Foundations of Operations Research

Master of Science in Computer Engineering

Roberto Cordone
roberto.cordone@unimi.it

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Thursday 10.15 - 13.15

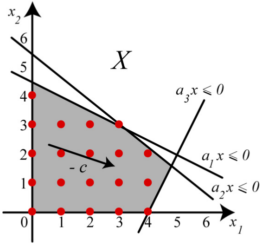
http://homes.di.unimi.it/~cordone/courses/2014-for/2014-for.html

Integer Linear Programming (*ILP*) problems are characterized by

1. an affine objective function $f = c^T x + d$
2. affine constraints $a_i^T x \leq b_i \ (i = 1, \ldots, m)$
3. integer variables: $x \geq 0$ and $x \in \mathbb{Z}^n \Rightarrow x \in \mathbb{N}^n$

$$\begin{aligned} \min f = c^T x &+ d \\ A x &\leq b \\ x &\in \mathbb{N}^n \end{aligned}$$

with c_j, d, a_{ij} and $b_i \in \mathbb{Z}$
(if rational numbers, rescale them)



Contrary to the appearance, they are nonlinear problems
This can be clarified by restating the integrality constraints:

$$x_j \in \mathbb{Z} \ \Leftrightarrow \ \sin(\pi x_j) = 0$$

Computational complexity

ILP can be

- generalized into Mixed Integer Linear Programming (*ILP*), where some variables are integer, but possibly not all: $x_j \in \mathbb{Z}$ for $j \in J' \subseteq J$
- specialized into Binary Linear Programming (*BLP*), or $0-1$ Linear Programming, where all variables are binary: $x_j \in \{0,1\}$ for all $j \in J$

The feasibility of *BLP* (hence, of *ILP* and *MILP*) is $\mathcal{NP}$-complete

Any *SAT* instance can be turned into a special *BLP* instance

- replace each Boolean variable with a binary one:

$$\xi_j \rightarrow x_j \qquad \text{and} \qquad \bar{\xi}_j \rightarrow 1 - x_j$$

- replace each Boolean clause with an affine constraint

$$\left(\bigvee_{j \in J^+} \xi_j\right) \bigvee \left(\bigvee_{j \in J^-} \bar{\xi}_j\right) \rightarrow \sum_{j \in J^+} x_j + \sum_{j \in J^-} (1 - x_j) \geq 1$$

Truth assignments and feasible solutions correspond one-to-one

Computational complexity: example

Given the Boolean function in Conjunctive Normal Form

$$\phi = \xi_1 \vee \left(\bar{\xi}_1 \wedge \bar{\xi}_2\right) \vee \left(\bar{\xi}_1 \wedge \xi_3\right)$$

find whether there exists a truth assignment ξ such that $\phi(\xi)$ is true

The only solution is $\xi_1 = \xi_3 =$ true and $\xi_2 =$ false

This is equivalent to finding whether the *BLP* problem

$$\begin{array}{rcl} x_1 & \geq & 1 \\ (1 - x_1) + (1 - x_2) & \geq & 1 \\ \bar{x}_1 \qquad\qquad + x_3 & \geq & 1 \\ x_1, x_2, x_3 & \in & \{0, 1\} \end{array}$$

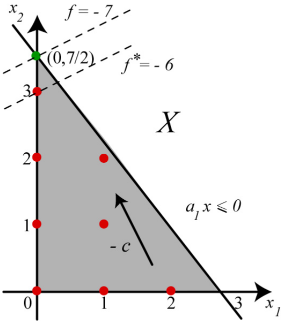
admits any feasible solution

The only solution is $x_1 = x_3 = 1$ and $x_2 = 0$

Every *ILP* instance trivially corresponds to its continuous relaxation, i. e. the *LP* instance obtained neglecting (relaxing) the integrality constraint

- the relaxation admits more solutions (the fractionary ones)
- the relaxed optimum is a bound on the original optimum (not worse)

$$\min f = x_1 - 2x_2$$
$$28x_1 + 22x_2 \leq 77$$
$$x_1, x_2 \in \mathbb{N}$$



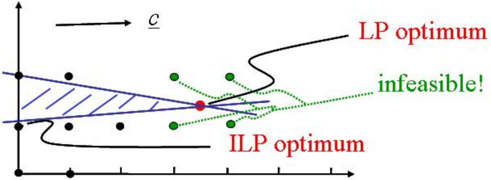
If the relaxed solution is integer, it is optimal for the *ILP* problem

Why not simply round the solution of the continuous relaxation?

ILP and rounded *LP* (1)

For many good reasons rounding is a bad algorithm for *ILP* (in general)

1. all rounded solutions could be unfeasible



2. the relaxation could be unable to suggest how to round

   Consider the independent set problem on an undirected graph

$$\max f = \sum_{v \in V} x_v$$

$$x_u + x_v \leq 1 \qquad [u, v] \in E$$

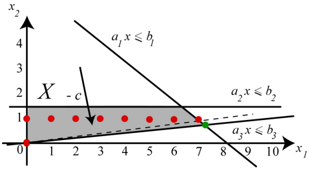
$$x_v \in \{0, 1\} \qquad v \in V$$

The optimal relaxed solution is $x_v = 1/2$ for all $v \in V$

ILP and rounded *LP* (2)

❸ the nearest rounded solution could be very bad

$$\begin{aligned} \min f = -10^9 x_1 + 8 \cdot 10^9 x_2 & & \\ 7x_1 + 8x_2 &\leq& 57 \\ x_2 &\leq& 3/2 \\ x_1 - 9x_2 &\leq& 0 \\ x_1, x_2 &\in& \mathbb{N} \end{aligned}$$



The relaxed solution is $(513/71, 57/71) \approx (7.23, 0.80)$
The rounded solution $(7, 1)$ costs $\hat{f} = 1$
The optimal solution $(0, 0)$ costs $f^* = 0$
Their ratio is infinite

Rounding is a good idea when the relaxed variables have very large values
In that case, a variation < 1 has little impact on the objective function and on the constraints

Total unimodularity

If the relaxed solution is integer, it is optimal for the *ILP* problem

For some special *ILP* problems, all relaxed basic solutions are integer

A coefficient matrix A is totally unimodular when all subsets of m columns of A have determinant in $\{-1, 0, +1\}$

If A and b are integer and A is totally unimodular, all basic solutions of $Ax = b$ are integer (thus, the optimal solution for any objective is integer)

Proof: *Consider any subset B of m columns extracted from A. If $|B| = 0$, B is not a basis, and does not identify a basic solution.*

If $|B| \in \{-1, +1\}$, then B is a basis. Its inverse is

$$B^{-1} = \frac{1}{|B|} \begin{bmatrix} \beta_{11} & \dots & \beta_{1m} \\ \dots & \dots & \dots \\ \beta_{m1} & \dots & \beta_{mm} \end{bmatrix}^T$$

where $\beta_{ij} = (-1)^{i+j} |M_{ij}|$ and M_{ij} is the submatrix obtained removing row i and column j from B. Consequently, B^{-1} is integer.

The corresponding basic solution is $x = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$, *which is also integer.*

Solving *ILP* problems

First check whether the given *ILP* problem provides integer solutions automatically due to its structure (assignment, transportation, flow, ...)

Excluding those easy cases, *ILP* problems can be attacked by

1. heuristics, which give no optimality guarantee:
   - greedy constructive methods, stingy destructive methods
   - local search heuristics and metaheuristics
   - population-based randomized methods
2. exponential exact algorithms, which repeatedly solve the relaxation and take care of fractionality introducing new constraints
   - cutting plane methods: a single relaxation is iteratively tightened and solved
   - branch-and-bound methods: several independent relaxations are recursively generated and solved