Solved exercises for the course of

Foundations of Operations Research

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Gomory cuts

Given the *ILP* problem

$$\begin{array}{rcl} \max f = 4x_1 + 3x_2 & & \\ 2x_1 + x_2 & \leq & 11 \\ -x_1 + 2x_2 & \leq & 6 \\ x_1, x_2 & \in & \mathbb{N} \end{array}$$

solve it with the Gomory cutting plane method, determining all possible cuts and applying one of them at each step.

The Gomory cutting plane algorithm

The Gomory cutting plane algorithm consists in solving the continuous relaxation of an *ILP* problem and deriving from its optimal solution one or more inequalities which are violated by the solution itself, but respected by any integer feasible solution of the problem. These inequalities are added to the problem, which is reoptimized. Then, the method is applied again to the new solution, proceeding as long as the optimal solution becomes integer, and therefore feasible and optimal also for the original *ILP* problem.

In order to generate the constraints, one combines the given linear constraints with the integrality constraint as follows. All constraints of a problem in basic canonical form are written as:

$$\sum_{j \in N} a_{ij} x_j + x_{j^*} = b_i \qquad i = 1, \ldots, m$$

that is exactly one of the occurring variables is basic, whereas the others are nonbasic.

We can obtain another constraint, which preserves all the *integer* feasible solutions, through two steps:

1. *relax* the constraint rounding dows its coefficients (replacing them with the maximum nonlarger integer)

$$\sum_{j \in N} \lfloor a_{ij} \rfloor x_j + x_{j^*} = b_i \qquad i = 1, \ldots, m$$

2. *tighten* the constraint rounding down the right-hand-side (replacing it with the maximum nonlarger integer)

$$\sum_{j \in N} \lfloor a_{ij} \rfloor x_j + x_{j^*} = \lfloor b_i \rfloor \qquad i = 1, \ldots, m$$

Geometrically speaking, the first step rotates the constraint (the angular coefficients of the separating hyperplane change), keeping all nonnegative solutions feasible. The second step transposes the constraint (the right-hand-side changes) so as to restrict the feasible region. If both the original and the derived constraint are maintained, the new fesible relaxed region becomes smaller, so that the continuous relaxation is tighter. However, no integer solution is lost.

Solution

First, solve the continuous relaxation of the problem with the simplex algorithm. It standard form is:

$$\begin{aligned} \min f' = -4x_1 - 3x_2 & \\ 2x_1 + x_2 + x_3 \quad &= 11 \\ -x_1 + 2x_2 \quad + x_4 &= 6 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The *tableaus* corresponding to the steps of the simplex method are the following (the *pivot* element is circled).

0	-4	-3	0	0
11	(2)	1	1	0
6	-1	2	0	1

22	0	-1	2	0
11/2	1	1/2	1/2	0
23/2	0	(5)/2	1/2	1

26+3/5	0	0	11/5	2/5
16/5	1	0	2/5	-1/5
23/5	0	1	1/5	2/5

The optimal relaxed solution is fractionary in all variables, and also the value of the objective function is fractional. One can therefore introduce a Gomory cut for each row of the *tableau*, that is three cuts, one of which deriving from row 0. Let us derive them, leaving the one derived from row 0 in the end, since it deserves some further remark.

Row 1 Constraint $x_1 + 2/5x_3 - 1/5x_4 = 16/5$, rounding down all coefficients, yields the looser constraint $x_1 - x_4 \leq 16/5$, which can be tightened rounding down the right-hand-side, based on the fact that all variables and coefficients are integer:

$$x_1 - x_4 \leq 3$$

Equivalently, subtracting this relation from the original constraint, one obtains the fractional form of the cut:

$$\frac{2}{5}x_3 + \frac{4}{5}x_4 \geq \frac{1}{5}$$

which can be simply added to the *tableau* under the form:

$$-\frac{2}{5}x_3 - \frac{4}{5}x_4 + x_5 = -\frac{1}{5}$$

so as to proceed with the dual simplex method.

To better understand the meaning of the introduction of the cut, let us retrieve its expression in terms of the natural variables x_1 and x_2, using the other constraints: since the second original constraint guarantees that $x_4 = 6+x_1-2x_2$, we conclude that the cut derived from row 1 imposes $x_2 \leq 9/2$.

Row 2 From constraint $x_2+1/5x_3+2/5x_4 = 23/5$ we obtain, rounding down the coefficients of all variables, the weaker constraint $x_2 \leq 23/5$, which can be strengthened rounding down also the right-hand-side, since variables and coefficients are all integer:

$$x_2 \leq 4$$

Equivalently, subtracting this relation from the original constraint, one obtains the cut in fractional form:

$$\frac{1}{5}x_3 + \frac{2}{5}x_4 \geq \frac{3}{5}$$

which can be simply added to the *tableau* under the form:

$$-\frac{1}{5}x_3 - \frac{2}{5}x_4 + x_5 = -\frac{3}{5}$$

so as to proceed with the dual simplex method. We denote the new *slack* variable as x_5, though it is a different variable from the one introduced considering row 1, because usually one introduces a single Gomory cut at a time, but nothing forbids to introduce more (and even all) cuts simultaneously, each with its own different *slack* variable.

The new cut, actually, resembles the previous one. To be precise, it dominates the previous one. Cuts generated from different rows can be identical, different (i. e. each cuts away a different region, with some solutions more and some solutions less than the other cuts), or some of them can dominate other ones.

It is a good heuristic rule to focus on the rows whose right-hand-side has the most fractionary value, that is has a fractionary part closest to 1/2. In fact, in this case row 1 is less fractionary than row 2 and provides a dominated but. This is not guaranteed: it is just a heuristic rule, since the strength of a cut depends on its interaction with all the constraints of the problem.

Row 0 Notice that, in the given case, $f = 4x_1 + 3x_2$, i. e. the cost coefficients are all integer. Therefore, also the objective function is, necessarily, integer.

If not only the decision variables, but also the objective is guaranteed to be integer, row 0 can be used to generate an additional Gomory cut. This is not always true, but it is a quite common case.

One should keep in mind that row 0 relates the variables x_j with the value of the objective function z. So, it can be seen as an equality constraint, and processed as the regular constraints, with a slight difference. While row 1 of the *tableau*

16/5	1	0	2/5	-1/5

simply stands for $x_1 + 2/5x_3 - 1/5x_4 = 16/5$, row 0 of the *tableau*

26+3/5	0	0	11/5	2/5

stands for $11/5x_3 + 2/5x_4 = z + 26 + 3/5$, and not $11/5x_3 + 2/5x_4 = 26 + 3/5$!

Hence, applying the usual procedure, one obtains first the weaker constraint $2x_3 \leq z + 26 + 3/5$ and then the stronger cut

$$2x_3 \leq z + 26$$

That means $z \geq 2x_3 - 26$, which actually preserves the integer solutions, but removes the optimal relaxed solution, for which $z = -26 - 3/5$.

Subtracting row 0 from the integer form of the cut, one obtains the fractionary form

$$1/5x_3 + 2/5x_4 \geq 3/5$$

which can be added to the *tableau* under the form $-1/5x_3 - 2/5x_4 + x_5 = -3/5$. Notice that, forgetting the term z, the integer form would be completely wrong, whereas the fractionary form does not present any interpretation problem: it can be mechanically derived from row 0 with the same rules used for the other rows (get the fractionary components of the coefficients and of the right-hand-side, reverse their sign and introduce a new *slack* variable).

The geometric meaning of the new cut $z \geq 2x_3 - 26$ can be deduced expressing z and x_3 in terms of the natural variables (from the given problem, $z = -4x_1 - 3x_2$ and $x_3 = 11 - 2x_1 - x_2$). The result is $-4x_1 - 3x_2 \geq 2\,(11 - 2x_1 - x_2) - 26$, and therefore $x_2 \leq 4$. So, in this specific problem two of the three Gomory cuts coincide and dominate the third one. In general, the cuts could also be independent.