5.1 Branch-and-Bound

Given the integer linear program

$$\begin{aligned} \max z = 3x_1 + 4x_2 & \\ 2x_1 + x_2 &\leq 6 \\ 2x_1 + 3x_2 &\leq 9 \\ x_1, x_2 &\geq 0, \text{ integer} \end{aligned}$$

solve it via the Branch-and-Bound method (solving graphically the continuous relaxation of each subproblem encountered in the enumeration tree). Branch on the fractional variable with fractional value closest to $\frac{1}{2}$. Among the set of active nodes, pick that with the most promising bound.

5.2 Branch-and-Bound for 0-1 knapsack

A bank has 14 million Euro, which can be invested into stocks of four companies (1, 2, 3, and 4). The table reports, for each company, the net revenue and the amount of money that must be invested into it.

Company	1	2	3	4
Revenue	16	22	12	8
Money	5	7	4	3

Given an integer linear programming formulation for the probelm of choosing a set of companies that maximizes the total revenue. Observe that no partial investment can be done, i.e., for each company we can either invest into it or not. Solve the problem with the Branch-and-Bound algorithm. Show that the solution to each continuous relaxation can be found with a greedy algorithm.

5.3 Cutting plane algorithm 1

Given the integer linear program

$$\begin{aligned} \min \quad & x_1 - 2x_2 \\ & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0, \text{ integer} \end{aligned}$$

solve it via Gomory's cutting plane method, solving the continuous relaxations graphically.

5.4 Cutting plane algorithm 2

Given the integer linear program

$$
\begin{aligned}
\min \quad & -x_2 \\
& 3x_1 + 2x_2 \leq 6 \\
& -3x_1 + 2x_2 \leq 0 \\
& x_1, x_2 \geq 0, \text{ integer}
\end{aligned}
$$

solve it via Gomory's cutting plane method, solving the continuous relaxations graphically.

SOLUTION

5.1 Branch-and-Bound

The enumeration tree is reported in Figure 1. The graphical solution of each subproblem is reported. The subproblems are solved in the following order: P1, P2, P3, P4, P5, P6, P7. Note that when the optimal value $\bar{z}$ of a subproblem is fractional, we can round the upper bound given by the subproblem to $\lfloor \bar{z} \rfloor$. For instance, in P1 we obtain the bound $\lfloor \frac{51}{4} \rfloor = 12$.

After solving P7, we observe that P6 yields an integer solution which is worse than that of P7, which is therefore discarded. We also observe that P2 yields an upper bound which is smaller than the value of the best feasible solution found (in P7). The node is therefore pruned. The optimal solution (found in P7) is $x^* = (0, 3)$, of value $z^* = 12$.

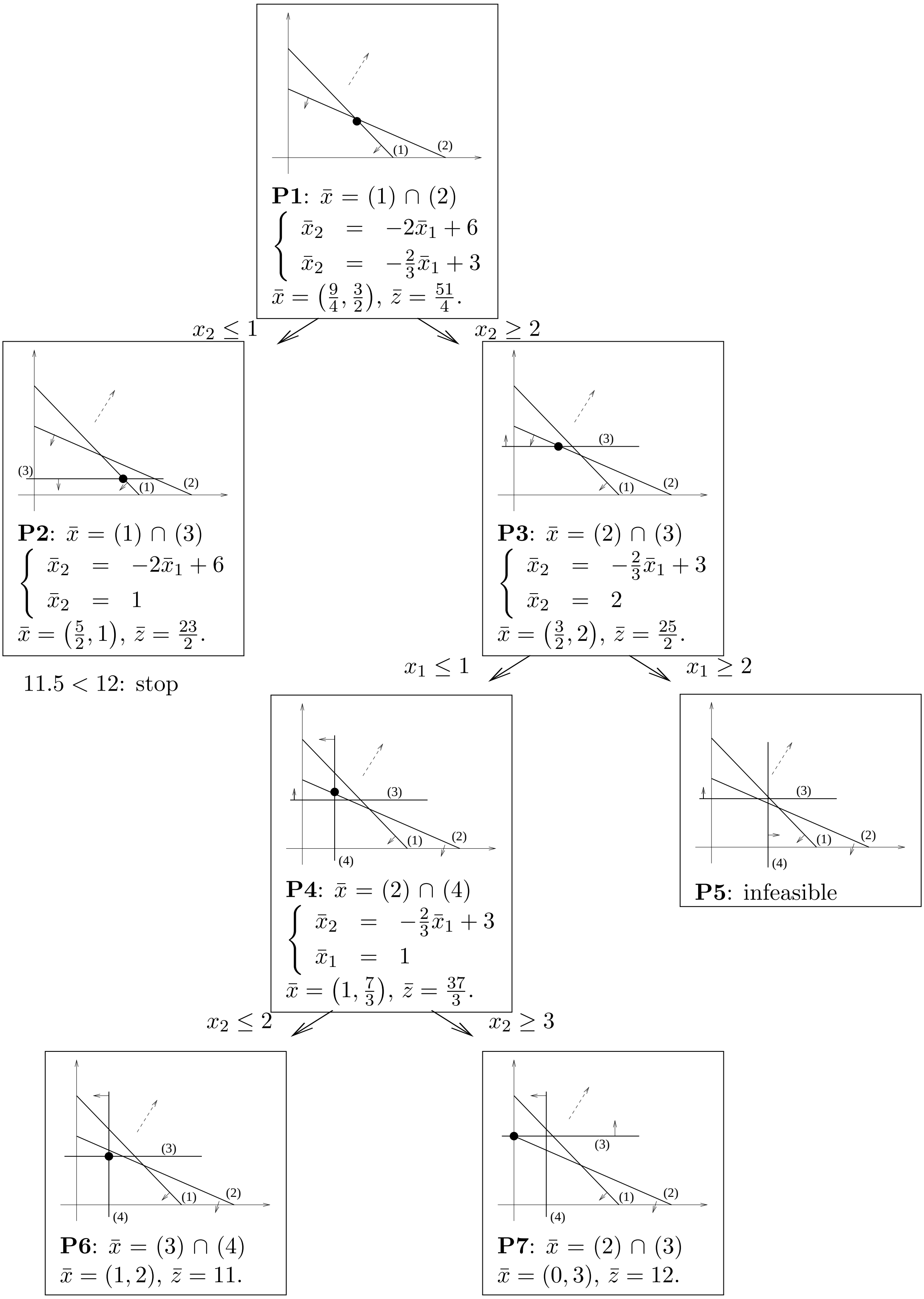

Figure 1: Enumeration tree for problem 5.2

5.2 Branch-and-Bound for 0-1 knapsack

The integer linear programming formulation for the problem is

$$\begin{aligned} \max 16x_1 + 22x_2 + 12x_3 + 8x_4 & \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 & \leq 14 \\ x_1, x_2, x_3, x_4 & \in \{0, 1\}. \end{aligned}$$

An optimal solution to its linear relaxation can be found as follows. First, sort the ratios between revenues and costs, obtaining

$$(16/5, 22/7, 12/4, 8/3) = (3.2, 3.14, 3, 2.66)$$

Then, put to 1 all the variables according to the ordering, until variable $i' : \sum_{i<i'} c_i \leq B$, and $\sum_{i<i'} c_i + c_{i'} > B$. Let $x_{i'} = \frac{c_{i'}}{B - \sum_{i<i'} c_i}$, and let all the other variables be equivalent to zero.

For instance, at node 1 we have: $x_1 = 1$, (it uses 5 units), $x_2 = 1$ (7 units), $x_3 = \frac{1}{2}$ (2/4=1/2 units). Since, at each branching iteration, we set a variable either to 0 or 1, this method can be applied in any node of the enumeration tree, by fixing the approriate variables.

The enumeration tree is given in Figure 2. Some observations:

- The index t indicates the order by which the subproblems are solved. l'ordine di risoluzione dei problemi.
- Since all variables are integer, whenever a subproblem yields a solution with fractional value, we round it to $\lfloor \bar{z} \rfloor$.
- The lower bounds (LB) is *not* computed at each node (to do this, a heuristic should be applied). We update it whenever a subproblem yields a feasible solution. Note that this value is NOT related to the specific subproblem, as it depends only on the iteration. Indeed, at each iteration t, LB corresponds to the value of the best feasible solution found in any part of the enumeration tree. For instance, in subproblem 4 we find a feasible solution of value $\bar{z} = 36$. Since it is the first that is found and LB still has the initial value of $+\infty$, we set LB to 36.
- In subproblem 6 an integer solution is found and the node is *pruned by feasibility.*
- Subproblem 7 is infeasible, since $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 1$ require a budget of $16 > 14$. The node is *pruned by infeasibility.*
- Subproblem 8 yields an upper bound of $\bar{z} = 38$ which is strictly smaller than the current LB of 42. The node is *pruned by bound.*
- The same happens for subproblem 9, where $\bar{z} = 42 + \frac{6}{7}$. The upper bound is $\lfloor \bar{z} \rfloor = 42$, which is strictly smaller than the current LB of value 42. Node 9 is *pruned by bound.*

The final optimal solution, which is found in node 9, is $x^* = (0, 1, 1, 1)$, of value 42.

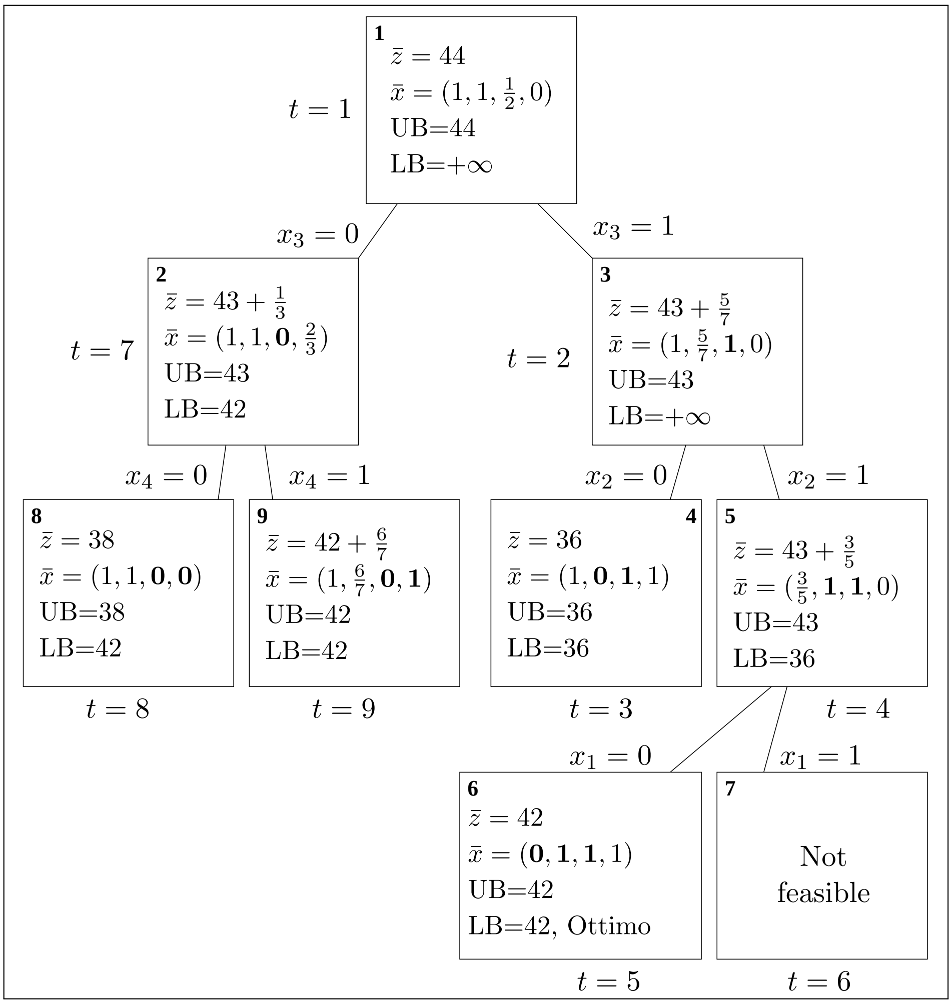

Figure 2: Enumeration tree for problem 5.3

5.3 Cutting plane algorithm 1

The continuous relaxation of the the problem at hand, reduced to standard form, reads

$$\begin{aligned} \min \quad & x_1 - 2x_2 \\ & -4x_1 + 6x_2 + x_3 = 9 \\ & x_1 + x_2 + x_4 \quad = 4 \\ & x_1, x_2, x_3, x_4 \quad \geq 0, \end{aligned}$$

were x_3, x_4 are slack variables.

We solve it via the simplex method. We obtain the following sequence of tableaux, where the pivot element is denoted by the symbol $\square$.

	x_1	x_2	x_3	x_4
0	1	-2	0	0
9	-4	$\boxed{6}$	1	0
4	1	1	0	1

	x_1	x_2	x_3	x_4
3	$-\frac{1}{3}$	0	$\frac{1}{3}$	0
$\frac{3}{2}$	$-\frac{2}{3}$	1	$\frac{1}{6}$	0
$\frac{5}{2}$	$\boxed{\frac{5}{3}}$	0	$-\frac{1}{6}$	1

	x_1	x_2	x_3	x_4
$\frac{7}{2}$	0	0	$\frac{3}{10}$	$\frac{1}{5}$
$\frac{5}{2}$	0	1	$\frac{1}{10}$	$\frac{2}{5}$
$\frac{3}{2}$	1	0	$-\frac{1}{10}$	$\frac{3}{5}$

The optimal solution to the relaxation is $\bar{x} = (\frac{3}{2}, \frac{5}{2})$, where $x_3 = x_4 = 0$ (see Figure 3).

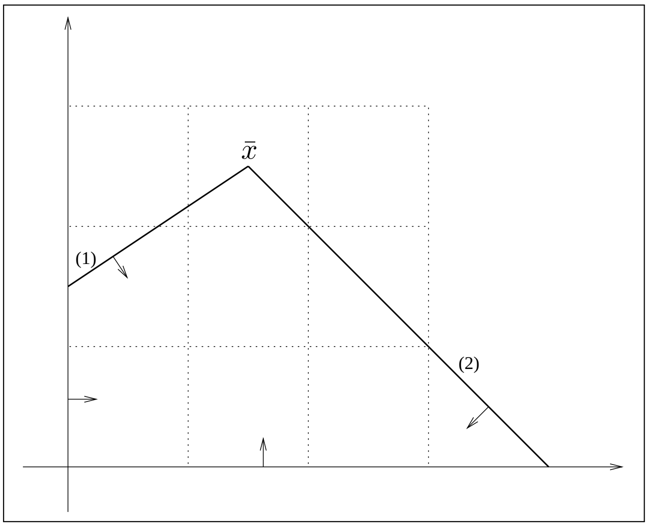

Figure 3: Graphical solution to problem 5.3

We derive a Gomory cut from the first row of the optimal tableau $x_2 + \frac{1}{10}x_3 + \frac{2}{5}x_4 = \frac{5}{2}$.

The cut is defined as

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor, \tag{1}$$

where N is the set of the indices of the nonbasic variables and i is the index of the basic variable corresponding to the tableau row that is chosen. We obtain the following cut (in integer form)

$$x_2 \leq 2,$$

see Figure 4 (constraint (3)). We can derive the fractional form of the cut by considering the row

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i$$

and subtracting from it the cut (1), obtaining

$$\sum_{j \in N} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq (\bar{b}_i - \lfloor \bar{b}_i \rfloor).$$

In our case, we have

$$\frac{1}{10}x_3 + \frac{2}{5}x_4 \geq \frac{1}{2}.$$

To iterate, we should add the newly found cut to the formulation and reoptimize it. In principle, without adopting any more sophisticated method, we should start a new optimization *ex novo*, solving the new problem which contains three inequalities. For 2-dimensional problems, like that at hand, rather than reoptimizing from scratch, we can save some computations by exploiting the graphical representation. Indeed, by looking at the new feasible region, as reported in Figure 4, we easily find the new solution $\tilde{x}$.. In such point, the set of nonbasic variables is $\{x_3, x_5\}$, whereas the basic ones are $\{x_1, x_2, x_4\}$.

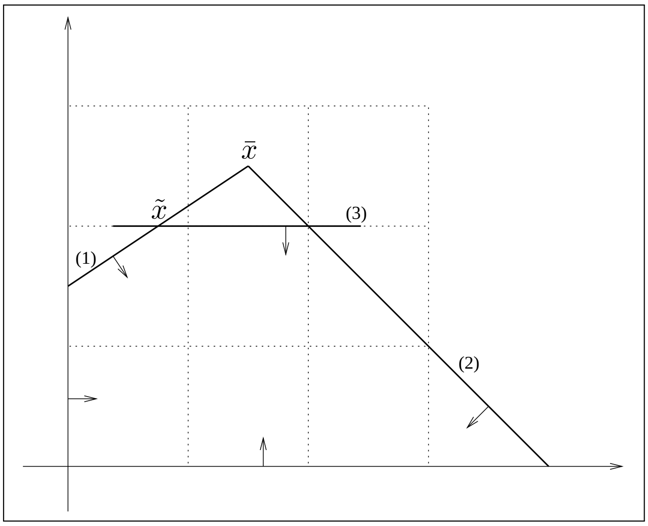

Figure 4: First Gomory cut for problem 5.3

We proceed as follows. First, we restate the new cut in standard form, by introducing a surplus variable $x_5 \geq 0$, obtaining

$$\frac{1}{10}x_3 + \frac{2}{5}x_4 - x_5 = \frac{1}{2}.$$

Observe that x_5 only occurs in the new row. Therefore, it is directly added to the set of basic variables. We multiply the cut by -1, obtaining

$$-\frac{1}{10}x_3 - \frac{2}{5}x_4 + x_5 = -\frac{1}{2},$$

and add it to the tableau, yielding

	x_1	x_2	x_3	x_4	x_5
$\frac{7}{2}$	0	0	$\frac{3}{10}$	$\frac{1}{5}$	0
$\frac{5}{2}$	0	1	$\frac{1}{10}$	$\frac{2}{5}$	0
$\frac{3}{2}$	1	0	$-\frac{1}{10}$	$\frac{3}{5}$	0
$-\frac{1}{2}$	0	0	$-\frac{1}{10}$	$-\frac{2}{5}$	1

Evidently, in this tableau (which yields an infeasible solution, with a negative right-hand side in the last row) we have $\{x_3, x_4\}$ nonbasic and $\{x_1, x_2, x_5\}$ basic. We need to swap x_4 with x_5, which is obtained by pivoting on the highlighted element

	x_1	x_2	x_3	x_4	x_5
$\frac{7}{2}$	0	0	$\frac{3}{10}$	$\frac{1}{5}$	0
$\frac{5}{2}$	0	1	$\frac{1}{10}$	$\frac{2}{5}$	0
$\frac{3}{2}$	1	0	$-\frac{1}{10}$	$\frac{3}{5}$	0
$-\frac{1}{2}$	0	0	$-\frac{1}{10}$	$\boxed{-\frac{2}{5}}$	1

obtaining the tableau

	x_1	x_2	x_3	x_4	x_5
$\frac{13}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{2}$
2	0	1	0	0	1
$\frac{3}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{3}{2}$
$\frac{5}{4}$	0	0	$\frac{1}{4}$	1	$-\frac{5}{2}$

which clearly yields $\tilde{x} = (\frac{3}{4}, 2)$ as a solution. We can now derive a new cut, by picking the second row

$$x_1 - \frac{1}{4}x_3 + \frac{3}{2}x_5 = \frac{3}{4},$$

from which we deduce the Gomory cut $x_1 - x_3 + x_5 \leq 0$ which, in the space of the original variables x_1, x_2, amounts to $-3x_1 + 5x_2 \leq 7$.

Adding the cut to the relaxation and reoptimizing (graphically), we obtain the new solution $x^* = (1, 2)$, as shown in Figure 5, which is integer. The method is halted.

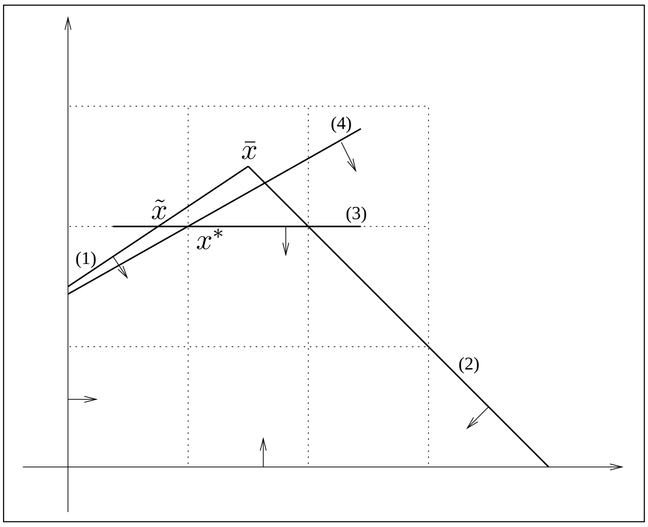

Figure 5: Last Gomory cut for problem 5.3

5.4 Cutting plane algorithm 2

The continuous relaxation of the problem, brought into standard form, reads

$$
\begin{aligned}
\min \quad & -x_2 \\
& x_1 + 2x_2 + x_3 = 6 \\
& -3x_1 + x_2 + x_4 = 0 \\
& x_1, x_2, x_3, x_4 \geq 0,
\end{aligned}
$$

where x_3, x_4 are slack variables. We solve this relaxation via the simplex method. We obtain the following sequence of tableaux, where the pivot element is denoted by the symbol $\square$.

		x_1	x_2	x_3	x_4
$-z$	0	0	-1	0	0
x_3	6	3	2	1	0
x_4	0	-3	$\boxed{2}$	0	1

		x_1	x_2	x_3	x_4
$-z$	0	$-\frac{3}{2}$	0	0	$\frac{1}{2}$
x_3	6	$\boxed{6}$	0	1	-1
x_2	0	$-\frac{3}{2}$	1	0	$-\frac{1}{2}$

		x_1	x_2	x_3	x_4
$-z$	$\frac{3}{2}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$
x_1	1	1	0	$\frac{1}{6}$	$-\frac{1}{6}$
x_2	$\frac{3}{2}$	0	1	$\frac{1}{4}$	$\frac{1}{4}$

The corresponding optimal solution (in the original variables) is $\underline{x} = (1, \frac{3}{2})$, of value $-\frac{3}{2}$. See Figure 6 for an illustration of the feasible region of the continuous relaxation and the corresponding solution (point A in the figure).

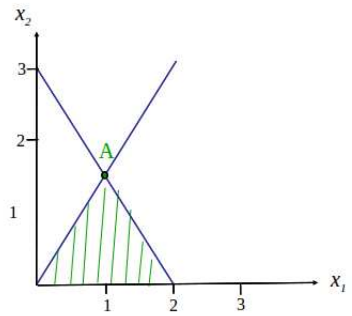

Figure 6: LP relaxation and corresponding solution for problem 5.4.

Since the solution is fractional, we can generate a Gomory cut. We consider the second row of the optimal tableau, which is

$$x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4 = \frac{3}{2},$$

which is the only row with a fractional value for the corresponding basic variable. The corresponding cut (in integer form)

$$x_2 \leq 1.$$

Reoptimizing, graphically, we obtain the result in Figure 7 with the corresponding solution $\underline{x} = (\frac{2}{3}, 1)$ (B in the picture).

Since the solution is not integer, we need to iterate the method by adding new cuts. This requires to optimize the new linear programming relaxation. As in the previous exercise, rather

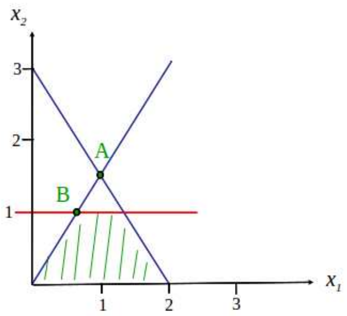

Figure 7: First Gomory cut and new LP solution for problem 5.4.

than solving a new LP from scratch, we update the tableau by exploiting the graphical representation. First, we write the fractional form of the cut

$$\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

and restate it into standard form (aftre multiplying it by -1)

$$-\frac{1}{4}x_3 - \frac{1}{4}x_4 + x_5 = -\frac{1}{2}.$$

We obtain the following tableau

	x_1	x_2	x_3	x_4	x_5	
$-z$	$\frac{3}{2}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0
x_1	1	1	0	$\frac{1}{6}$	$-\frac{1}{6}$	0
x_2	$\frac{3}{2}$	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0
x_5	$-\frac{1}{2}$	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	1

where the basic variables are $\{x_1, x_2, x_5\}$, whereas the nonbasic ones are $\{x_3, x_4\}$. Looking at point B in the figure, we see that it corresponds to the solution where $\{x_4, x_5\}$ are nonbasic and $\{x_1, x_2, x_3\}$ are basic. Therefore, we update the tableau by performing a swap between x_3 and x_5, pivoting on the highlighted element

		x_1	x_2	x_3	x_4	x_5
$-z$	$\frac{3}{2}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0
x_1	1	1	0	$\frac{1}{6}$	$-\frac{1}{6}$	0
x_2	$\frac{3}{2}$	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0
x_5	$-\frac{1}{2}$	0	0	$\boxed{-\frac{1}{4}}$	$-\frac{1}{4}$	1

and obtaining

		x_1	x_2	x_3	x_4	x_5
$-z$	1	0	0	0	0	1
x_1	$-\frac{2}{3}$	1	0	0	$-\frac{1}{3}$	$\frac{2}{3}$
x_2	1	0	1	0	0	1
x_3	2	0	0	1	1	-4

which corresponds to the new basic solution B. We can now deduce a new Gomory cut from the row

$$x_1 - \frac{1}{3}x_4 + \frac{2}{3}x_5 = \frac{2}{3},$$

which, in integer form, reads

$$x_1 - x_4 \leq 0.$$

We express it in the x_1, x_2-space by susbstituting the expression for x_4

$$x_4 = 3x_1 - 2x_2,$$

thus obtaining

$$-x_1 + x_2 \leq 0.$$

Adding it to the relaxation and reoptimizing (graphically), we obtain the solution $\underline{x} = (1, 1)$, reported in Figure 8 (point C), which is integer. The method is therefore stopped.

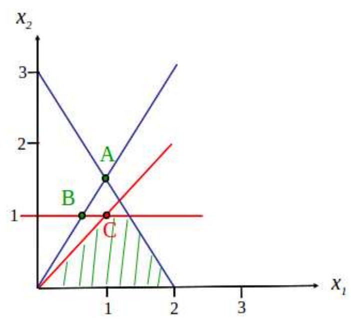

Figure 8: Second Gomory cut and integer optimal solution for problem 5.4.