Solved exercises for the course of

Foundations of Operations Research

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Branch-and-bound algorithm based on the continuous relaxation

Given the following problem

$$\begin{aligned} \min z = -2x_1 - 3x_2 & \\ x_1 + 2x_2 &\leq 9 \\ 6x_1 + 7x_2 &\leq 39 \\ 2x_1 - x_2 &\leq 8 \\ x_1, x_2 &\in \mathbb{N} \end{aligned}$$

a) solve it with the branch-and-bound algorithm based on the continuous relaxation solved graphically;

b) replace the objective function z with $z' = x_1 - 2x_2$ and solve again;

c) do the same with objective function $z'' = -3x_1 + x_2$.

The branch-and-bound-algorithm

The branch-and-bound-algorithm is a general technique to enumerate solutions implicitly. In other words, it evaluates all solutions of the problem, taking them into account not one by one but in subsets, for which it is easy to determine the optimal solution, or to prove that they contain no optimal solution.

It is a *divide et impera* (*divide and conquer*) method: the given problem (and its feasible region) is divided into subproblems (and subregions) which are separately tackled. One maintains a list $\mathcal{P}$ of *open* subproblems, that is subproblems not yet processed, from which one problem $P^{(r)}$ at a time is extracted and processed.

- If $P^{(r)}$ has no feasible solution, it is *closed*, or *pruned*.
- If $P^{(r)}$ can be solved to optimality, it is *closed*, and its optimal solution is compared to the best known solution possibly replacing it (if better); the value z_E^* of the best known solution is a *suboptimal estimate* of the global optimum. Other estimates can derive from the application of heuristic algorithms combined with the branch-and-bound mechanism. As long as no solution is known, $z_E^* = +\infty$ in minimization problem and $z_E^* = -\infty$ in maximization problem.
- If it can be proved that $P^{(r)}$ contains no solution strictly better than the best known one, P_r is closed. Such a proof can be obtained computing a *superoptimal estimate* $z_B^{(r)}$ of the best solution of P_r (local optimum). Such

an estimate is usually obtained solving a relaxation of P_r, i. e. an auxiliary problem whose optimal solution is certainly better than that of the given one. If the estimate $z_B^{(r)}$ is not strictly better than z_E^*, all solutions of $P(r)$ are dominated by the best known solution, and therefore it is unnecessary to solve P_r exactly. Notice that $z_B^{(r)}$ is a superoptimal estimate only for subproblem $P(r)$, while z_E^* is a suboptimal estimate for the whole problem P.

- If the three previous cases are not verified, $P^{(r)}$ is split into subproblems, which are introduced in the list of open problems $\mathcal{P}$.

The algorithm terminates when the list is empty; at that point, the best solution found is certainly optimal.

The process can be modelled on one side as the partition of the feasible region S into subsets S_r subsequently processed and removed, on the other side as the creation and visit of a tree, denoted as *branching tree*, whose nodes correspond to the subproblems subsequently processed. In particular, the leaves correspond to the closed problems (see Figure 1).

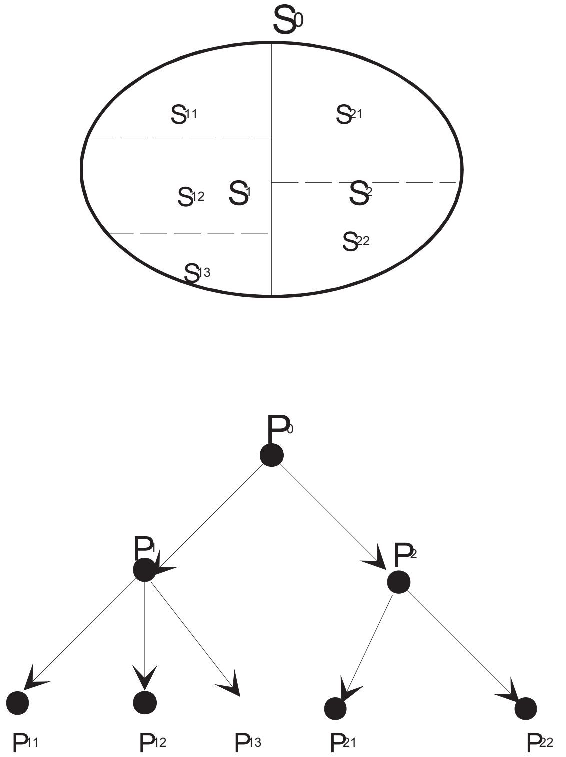

Figura 1: Scheme of the branch-and-bound algorithm

This is a general scheme, that includes very different branch-and-bound algorithms. The elements that characterize a specific branch-and-bound algorithm:

- the *visit strategy* of the branching tree, that is the criterium to select from $\mathcal{P}$ the subproblem P to be processed;
- the *bounding technique*, that is the procedure to evaluate the superoptimal estimate $z_B^{(r)}$;
- the *auxiliary heuristic*, that is the procedure to evaluate the suboptimal estimate z_E: this can be a true heuristic, or simply consist in saving the solutions of the subproblems solved to optimality;
- the *branching rule*, that is the way to split the current problem $P^{(r)}$ into reduced subproblems.

In the following, we refer to the following version of the branch-and-bound general framework:

- the tree is visited with the *depth-first* strategy, which always visits the subproblem generated most recently, so as to work on strongly constrained subproblems and to get rapidly to an integer solution;
- the *bounding* technique is the continuous relaxation ($x_i \in \mathbb{N} \to x_i \geq 0$);
- the auxiliary heuristic reduces to keeping track of the best integer solution generated so far by the continuous relaxations (at first, the suboptimal estimate, or upper bound, UB is set to $+\infty$);
- the *branching* rule consists in selecting a fractionary variable (typically *the most fractionary* one, that is the variable x_i whose fractionary part $x_i - \lfloor x_i \rfloor$ is closest to 0.5) and generate two subproblems, setting $x_i \leq \lfloor x_i^* \rfloor$ in one and $x_i \geq \lfloor x_i^* \rfloor + 1$ in the other.

Solution with the first objective function

The problem is a minimization problem. Therefore, the superoptimal estimate $z_B^{(0)}$ obtained with the continuous relaxation is a *lower bound* on the optimum, whereas the value z_E^* of the best heuristic solution found during the process is an *upper bound*. The continuous relaxation of the problem is obtained replacing constraints $x_1 \in \mathbb{N}$ and $x_2 \in \mathbb{N}$ with $x_1 \geq 0$ and $x_2 \geq 0$.

$$
\begin{aligned}
\min z = -2x_1 - 3x_2 & \\
x_1 + 2x_2 &\leq 9 \\
6x_1 + 7x_2 &\leq 39 \\
2x_1 - x_2 &\leq 8 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

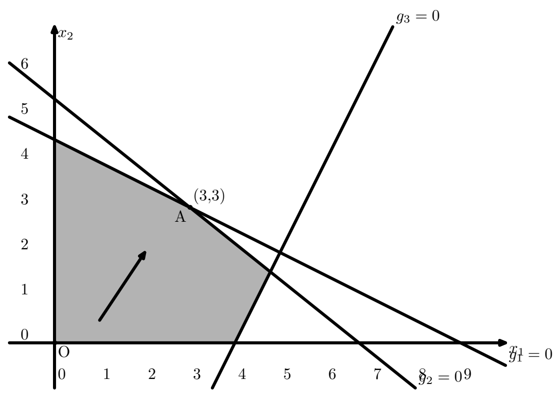

Figura 2: Graphical solution of the continuous relaxation of $P = P^{(0)}$ with the first objective function

Figure 2 reports the graphical resolution of the continuous relaxation of problem $P^{(0)}$. Its optimal solution is $x^{*(0)} = A = (3, 3)$. This provides a lower bound $z_B^{(0)} = -15$ of the optimum of $P = P^{(0)}$. In this case, however, the solution is also feasible for the integrality constraint, though the latter has been relaxed. Consequently, the current problem $P^{(0)}$ is solved to optimality. We update the best known heuristic solution, setting $z_E^* = -15$. As there is no other open problem, we have determined the optimal solution $x^* = (3, 3)$ and the optimum $z^* = -15$ of the whole problem.

Solution with the second objective function

Problem $P^{(0)}$: Figure 3 reports the graphical resolution of the continuous relaxation of $P^{(0)}$. Its optimal solution is $x^{*(0)} = B = (0, 9/2)$, from which we derive a lower bound $z_B^{(0)} = -9$. Since the solution is fractionary, neither has the problem been solved, nor can we update the suboptimal estimate z_E^*, which remains $+\infty$. This is clearly worse than $z_B^{(0)}$, so that we are forced to decompose the problem into subproblems with a *branching* operation. Given that the only fractionary component of $x^{*(0)}$ is x_2, we decompose the problem with respect to the branching variable $x_2^{*(0)}$:

- $P^{(1)}$: add to $P^{(0)}$ constraint $x_2 \leq \lfloor x_2^* \rfloor = 4$
- $P^{(2)}$: add to $P^{(0)}$ constraint $x_2 \geq \lfloor x_2^* \rfloor + 1 = 5$

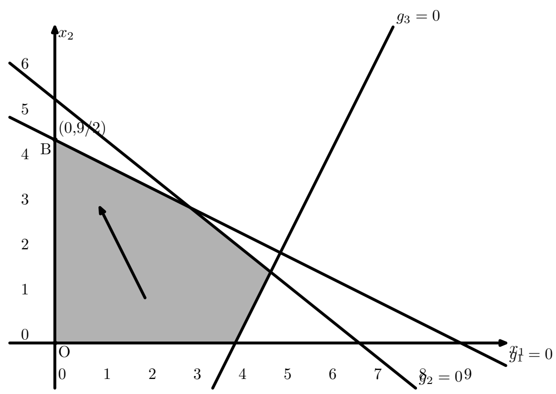

Figura 3: Graphical resolution of the continuous relaxation of $P = P^{(0)}$ with the second objective function

Problem $\mathbf{P^{(1)}}$:

$$\begin{aligned} \min z' = x_1 - 2x_2 & \\ x_1 + 2x_2 &\leq 9 \\ 6x_1 + 7x_2 &\leq 39 \\ 2x_1 - x_2 &\leq 8 \\ x_2 &\leq 4 \\ x_1, x_2 &\in \mathbb{N} \end{aligned}$$

Figure 4 reports the graphical resolution of the continuous relaxation of $P^{(1)}$: its optimal solution is $x^{*(1)} = C = (0, 4)$, and its value is $z_B^{(1)} = -8$. This is an

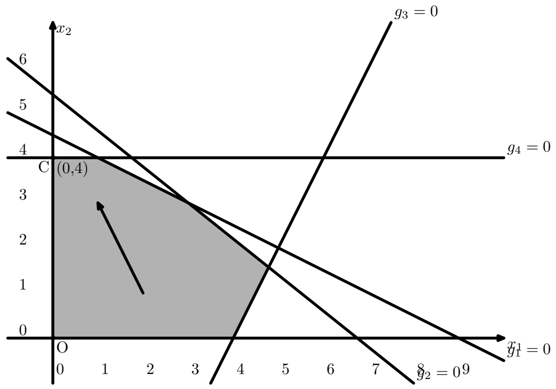

Figura 4: Graphical resolution of the continuous relaxation of $P^{(1)}$ with the second objective function

integer solution, so that the problem is solved. Moreover, it provides a heuristic integer solution, which improves the suboptimal estimate z_E^* from $+\infty$ to -8.

As an exercise, one can solve this problem with the simplex method, instead of graphically. The *tableau* is identical to that of the parent node $P^{(0)}$, with an additional row associated to the branching constraint. If one starts from the optimal *tableau* of problem $P^{(0)}$ and adds the new row, it is first of all necessary to get back to a basic canonical form with a simple *pivot* operation that concerns only the new row. Then, the reduced costs remain nonnegative, but the right-hand-side of the new constraint is negative, because the new constraint is violated by the current optimal solution of the continuous relaxation. Thus, it is licit (and profitable) to apply the dual simplex method, and to reoptimiza the solution without starting from scratch or using the two-phase method.

Problem $\mathbf{P^{(2)}}$: Let us consider the following subproblem in the list, that is $P^{(2)}$.

$$
\begin{aligned}
\min z' = x_1 - 2x_2 & & \\
x_1 + 2x_2 & \leq & 9 \\
6x_1 + 7x_2 & \leq & 39 \\
2x_1 - x_2 & \leq & 8 \\
x_2 & \geq & 5 \\
x_1, x_2 & \in & \mathbb{N}
\end{aligned}
$$

This problem has no feasible solution. If one applies the simplex method, after adding the new row and getting back to a basic canonical form, one obtains a row with all positive coefficients and with a negative right-hand-side. This is an obvious unfeasibility condition, so the problem is closed. Notice that, since the reduced costs are nonnegative, the corresponding dual problem is unbounded.

Since there are no more subproblems to solve, we have obtained the optimum: the optimal solution is the best heuristic solution found during the process, that is $x^* = (0, 4)$, with $z^* = -8$.

Solution with the third objective function

$$
\begin{aligned}
\min z'' = -3x_1 + x_2 & & \\
x_1 + 2x_2 & \leq & 9 \\
6x_1 + 7x_2 & \leq & 39 \\
2x_1 - x_2 & \leq & 8 \\
x_1, x_2 & \in & \mathbb{N}
\end{aligned}
$$

Figure 5 reports the graphical resolution of the continuous relaxation: its optimal solution is $x^{*(0)} = D = (19/4, 3/2)$, that is fractionary. Consequently, it provides a lower bound on the optimum $z_B^{(0)} = -51/4 = -12.75$ and suggests to use x_1 as a *branching* variable. The value of x_1^* is 19/4 and its distance from the middle point between 4 and 5 is 1/4, whereas the value of x_2^* is 3/2, which is exactly the middle point between 1 and 2. Therefore, the problem is branched with respect to x_2:

- $P^{(1)}$: add to the formulation of $P^{(0)}$ constraint $x_2 \leq \lfloor x_2^* \rfloor = 1$
- $P^{(2)}$: add to the formulation of $P^{(0)}$ constraint $x_2 \geq \lfloor x_2^* \rfloor + 1 = 2$

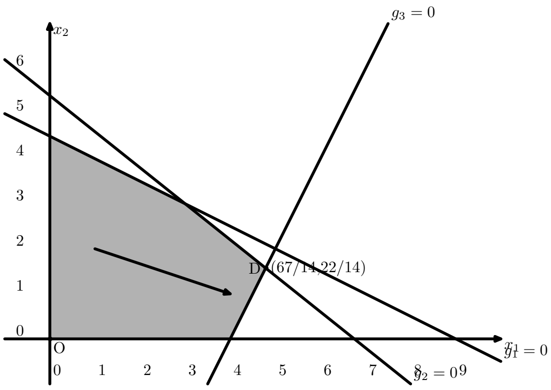

Figura 5: Graphical resolution of the continuous relaxation of $P = P^{(0)}$ with the third objective function

Problem P$^{(1)}$:

$$
\begin{aligned}
\min z'' = -3x_1 + x_2 & \\
x_1 + 2x_2 &\leq 9 \\
6x_1 + 7x_2 &\leq 39 \\
2x_1 - x_2 &\leq 8 \\
x_2 &\leq 1 \\
x_1, x_2 &\in \mathbb{N}
\end{aligned}
$$

The optimal solution of the continuous relaxation of $P^{(1)}$ is $x^{*(1)} = E = (9/2, 1)$, which provides a lower bound equal to $z_B^{(1)} = -25/2 = -12.5$ (see Figure 6).

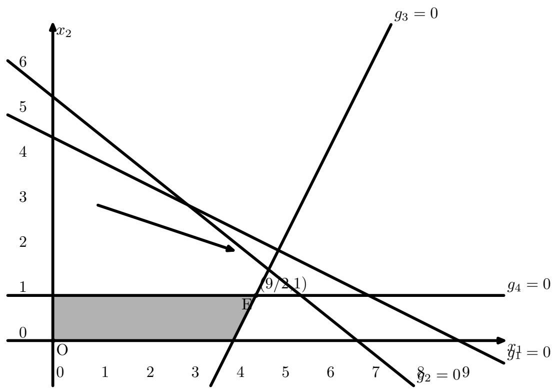

Figura 6: Risoluzione grafica del rilassamento lineare di $P^{(1)}$

The solution is fractionary, but only with respect to x_1, so we must decompose the problem branching on x_1:

- $P^{(3)}$: add to the formulation of $P^{(1)}$ constraint $x_1 \leq \lfloor x_1^* \rfloor = 4$
- $P^{(4)}$: add to the formulation of $P^{(1)}$ constraint $x_1 \geq \lfloor x_1^* \rfloor + 1 = 5$

Problem P$^{(3)}$: The following problem in the list is $P^{(3)}$.

$$
\begin{aligned}
\min z'' = -3x_1 + x_2 & \\
x_1 + 2x_2 &\leq 9 \\
6x_1 + 7x_2 &\leq 39 \\
2x_1 - x_2 &\leq 8 \\
x_2 &\leq 1 \\
x_1 \quad &\leq 4 \\
x_1, x_2 &\in \mathbb{N}
\end{aligned}
$$

The optimal solution of its continuous relaxation is $x^{*(3)} = F = (4,0)$, which provides a lower bound equal to $z_B^{(1)} = -12$ (see Figure 7). Being an integer solution, the superoptimal estimate is actually optimal: the problem is closed and the best known heuristic solution is updated, together with the corresponding suboptimal estimate z_E^*. This was $+\infty$, since we had not yet determined any integer solution, and now becomes $z_E^* = -12$.

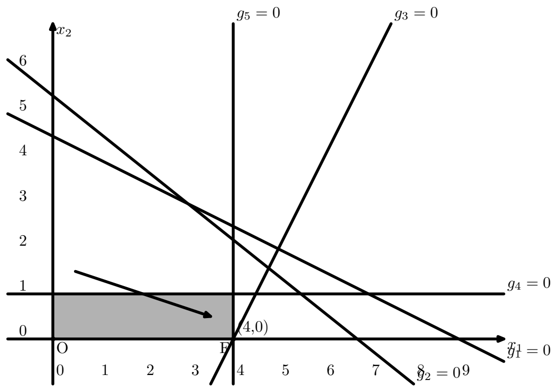

Figura 7: Graphical resolution of the continuous relaxation of $P^{(3)}$ with the third objective function

Problema $\mathbf{P^{(4)}}$: Let us consider problem $P^{(4)}$.

$$\begin{aligned} \min z'' = -3x_1 + x_2 & \\ x_1 + 2x_2 &\leq 9 \\ 6x_1 + 7x_2 &\leq 39 \\ 2x_1 - x_2 &\leq 8 \\ x_2 &\leq 1 \\ x_1 \quad &\geq 5 \\ x_1, x_2 &\in \mathbb{N} \end{aligned}$$

Its continuous relaxation is unfeasible. The original problem, which is tighter, is necessarily also unfeasible. So, it is closed. Some authors use to say, informally, that both the suboptimal and the superoptimal estimate are equal to $+\infty$: this has no rigorous meaning, but allows to say that the problem does not improve the best known heuristic solution and that it must be closed. Moreover, it extends to unfeasible nodes the general property that the superoptimal estimate is always getting worse as one goes down the branching tree (due to the additional constraints).

Problem $\mathbf{P^{(2)}}$:

$$\begin{aligned} \min z'' = -3x_1 + x_2 & \\ x_1 + 2x_2 &\leq 9 \\ 6x_1 + 7x_2 &\leq 39 \\ 2x_1 - x_2 &\leq 8 \\ x_2 &\geq 2 \\ x_1, x_2 &\in \mathbb{N} \end{aligned}$$

The optimal solution of the continuous relaxation is $x^{*(2)} = G = (17/4, 2)$ (see Figure 8) and provides a superoptimal estimate equal to $z^{(2)_B} = -47/4 = -11.75$, which is worse than the best known suboptimal estimate $z_E^* = -12$. Therefore, the node is closed by dominance.

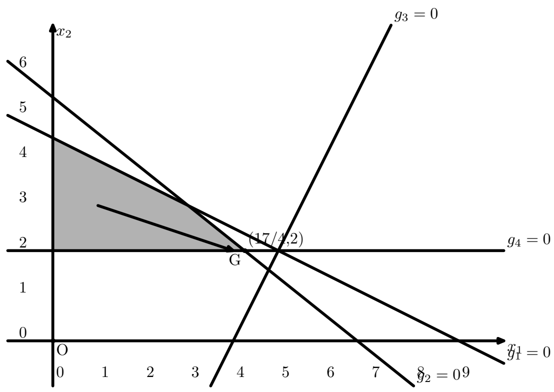

Figura 8: Graphical resolution of the continuous relaxation of $P^{(2)}$ with the third objective function

Problem $\mathbf{P^{(3)}}$:

$$\begin{aligned} \min z'' = -3x_1 + x_2 & \\ x_1 + 2x_2 &\leq 9 \\ 6x_1 + 7x_2 &\leq 39 \\ 2x_1 - x_2 &\leq 8 \\ x_2 &\leq 1 \\ x1 \quad &\geq 5 \\ x_1, x_2 &\in \mathbb{N} \end{aligned}$$

The subproblem has no feasible solution, and it is therefore closed.

Summary

Figure 9 sums up the process: notice that along each path going down from the root to a leaf of the branching tree the value of the lower bound z_B keeps increasing (not decreasing, at least). This is a general property, deriving from the fact that, as one goes down the tree, more and more *branching* constraints are added to the original formulation.

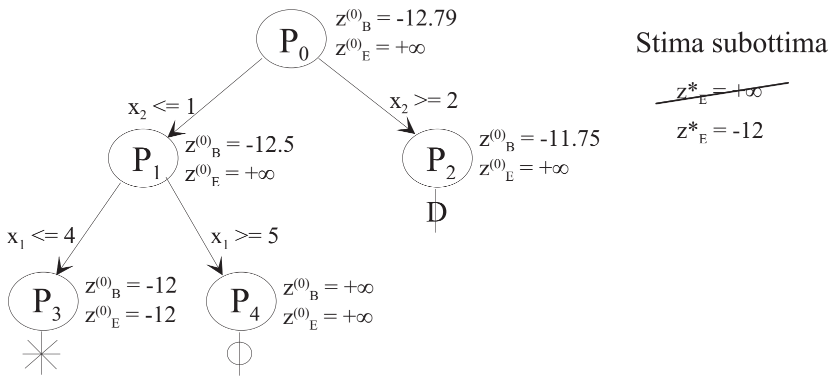

Figura 9: Branching tree with the third objective function