# Foundations of Operations Research

## Master of Science in Computer Engineering

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# Summary

- Any *LP* problem (primal problem) can be transformed with mechanical rules into another *LP* problem (dual problem)
- The relation between the two problems is a duality:
  applying the rules to the dual problem one obtains the primal
- The two problems are not equivalent: they have different variables, objective, constraints, feasible sets, etc...
- But they are strongly related, with relevant practical consequences
  - their basic solutions correspond one-to-one
  - their optimal basic solutions correspond one-to-one
  - their optimal value is the same
  - all feasible values of one are bounds for the other
  - ...

# Estimation of bounds on the optimum

Given a *LP* problem in the following form

$$\begin{aligned} \max f = c^T x &+ d \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

how to compute bounds on its optimum?

Lower bounds are provided by the values of feasible solutions

$$\begin{aligned} \max f = 4\,x_1 + x_2 + 5\,x_3 + 3\,x_4 & \\ x_1 - x_2 - x_3 + 3\,x_4 &\leq 1 \\ 5\,x_1 + x_2 + 3\,x_3 + 8\,x_4 &\leq 55 \\ -x_1 + 2\,x_2 + 3\,x_3 - 5\,x_4 &\leq 3 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$x' = (0, 0, 1, 0) \Rightarrow f^* \geq f(x') = 5$
$x'' = (2, 1, 1, 1/3) \Rightarrow f^* \geq f(x'') = 15$
$x''' = (3, 1, 1, 0) \Rightarrow f^* \geq f(x''') = 18$

*What about upper bounds?*

# Estimation of bounds on the optimum

Upper bounds are provided by expressions derived aggregating the constraints so as to dominate the objective function

$$
\begin{aligned}
\max f = 4\,x_1 + x_2 + 5\,x_3 + 3\,x_4 & & & \\
x_1 - x_2 - x_3 + 3\,x_4 &\leq& 1 & \qquad (1)\\
5\,x_1 + x_2 + 3\,x_3 + 8\,x_4 &\leq& 55 & \qquad (2)\\
-x_1 + 2\,x_2 + 3\,x_3 - 5\,x_4 &\leq& 3 & \qquad (3)\\
x_1, x_2, x_3, x_4 &\geq& 0 &
\end{aligned}
$$

1. multiply Constraint (2) by $5/3$ and compare the result with the objective

$$
\begin{aligned}
g(x) &= \frac{5}{3} \cdot (5\,x_1 + x_2 + 3\,x_3 + 8\,x_4) \leq \frac{5}{3} \cdot 55 \Rightarrow \\
\Rightarrow g(x) &= \frac{25}{3}\,x_1 + \frac{5}{3}\,x_2 + 5\,x_3 + \frac{40}{3}\,x_4 \leq \frac{275}{3}
\end{aligned}
$$

For all $x \geq 0$, each coefficient of $g$ dominates the corresponding one of $f$:

$$4x_1 \leq \frac{25}{3}x_1 \qquad x_2 \leq \frac{5}{3}x_2 \qquad 5\,x_3 \leq 5\,x_3 \qquad 3x_4 \leq \frac{40}{3}x_4$$

Hence, $f(x) \leq g(x) \leq \frac{275}{3}$ for all $x \geq 0$ that satisfy Constraint (2)

# Estimation of bounds on the optimum

Upper bounds are provided by expressions derived aggregating the constraints so as to dominate the objective function

$$\min f = 4\,x_1 + x_2 + 5\,x_3 + 3\,x_4$$

$$\begin{aligned} x_1 - x_2 - x_3 + 3\,x_4 &\leq 1 && (1) \\ 5\,x_1 + x_2 + 3\,x_3 + 8\,x_4 &\leq 55 && (2) \\ -x_1 + 2\,x_2 + 3\,x_3 - 5\,x_4 &\leq 3 && (3) \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

2. add Constraints (2) and (3) and compare the result with the objective

$$g'(x) = 4\,x_1 + 3\,x_2 + 6\,x_3 + 3\,x_4 \leq 58$$

For all $x \geq 0$, each coefficient of $g'$ dominates the corresponding one of $f$:

$$4x_1 \leq 4x_1 \qquad x_2 \leq 3x_2 \qquad 5\,x_3 \leq 6\,x_3 \qquad 3x_4 \leq 3x_4$$

Hence, $f(x) \leq g'(x) \leq 58$ for all $x \geq 0$ that satisfy Constraints (2) and (3)

Any combination of the constraints with nonnegative values provides a bound

# A general bounding strategy

1. Linearly combine the constraints with nonnegative multipliers $y_i \geq 0$

$$\sum_{i=1}^{m} y_i \left( \sum_{j=1}^{n} a_{ij}\, x_j \right) \leq \sum_{i=1}^{m} y_i\, b_i \qquad (x \geq 0, y_i \geq 0 \text{ for } i = 1, \ldots, m)$$

2. Choose $y_i$ so that each coefficient on the left-hand-side is larger than the corresponding one in the objective function

$$\sum_{i=1}^{m} y_i\, a_{ij} \geq c_j \qquad (j = 1, \ldots, n)$$

3. The combination of the right-hand-sides is an upper bound on the optimum (multiply by $x_j^*$, sum over $j$ and apply the first thesis)

$$f^* \leq \sum_{i=1}^{m} y_i\, b_i$$

4. Compute the tightest possible upper bound (i. e., the smallest one)

$$\min \phi = \sum_{i=1}^{m} b_i\, y_i \quad \text{subject to the second thesis and } y \geq 0$$

# Matrix form of the dual

The dual problem is the problem of finding the costraint multipliers which provide the tightest bound on the primal optimum

$$\begin{aligned} \min \phi = \sum_{i=1}^{m} b_i \, y_i \quad &+ \quad d \\ \sum_{i=1}^{m} a_{ij} \, y_i \quad &\geq \quad c_j \qquad j = 1, \ldots, n \\ y_i \quad &\geq \quad 0 \qquad i = 1, \ldots, m \end{aligned}$$

The constant term $d$ has no impact on the optimization process

| Primal problem | Dual problem |
| --- | --- |
| $\max f = c^T x \quad + \quad d$ | $\min \phi = b^T y \quad + \quad d$ |
| $Ax \quad \leq \quad b$ | $A^T y \quad \geq \quad c$ |
| $x \quad \geq \quad 0$ | $y \quad \geq \quad 0$ |

*The two problems have the same elements rearranged differently*

# Example

Compute the dual of the following problem

$$
\begin{aligned}
\max f = \ & x_1 + 2\,x_2 \\
& 2\,x_1 && \leq 5 \\
& -x_1 + x_2 && \leq 2 \\
& x_2 && \leq 4 \\
& x_1, x_2 && \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min \phi = \ & 5\,y_1 + 2\,y_2 + 4\,y_3 \\
& 2\,y_1 - y_2 && \geq 1 \\
& y_2 + y_3 && \geq 2 \\
& y_1, y_2, y_3 && \geq 0
\end{aligned}
$$

# The dual of the dual

The dual of the dual problem is the primal problem

First, put it into the required form

$$\begin{aligned} \min \phi = b^T y \quad &+ \quad d \\ A^T y \quad &\geq \quad c \\ y \quad &\geq \quad 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -\max \phi' = -b^T y \quad &- \quad d \\ -A^T y \quad &\leq \quad -c \\ y \quad &\geq \quad 0 \end{aligned}$$

Then, apply the rules to build the dual problem

$$\begin{aligned} -\min f' = -c^T x \quad &- \quad d \\ -Ax \quad &\geq \quad -b \\ x \quad &\geq \quad 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max f = c^T x \quad &+ \quad d \\ Ax \quad &\leq \quad b \\ x \quad &\geq \quad 0 \end{aligned}$$

# Dual of the standard form

If the problem is in standard form, the transformation is slightly more involved

$$\begin{array}{rcl} \min f = c^T x & + & d \\ Ax & = & b \\ x & \geq & 0 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} -\max f' = -c^T x & - & d \\ \begin{bmatrix} A \\ -A \end{bmatrix} x & \leq & \begin{bmatrix} b \\ -b \end{bmatrix} \\ x & \geq & 0 \end{array}$$

The dual problem is

$$\begin{array}{rcl} -\min \phi' = \left[ b^T \mid -b^T \right] \begin{bmatrix} y^- \\ y^+ \end{bmatrix} & - & d \\ \left[ A^T \mid -A^T \right] \begin{bmatrix} y^- \\ y^+ \end{bmatrix} & \geq & -c \\ y^-, y^+ & \geq & 0 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} -\min \phi' = -b^T \left( y^+ - y^- \right) & - & d \\ -A^T \left( y^+ - y^- \right) & \geq & -c \\ y^+, y^- & \geq & 0 \end{array}$$

that is (replacing $y^+ - y^-$ with $y$)

$$\begin{array}{rcl} \max \phi = b^T y & + & d \\ A^T y & \leq & c \\ y & \in & \mathbb{R}^m \end{array}$$

Equalities correspond to free variables

Similar derivations can be devised for other *LP* forms (nonpositive variables, "unnatural" constraints, etc...)

By "natural" constraints we mean $\leq$ for max, $\geq$ for min problems

# Transformation rules

The dual can be built directly, instead of using an intermediate form

| Primal problem (min) | Dual problem (max) |
|---|---|
| $m$ constraints | $m$ variables |
| right-hand-sides | objective coefficients |
| $\geq / \leq$ inequalities | $\geq / \leq$ variables |
| equalities | free variables |
| $n'$ variables | $n'$ constraints |
| objective coefficients | right-hand-sides |
| $\geq / \leq$ variables | $\leq / \geq$ inequalities |
| free variables | equalities |
| $A$ | $A^T$ |

Nonnegative variables correspond to "natural" constraints
($\leq$ for max, $\geq$ for min)

Both problems have at most $\binom{n'+m}{m} = \binom{m+n'}{n'}$ basic solutions

- the primal problem has $n' + m$ variables, $m$ constraints
- the dual problem has $n' + m$ variables, $n'$ constraints

# Example

Compute the dual of the following problem

$$\begin{array}{rcrcrcl} \min f = 10\,x_1 &+& 20\,x_2 &+& 30\,x_3 & & \\ 2\,x_1 &-& x_2 & & & \geq & 1 \\ & & x_2 &+& x_3 & = & 2 \\ x_1 & & & - & x_3 & \leq & 3 \end{array}$$

$$x_1 \geq 0,\ x_2 \leq 0,\ x_3 \in \mathbb{R}$$

$$\begin{array}{rcrcrcl} \max \phi = y_1 &+& 2\,y_2 &+& 3\,y_3 & & \\ 2\,y_1 & & & + & y_3 & \leq & 10 \\ -y_1 &+& y_2 & & & \geq & 20 \\ & & y_2 &-& y_3 & = & 30 \end{array}$$

$$y_1 \geq 0,\ y_2 \in \mathbb{R},\ y_3 \leq 0$$

# Weak duality theorem

Given a feasible primal problem and a feasible dual problem

$$\min \left\{ c^T x : x \in X \right\} \text{ with } X = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\} \neq \emptyset$$

$$\max \left\{ b^T y : y \in Y \right\} \text{ with } Y = \left\{ y \in \mathbb{R}^m : A^T y \leq c, y \geq 0 \right\} \neq \emptyset$$

the value of each feasible primal solution dominates the value of each feasible dual solution

$$b^T y \leq c^T x \text{ for all } x \in X, y \in Y$$

Proof: *Let $x \in X$ and $y \in Y$ be two feasible solutions of the two problems.*

*Using the dual values $y_i \geq 0$ as multipliers to combine the primal constraints $b_i \leq A_i x$, we obtain other valid inequalities $y^T b \leq y^T A x \Leftrightarrow b^T y \leq x^T A^T y$.*

*By definition, $x \geq 0$ and $A^T y \leq c \Rightarrow b^T y \leq x^T c = c^T x$, that is the thesis.*

Corollary: If there exist $x^* \in X$, $y^* \in Y$ such that $b^T y^* = c^T x^*$, $x^*$ is optimal for the primal problem and $y^*$ is optimal for the dual problem

This is a generalization of the maximum flow and minimum cut property: in fact, they are dual problems

# Strong duality theorem

Given a feasible primal problem and a feasible dual problem

$$\min\left\{c^T x : x \in X\right\} \text{ with } X = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\} \neq \emptyset$$

$$\max\left\{b^T y : y \in Y\right\} \text{ with } Y = \left\{y \in \mathbb{R}^m : A^T y \leq c, y \geq 0\right\} \neq \emptyset$$

there exist $x^* \in X$, $y^* \in Y$ such that $b^T y^* = c^T x^*$. (*Hence, they are optimal*)

Proof: *We derive the optimal dual solution from the optimal primal solution (constructive proof).*

*If the dual is feasible, the primal is bounded (dual values provide bounds). The converse is also true. Thus, both problems have an optimum.*

*Introduce surplus variables to put the primal problem into standard form.*

*Let B be an optimal basic submatrix of A in the primal problem.*

*Then* $x^* = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$ *is the corresponding optimal primal solution.*

*Write the dual problem, considering that* $A = [\, B \mid N \,]$

$$\begin{aligned} \max \phi = y^T b \quad &+ \quad d \\ y^T B \quad &\leq \quad c_B^T \\ y^T N \quad &\leq \quad c_N^T \\ y \quad &\in \quad \mathbb{R}^m \end{aligned}$$

# Strong duality theorem

$$\begin{aligned} \max \phi = y^T b &+ d \\ y^T B &\leq c_B^T \quad (1) \\ y^T N &\leq c_N^T \quad (2) \\ y &\in \mathbb{R}^m \end{aligned}$$

*Now, we show that, if* $y = \left(c_B^T B^{-1}\right)^T$ *is an optimal dual solution.*

*First, it is feasible; remind that the primal problem in optimal basic canonical form has zero basic reduced costs and nonnegative nonbasic reduced costs*

$$\bar{c}_N^T = c_N^T - c_B^T B^{-1} N \geq 0 \Rightarrow c_N^T \geq y^T N \qquad \text{i. e. } (2)$$

$$y^T = c_B^T B^{-1} \Rightarrow y^T B = c_B^T \qquad \text{implying } (1)$$

*Then,* $y$ *is optimal. In fact, its value is*

$$\phi^* = y^{*T} b + d = c_B^T B^{-1} b + d = c_B^T x_B^* + d = c^T x^* + d = f^*$$

*Two feasible solutions of dual problems having the same value are optimal.*

# Possible cases

Three cases are possible for any *LP* problem: Unbounded, Finite, Unfeasible

Nine cases are possible for a general pair of *LP* problems

But only four cases are possible for a primal-dual pair of *LP* problems

- weak duality forbids the (Unbounded,Finite), (Unbounded,Unbounded) and (Finite,Unbounded) cases:

$$P \text{ Unbounded} \Rightarrow D \text{ Unfeasible}$$
$$D \text{ Unbounded} \Rightarrow P \text{ Unfeasible}$$

- strong duality forbids the (Unfeasible,Finite) and (Finite,Unfeasible) cases:

$$P \text{ Finite} \Leftrightarrow D \text{ Finite}$$

| | Dual | | |
|---|---|---|---|
| Primal | Unbounded | Finite | Unfeasible |
| Unbounded | NO | NO | YES |
| Finite | NO | $f^* = \phi^*$ | NO |
| Unfeasible | YES | NO | YES |

# Economic interpretation

In practice, the primal and dual problem often correspond to complementary points of view on the same situation

- the diet problem: select food amounts to build a minimum cost diet satisfying nutritional constraints
- the "synthetic diet" problem: replace each food with a cheaper mix of synthetic nutrients, selecting their prices so as to maximize income

$x_j$: amount of food $j \in F$

$$\min f = \sum_{j \in F} c_j x_j$$

$$\begin{aligned} \sum_{j \in F} a_{ij} x_j &\geq b_i \quad i \in N \\ x_j &\geq 0 \quad j \in F \end{aligned}$$

$y_i$: price of synthetic nutrient $i \in N$

$$\max \phi = \sum_{i \in N} b_i y_i$$

$$\begin{aligned} \sum_{i \in N} a_{ij} y_i &\leq c_j \quad j \in F \\ y_i &\geq 0 \quad i \in N \end{aligned}$$

The two optimal solutions have the same value:
the optimal natural and synthetic diets have the same cost

# Economic interpretation

In practice, the primal and dual problem often correspond to complementary points of view on the same situation

- the transportation problem: decide how many items to transport from each source to each destination respecting the source capacities and the destination demands at minimum total cost
- the "outsourced" transportation problem: replace the transport between each pair with a cheaper alternative, selecting the purchase price at each source and selling price at each destination so as to maximize income

$x_{ij}$: items transported from $i \in S$ to $j \in D$

$y_i$: purchase price in $i \in S$
$y'_j$: selling price in $j \in D$

$$
\begin{aligned}
\min f &= \sum_{i \in S} \sum_{j \in D} c_{ij} x_{ij} \\
-\sum_{j \in D} x_{ij} &\leq -c_i \quad i \in S \\
\sum_{i \in S} x_{ij} &\geq d_j \quad j \in D \\
x_{ij} &\geq 0 \quad i \in S, j \in D
\end{aligned}
$$

$$
\begin{aligned}
\max \phi &= \sum_{j \in D} d_j y'_j - \sum_{i \in S} c_i y'_i \\
y'_j - y_i &\leq c_{ij} \quad i \in S, j \in D \\
y_i &\geq 0 \quad i \in S \\
y'_j &\geq 0 \quad j \in D
\end{aligned}
$$

The two optimal solutions have the same value:
the optimal insource and outsource transport plans have the same cost

# Optimality conditions

Remember that feasibility and optimality are equivalent from a complexity point of view, because

- feasibility can be solved by applying an algorithm for optimality
- optimality can be solved by repeatedly applying an algorithm for feasibility with dycothomic search

For *LP* problems dycothomic search is unnecessary

Strong duality allows to find an optimal solution for a *LP* problem in one step, by finding a feasible solution for an auxiliary *LP* problem

$$\begin{array}{rcl} \min f = c^T x & + & d \\ Ax & \geq & b \\ x & \geq & 0 \end{array} \quad \rightarrow \quad \begin{array}{rcl} Ax & \geq & b \\ x & \geq & 0 \\ A^T y & \leq & c \\ y & \geq & 0 \\ c^T x & = & b^T y \end{array}$$

# Complementary slackness conditions

We have already noticed that, given a pair of dual *LP* problems

- the natural variables of the primal problem correspond to the auxiliary (*slack* or *surplus*) variables of the dual
- the auxiliary variables of the primal problem correspond to the natural variables of the dual

Given a feasible primal problem and a feasible dual problem

$$\min \left\{ c^T x : x \in X \right\} \text{ with } X = \{ x \in \mathbb{R}^n : Ax \geq b, x \geq 0 \} \neq \emptyset$$

$$\max \left\{ b^T y : y \in Y \right\} \text{ with } Y = \left\{ y \in \mathbb{R}^m : A^T y \leq c, y \geq 0 \right\} \neq \emptyset$$

the corresponding variables have zero product in the optimal solution.

Proof: *Let $x^* \in X$ and $y^* \in Y$ be the corresponding optimal solutions of the two problems according to strong duality.*

*Since $b \leq Ax^*$ and $y^{*T} A \leq c^T$, weak duality states that*

$$y^{*T} b \leq y^{*T} A x^* \leq c^T x^*$$

*Since strong duality states that $y^{*T} b = c^T x^*$, necessarily*

$$\begin{cases} y^{*T} A x^* = c^T x^* \\ y^{*T} b = y^{*T} A x^* \end{cases} \Rightarrow \begin{cases} \left( c^T - y^{*T} A \right) x^* = 0 \\ y^{*T} \left( A x^* - b \right) = 0 \end{cases}$$

# Complementary slackness conditions

Modify the indices so that corresponding variables have the same index:

- the *surplus* primal variable $x^*_{n'+i}$ corresponds to the natural dual variable $y_{n'+i}$
$$A_i x^* - b_i = x^*_{n'+i} \quad \longleftrightarrow \quad y_{n'+i} \qquad (i = 1, \dots, m)$$
- the natural primal variable $x_j$ corresponds to the *slack* dual variable $y_j$
$$x_j \quad \longleftrightarrow \quad y_j = c_j - A_j^T y^* \qquad (j = 1, \dots, n')$$

Corresponding variables in the two problems are complementary (zero product)

$$\begin{cases} \left(c^T - {y^*}^T A\right) x^* = 0 \\ {y^*}^T \left(Ax^* - b\right) = 0 \end{cases} \Rightarrow \begin{cases} \left(c_j - A_j^T y^*\right) x_j^* = y_j^* x_j^* = 0 & (j = 1, \dots, n') \\ y^*_{n'+i} \left(A_i x^* - b_i\right) = y^*_{n'+i} x^*_{n'+i} = 0 & (i = 1, \dots, m) \end{cases}$$

that is

$$y_j^* x_j^* = 0 \qquad (j = 1, \dots, n)$$

These conditions allow to derive the optimal solution of the dual from the optimal solution of the primal, and vice versa

# Example

Given the optimal solution $x^* = (1, 0, 1)$, find the optimal dual one for

$$\begin{aligned} \min f = 13\,x_1 + 10\,x_2 + 6\,x_3 & & \\ 5\,x_1 + \ \ x_2 + 3\,x_3 & \geq & 8 \\ 3\,x_1 + \ \ x_2 \qquad & = & 3 \\ x_1, x_2, x_3 & \geq & 0 \end{aligned}$$

The complete optimal primal solution is $x^* = (1, 0, 1, 0, 0)$

The dual problem is

$$\begin{aligned} \max \phi = 8\,y_4 + 3\,y_5 & & \\ 5\,y_4 + 3\,y_5 & \leq & 13 \\ y_4 + \ \ y_5 & \leq & 10 \\ 3\,y_4 \qquad & \leq & 6 \\ y_4 \geq 0, y_5 & \in & \mathbb{R} \end{aligned}$$

# Example

The $n' + m = 3 + 2 = 5$ complementary slackness conditions state that

$$\begin{cases} y_1^* x_1^* = y_1^* \cdot 1 = 0 \\ y_2^* x_2^* = y_2^* \cdot 0 = 0 \\ y_3^* x_3^* = y_3^* \cdot 1 = 0 \\ y_4^* x_4^* = y_4^* \cdot 0 = 0 \\ y_5^* x_5^* = y_5^* \cdot 0 = 0 \end{cases} \Rightarrow y_1^* = y_3^* = 0$$

Together with the dual constraints, they imply

$$\begin{cases} 5y_4^* + 3y_5^* + y_1^* = 13 \\ y_4^* + y_5^* + y_2^* = 10 \\ 3y_4^* + y_3^* = 6 \end{cases} \Rightarrow y_4^* = 2, y_5^* = 1 \text{ and } y_2^* = 7 \Rightarrow y^* = (0, 7, 0, 2, 1)$$

# Correspondence between basic solutions

We know that two dual problems have the same number of basic solutions

In general, given a basic solution of the primal, the complementary slackness conditions allow to derive a basic solution of the dual, and vice versa

- if the former is optimal, the latter is optimal
- if the former is superoptimal, i. e. has nonnegative reduced costs, the latter is feasible (though in general nonoptimal)
- if the former is feasible, the latter is superoptimal (though in general unfeasible)

This correspondence can be exploited to build a dual simplex algorithm, which

- visits basic dual solutions, instead of primal
- achieves first superoptimality, instead of feasibility, and then optimality

This algorithm applies different rules to select the *pivot* element

Since the average complexity is proportional to $m$, the dual simplex algorithm is more efficient than the standard one for problems with many constraints and few variables