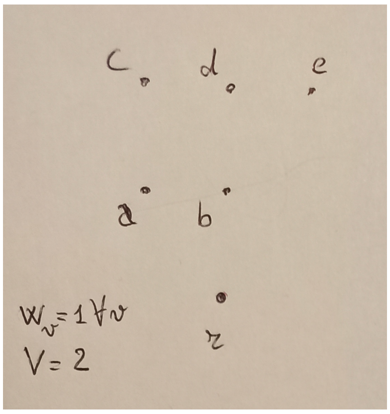
**Scatter Search**

**Exercise** Consider the following instance of the Capacitated Minimum Spanning Tree ($CMST$) problem, that is a complete graph with 5 vertices of unitary weight $w_v = 1$ for all $v \in \{a, b, c, d, e\}$, a root vertex, capacity $V = 2$




and the following cost matrix

| | $r$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| $r$ | - | 14 | 11 | 25 | 26 | 24 |
| $a$ | 14 | - | 15 | 13 | 16 | 27 |
| $b$ | 11 | 15 | - | 15 | 12 | 18 |
| $c$ | 25 | 13 | 15 | - | 14 | 20 |
| $d$ | 26 | 16 | 12 | 14 | - | 10 |
| $e$ | 24 | 27 | 18 | 20 | 10 | - |

Given solutions

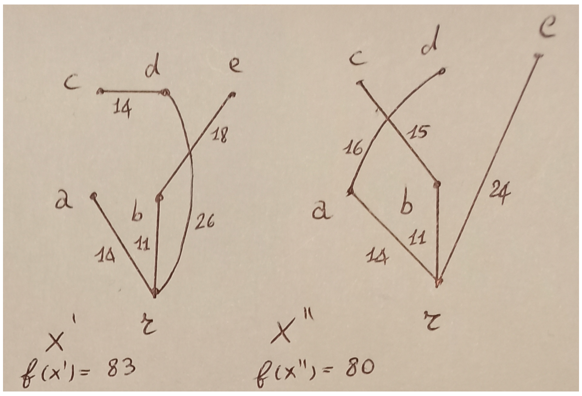
- $x' = \{(r, a), (r, b), (r, d), (b, e), (c, d)\}$
- $x'' = \{(r, a), (r, b), (r, e), (a, d), (b, c)\}$

apply the following recombination procedures:

a) alternated greedy extraction based on Kruskal's algorithm (cheapest edge not inducing cycles or unfeasible trees) starting from $x'$;

b) alternated random extraction with uniform probability assuming the pseudorandom number sequence $0.1, 0.6, 0.8, 0.5, 0.3$.

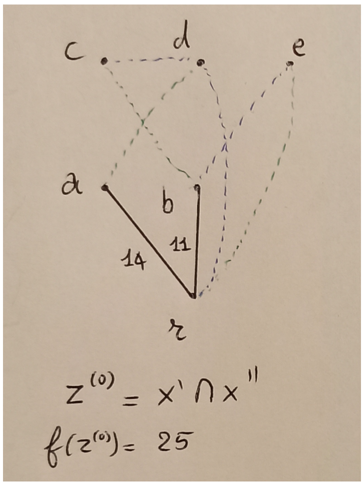
**Solution** The following picture represents the two given solutions.




Both recombination procedures suggest to start from $z^{(0)} = x' \cap x'' = \{(r,a),(r,b)\}$.

**Part a)** The alternate greedy recombination considers as acceptable all edges that produce forests (no cycles) whose trees have weight not larger than $V = 2$, and that belong, respectively, to $x' \setminus z^{(t)}$ in the odd iterations and to $x'' \setminus z^{(t)}$ in the even iterations. Consequently, the operations comply with the following sequence:

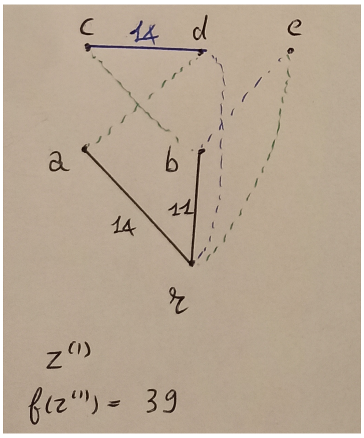
1. start from $z^{(0)} = x' \cap x'' = \{(r,a),(r,b)\}$;




2. all edges of $x' \setminus z^{(0)} = \{(r,d),(b,e),(c,d)\}$ are compatible with the constraints (forests with trees of weight $\geq V = 2$); add the cheapest one, that is $(c,d)$;

| Add | Evaluation |
|---|---|
| $(r,d)$ | $\delta f = 26$ |
| $(b,e)$ | $\delta f = 18$ |
| $(c,d)$ | $\delta f = 14$ |

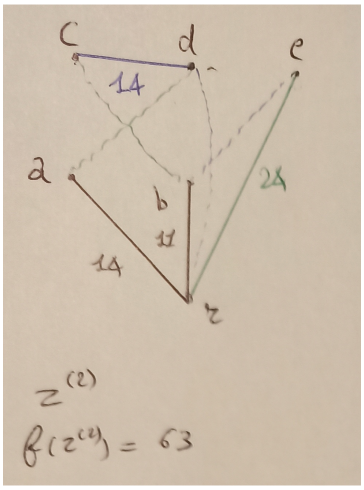
now $z^{(1)} = \{(r,a),(r,b),(c,d)\}$.



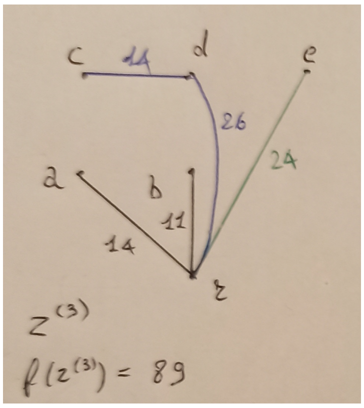

3. in set $x'' \setminus z^{(1)} = \{(r, e), (a, d), (b, c)\}$ only edge $(r, e)$ is compatibile with the constraints, because $(a, d)$ and $(b, c)$ would yield trees of weight 3;

| Add | Evaluation |
|---|---|
| $(r, e)$ | $\delta f = 24$ |
| $(a, d)$ | Unfeasible (capacity) |
| $(b, c)$ | Unfeasible (capacity) |

now $z^{(2)} = \{(r, a), (r, b), (c, d), (r, e)\}$.

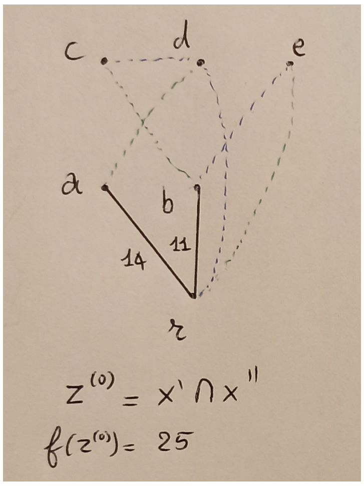



4. finally, in set $x' \setminus x = \{(r, d), (b, e)\}$ only edge $(r, d)$ is compatibile with the constraints, because $(b, e)$ closes a loop; now $z^{(3)} = \{(r, a), (r, b), (c, d), (r, e), (r, d)\}$ is a feasible solution.




**Part b)** The alternate random strategy extracts one edge from $x'$ and one from $x''$ alternatively, selecting it based on the given pseudorandom number sequence. If this generates an incomplete solution, further edges from the overall graph must be added, possibly with a greedy constructive heuristic. In the specific case:

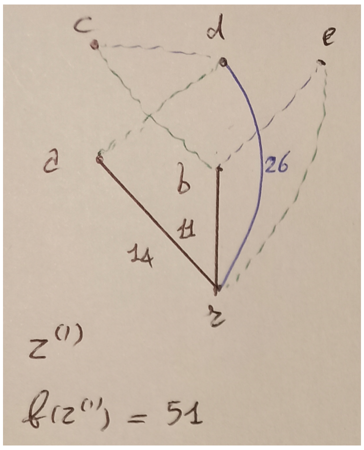
1. start from $z^{(0)} = x' \cap x'' = \{(r,a),(r,b)\}$;




2. in $x' \setminus z^{(0)} = \{(r,d),(b,e),(c,d)\}$ all edges are compatible with the constraints; we select $(r,d)$ because there are three alternatives and 0.1 selects the first one (the probabilities are uniform);

| Add | Evaluation | Cumulated probability |
|---|---|---|
| $(r,d)$ | $\delta f = 26$ | $0.\bar{3}$ |
| $(b,e)$ | $\delta f = 18$ | $0.\bar{6}$ |
| $(c,d)$ | $\delta f = 14$ | 1.0 |

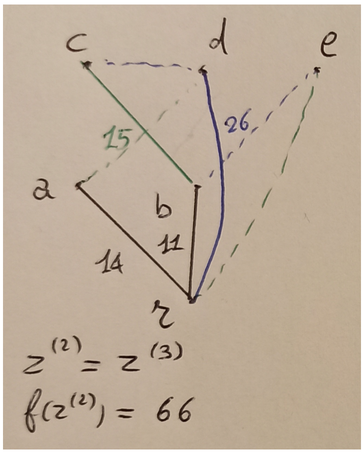
now $z^{(1)} = \{(r,a),(r,b),(r,d)\}$.




3. in $x'' \setminus z^{(1)} = \{(r,e),(a,d),(b,c)\}$ only the edges $(r,e)$ and $(b,c)$ are compatible with the constraints because $(a,d)$ would close a loop, and 0.6 selects the second one;

| Add | Evaluation | Cumulated probability |
|---|---|---|
| $(r,e)$ | $\delta f = 24$ | 0.5 |
| $(a,d)$ | Unfeasible (cycle) | |
| $(b,c)$ | $\delta f = 15$ | 1.0 |

now $z^{(2)} = \{(r,a),(r,b),(r,d),(b,c)\}$.

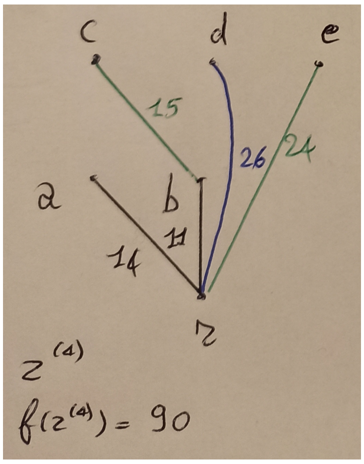



4. in $x' \setminus z^{(2)} = \{(b,e),(c,d)\}$ no edge is compatible with the constraints, because $(b,e)$ would yield a tree with weight $> V = 2$ and $(c,d)$ would close a loop;

| Add | Evaluation |
|---|---|
| $(b,e)$ | Unfeasible (capacity) |
| $(c,d)$ | Unfeasible (cycle) |

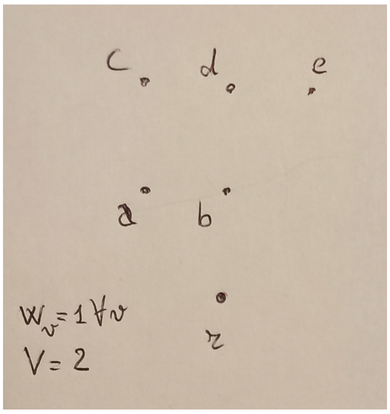
the solution does not change: $z^{(3)} = z^{(2)} = \{(r,a),(r,b),(r,d),(b,c)\}$.

5. in $x'' \setminus z^{(3)} = \{(r,e),(a,d)\}$ only edge $(r,e)$ is compatible with the constraints because $(a,d)$ would close a loop; now $z^{(4)} = \{(r,a),(r,b),(r,d),(b,c),(r,e)\}$ is a feasible solution.


c
d
e
15
26
24
a
b
11
14
z
$z^{(4)}$
$f(z^{(4)}) = 90$

## Path Relinking

**Exercise** Consider the following instance of the Capacitated Minimum Spanning Tree (*CMST*) problem, that is a complete graph with 5 vertices of unitary weight $w_v = 1$ for all $v \in \{a, b, c, d, e\}$, a root vertex, capacity $V = 2$




and the following cost matrix

| | $r$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| $r$ | - | 14 | 11 | 25 | 26 | 24 |
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| $c$ | 25 | 13 | 15 | - | 14 | 20 |
| $d$ | 26 | 16 | 12 | 14 | - | 10 |
| $e$ | 24 | 27 | 18 | 20 | 10 | - |

Given solutions

- $x' = \{(r, a), (r, b), (r, d), (b, e), (c, d)\}$
- $x'' = \{(r, a), (r, b), (r, e), (a, d), (b, c)\}$

apply a Path Relinking procedure from $x'$ to $x''$ using the one-swap neighbourhood $\mathcal{N}_{S_1}$ with respect to the edges of the graph.

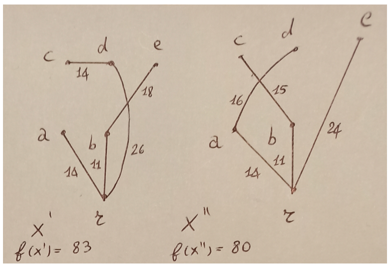
**Solution** The problem and the two starting solutions are the same as for the *Scatter Search* exercise. In order to reduce the Hamming distance from the current solution $x$ to the final one, $x''$, every exchange must:

- delete an edge from $z \setminus x''$;
- add an edge from $x'' \setminus z$.

At the first step, $z^{(0)} = x'$ and the possible swaps require to:

- delete an edge from $z^{(0)} \setminus x'' = \{(r, d), (b, e), (c, d)\}$;
- add an edge from $x'' \setminus z^{(0)} = \{(r, e), (a, d), (b, c)\}$.

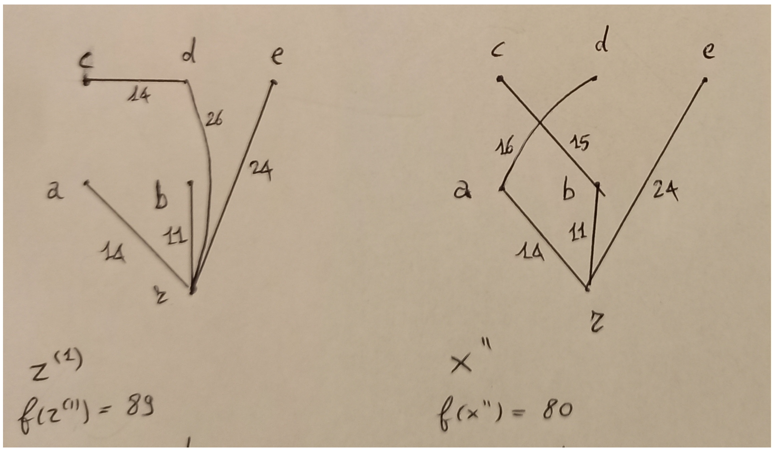
The two extreme solutions are represented in the following picture.




The possible swaps are evaluated in the following table.

| Add | Delete | Evaluation |
|---|---|---|
| | $(r, d)$ | Unfeasible (cycle and disconnection) |
| $(r, e)$ | $(b, e)$ | $\delta f = 24 - 8 = 16$ |
| | $(c, d)$ | Unfeasible (cycle and disconnection) |
| | $(r, d)$ | Unfeasible (capacity) |
| $(a, d)$ | $(b, e)$ | Unfeasible (cycle and disconnection) |
| | $(c, d)$ | Unfeasible (cycle and disconnection) |
| | $(r, d)$ | Unfeasible (capacity) |
| $(b, c)$ | $(b, e)$ | Unfeasible (cycle and disconnection) |
| | $(c, d)$ | Unfeasible (capacity) |

Since only swapping $(r, e)$ and $(b, e)$ is feasible and reduces the Hamming distance, the next solution is $z^{(1)} = \{(r, a), (r, b), (r, d), (r, e), (c, d)\}$ and its cost is $f\left(z^{(1)}\right) = 89$



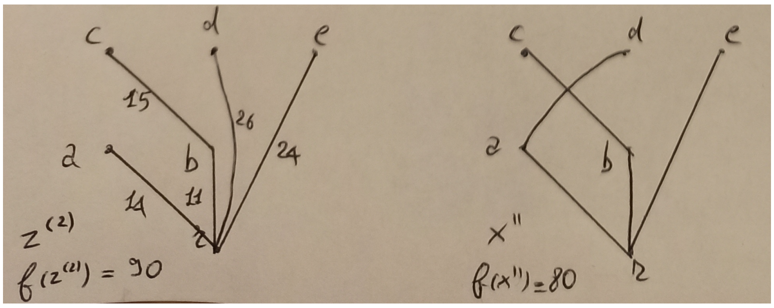

The possible swaps from $z^{(1)}$ towards $x''$ require to:

- delete an edge from $z^{(1)} \setminus x'' = \{(r, d), (c, d)\}$;
- add an edge from $x'' \setminus z^{(1)} = \{(a, d), (b, c)\}$.

and are evaluated in the following table.

| Add | Delete | Evaluation |
|---|---|---|
| $(a, d)$ | $(r, d)$ | Unfeasible (capacity) |
| | $(c, d)$ | Unfeasible (cycle and disconnection) |
| $(b, c)$ | $(r, d)$ | Unfeasible (capacity) |
| | $(c, d)$ | $\delta f = 15 - 14 = 1$ |

The only feasible exchange yields $z^{(2)} = \{(r,a),(r,b),(r,d),(r,e),(c,d)\}$, whose cost is $f\left(z^{(2)}\right) = 90$.




A final swap between $(a,d)$ and $(r,d)$ would allow to reach $x''$, but this is not required because that solution is already known.

The best solution found along the path is $z^{(1)}$, and it will be tested for insertion in the reference set. Probably, it is not a very promising starting point for further search, but this is mainly due to the very small size of the instance.