ij}^{k}}\right\} = \max\left\{0, \lambda_{ij}^{k} + \alpha (x_{ij}^{k} - y_{ij}^{h_k})\right\}$$

$$\mu_{ij}^{k+1} = \max\left\{0, \mu_{ij}^{k} + \alpha \frac{\partial L(\lambda,\mu)}{\partial \mu_{ij}^{k}}\right\} = \max\left\{0, \mu_{ij}^{k} + \alpha \left(\frac{w_j^k - w_i^k + M y_{ij}^{h_k}}{c_{ij}^{k} + M} \right)\right\}$$

where $\alpha = \dfrac{\beta(R_{UB} - L(\lambda,\mu))}{\left\| \nabla L(\lambda,\mu) \right\|^2}$

$\beta = 2$ and halved every $n = 30$ iterations

Lagrangean heuristic

$$x_{ij}^{k} \le y_{ij}^{h_k} \quad \forall (i,j) \in T, \forall k = 1,\dots,s \quad (6)$$

Relaxed solution may generate paths passing through tunnels closed

$$w_j^{k} - w_i^{k} \le c_{ij}^{k} + M(1 - y_{ij}^{h_k}) \quad \forall (i,j) \in T, \forall k = 1,\dots,s \quad (8)$$

Relaxed solution may generate paths with non minimum cost

1. $\forall h \in H$ we consider closed each tunnel (i,j) s.t. $y_{ij}^{h} = 0$ in the solution of S1
2. For each commodity k we solve a shortest path problem on a graph where all tunnels closed for category h_k are eliminated

Some computational results

- PC Intel Xeon 2.80 GHz and 512KB L2 cache, 2GB RAM
- For practical reasons S_2^k and Lagrangean heuristic min path problems are solved by AMPL-CPLEX 11.0
- **Lecco instance**:

  12 o-d pairs (12·3=36 shipment requests)

  22 iterations of subgradient method (CPU time limit=24h)

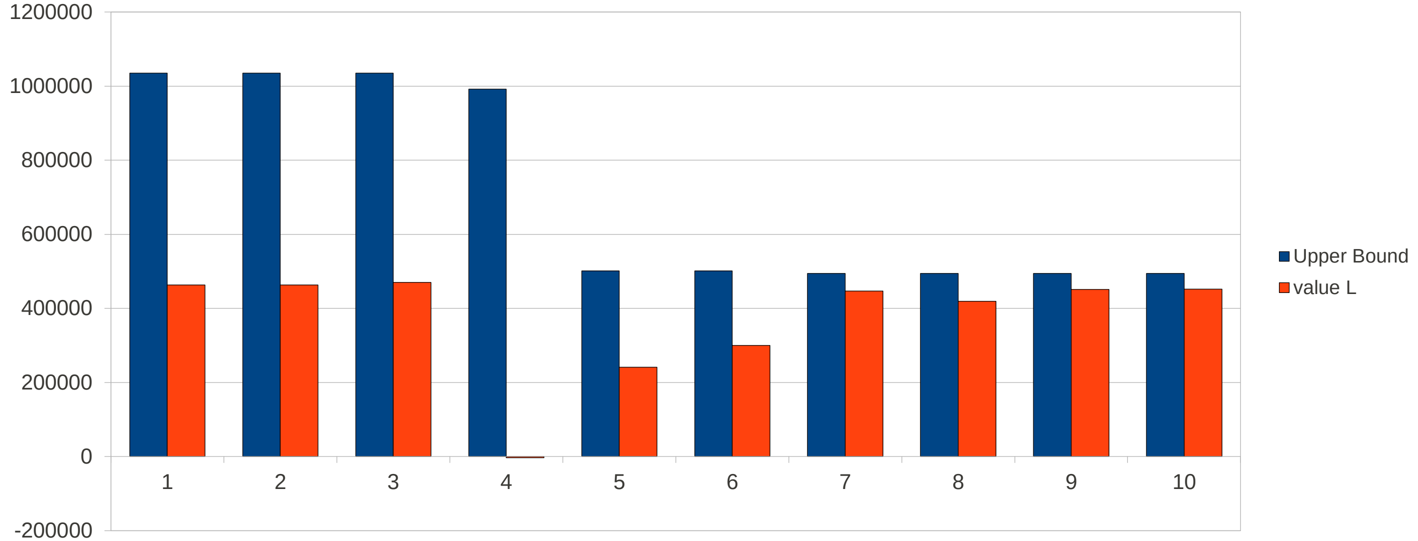
R^*	R^{heur}	L	$(R^{heur} - R^*)/R^*$	$(R^* - L)/R^*$
98907	102049	93869	3.18%	5.40%

Risk reduction of 16.3% compared to the unregulated scenario

Some computational results

- **Brescia instance**:

  32 o-d pairs (32·3=96 shipment requests)



R^{heur}	L	$(R^{heur}-L)/L$
493880	451926	8.49%

(CPU time limit=24h)

Risk reduction of 52.3% compared to the deregulated scenario!

Dantzig-Wolfe decomposition

- Suppose a COP is modeled this way:

$$
\begin{aligned}
z_P = \min\ & \mathbf{c}_1\mathbf{x} + \mathbf{c}_2\mathbf{y} && (5.1)\\
s.t.\ & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \geq \mathbf{b} && (5.2)\\
& \mathbf{D}\mathbf{y} \geq \mathbf{d} && (5.3)\\
& \mathbf{x} \geq \mathbf{0} && (5.4)\\
& \mathbf{y} \geq \mathbf{0} \text{ and integer} && (5.5)
\end{aligned}
$$

- Suppose that (5.2) are the only difficult constraints

- Suppose the feasible region is non empty and bounded

Dantzig-Wolfe decomposition

- Let $F = \{(x, y): Dy \geq d, x \geq 0, y \geq 0 \text{ and integer}\}$ which we assume bounded and non empty

- Let $\{(x^t, y^t): t = 1, \ldots, T\}$ be the extreme points of F

- **Main idea:**
  - With the ***master problem*** we compute the best convex combination of the current extreme points that also satisfy the relaxed constraints (5.2)
  - With a ***subproblem*** we identify a possible less expensive extreme point of *F* computing the reduced costs

Dantzig-Wolfe decomposition

- **Master problem**:

$$z_{MDW} = \min \sum_{t=1}^{T} (\mathbf{c}_1 \mathbf{x}^t + \mathbf{c}_2 \mathbf{y}^t)\mu_t \quad (5.15)$$

$$s.t. \sum_{t=1}^{T} (\mathbf{A}\mathbf{x}^t + \mathbf{B}\mathbf{y}^t)\mu_t \geq \mathbf{b} \quad (5.16)$$

$$\sum_{t=1}^{T} \mu_t = 1 \quad (5.17)$$

$$\mu_t \geq 0, \qquad t = 1, \ldots, T \quad (5.18)$$

- **Corresponding subproblem**:

$$z_{SDW}(\mathbf{u}, \alpha) = \min\ (\mathbf{c}_1 - \mathbf{u}\mathbf{A})\mathbf{x} + (\mathbf{c}_2 - \mathbf{u}\mathbf{B})\mathbf{y} - \alpha \quad (5.19)$$

$$s.t.\ \mathbf{D}\mathbf{y} \geq \mathbf{d} \quad (5.20)$$

$$\mathbf{x} \geq \mathbf{0} \quad (5.21)$$

$$\mathbf{y} \geq \mathbf{0} \text{ and integer} \quad (5.22)$$

where ***u*** and *α* are the dual variables associated with (5.16) and (5.17), respectively

Dantzig-Wolfe decomposition matheur.

Algorithm 2: DWHEURISTIC

1 identify a master MDW and an "easy" subproblem SDW($\mathbf{u}$,α), set T=0
2 **repeat**
3   solve master problem MDW
4   given the solution $\boldsymbol{\mu}$ of MDW define $(\mathbf{x}, \mathbf{y}) = \sum_{t=1}^{T} (\mathbf{x}^t, \mathbf{y}^t)\mu_t$
5   solve problem SDW($\mathbf{u}$,α), where ($\mathbf{u}$,α) is the dual solution of MDW
6   construct feasible solutions using $(\mathbf{x}, \mathbf{y})$ and/or ($\mathbf{u}$,α), generated by MDW,
7   and/or $(\mathbf{x}', \mathbf{y}')$, generated by SDW($\mathbf{u}$,$\alpha$)
8   **if** *(no more columns can be added)* **then**
9   STOP
10   **else**
11   set $T = T + 1$
12   add the column $(\mathbf{x}', \mathbf{y}')$ generated at step 5
13 **until** *(end_condition)* ;

Application of Dantzig-Wolfe to SSCFLP

- **Single Source Capacitated Facility Location Problem (SSCFLP):**
  Given *n* customers and *m* possible facility locations, each customer *j* has an associated demand, q_j, that must be served by a single facility, each facility *i* has an overall capacity Q_i. The costs are composed of a cost c_{ij} for supplying the demand of a customer j from a facility established at location *i* and of a fixed cost, f_i, for opening a facility at location *i*.
  We want to decide which facilities opening and how to assign the customers to the facilities so that the overall cost is minimized.

$$\min \sum_{i\in I, j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i \qquad (5.50)$$

$$s.t. \sum_{i\in I} x_{ij} = 1, \qquad j \in J \qquad (5.51)$$

$$\sum_{j\in J} q_j x_{ij} \le Q_i y_i, \qquad i \in I \qquad (5.52)$$

$$x_{ij} \in \{0,1\}, \qquad i \in I, j \in J \qquad (5.53)$$

$$y_i \in \{0,1\}, \qquad i \in I \qquad (5.54)$$

Application of Dantzig-Wolfe to SSCFLP

- **Master problem**:

$$z_{MDW} = \min \sum_{k=1}^{t} \left(\sum_{i \in I, j \in J} c_{ij} x_{ij}^k + \sum_{i \in I} f_i y_i^k \right) \lambda_k \tag{5.64}$$

$$\text{s.t.} \sum_{k=1}^{t} \left(\sum_{i \in I} x_{ij}^k \right) \lambda_k = 1, \qquad j \in J \tag{5.65}$$

$$\sum_{k=1}^{t} \lambda_k = 1 \tag{5.66}$$

$$\lambda_k \geq 0, \qquad k = 1, \ldots, t \tag{5.67}$$

- **Corresponding subproblem**:

$$z_{SDW}(\mathbf{u}, \alpha) = \min \sum_{i \in I, j \in J} (c_{ij} - u_j) x_{ij} + \sum_{i \in I} f_i y_i - \alpha \tag{5.68}$$

$$\text{s.t.} \sum_{j \in J} q_j x_{ij} \leq Q_i y_i, \qquad i \in I \tag{5.69}$$

$$x_{ij} \in \{0, 1\}, \qquad i \in I, j \in J \tag{5.70}$$

$$y_i \in \{0, 1\}, \qquad i \in I \tag{5.71}$$

Application of Dantzig-Wolfe to SSCFLP

2. Solve subproblem SDW by solving |I| knapsack problems separately:

$$z_{SDW}(\boldsymbol{u}, \alpha) = \min \sum_{j \in J} (c_{ij} - u_j) x_{ij}$$

$$\sum_{j \in J} q_j x_{ij} \leq Q_i, \quad x_{ij} \in \{0,1\}$$

3. For each $i \in I$, if $z_{SDW}(\boldsymbol{u}, \alpha) < -f_i \Rightarrow y_i = 1,$ otherwise $y_i = 0$

Application of Dantzig-Wolfe to SSCFLP

4. Check for unsatisfied constraints: the solution obtained may have cutomers assigned to multiple or no location. This can be detected by inspection. If the solution is feasible go to step 6, otherwise go to step 5.

5. Build a feasible solution: let $\bar{I}$ be the set of locations chosen in step 3. Solve the following GAP:

$$
\begin{aligned}
z_{GAP} = \min \sum_{i \in \bar{I}, j \in J} & c_{ij} x_{ij} \\
s.t. \sum_{i \in \bar{I}} x_{ij} &= 1, && j \in J \\
\sum_{j \in J} q_j x_{ij} &\leq Q_i, && i \in \bar{I} \\
x_{ij} &\in \{0,1\}, && i \in \bar{I}, j \in J
\end{aligned}
$$

6. Stop condition: if $z_{SDW}(\boldsymbol{u}, \alpha) \geq 0 \Rightarrow \text{STOP}$, otherwise add the new column of SDW to the master problem MDW and solve it again.

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