## 4.4 Linear programming duality

Write the dual of the following linear program

$$
\begin{array}{rrrrrrrrrl}
\min & & & 2x_2 & + & x_3 & - & 3x_4 & & \\
& x_1 & - & x_2 & & & + & 2x_4 & \geq & 2 \\
& & & 2x_2 & + & x_3 & & & = & 4 \\
& 2x_1 & & & - & x_3 & + & x_4 & \leq & 1
\end{array}
$$

$$x_1 \geq 0 \qquad x_2 \geq 0$$

$$x_3, x_4 \quad \text{unrestricted}$$

## 4.5 Dual of the transportation problem

A company owns $m$ production factories, and produces goods for $n$ warehouses. Each warehouse $j \in \{1, \ldots, n\}$ requires $d_j$ units of product, while $b_i$ units are available at each production factory $j \in \{1, \ldots, m\}$. Let $c_{ij}$ bet the unit transportation cost from factory $i$ to warehouse $j$.

Consider the problem of determining a transportation plan that minimizes the total cost, satisfying the demands and the availability constraints. Let $x_{ij}$ be the quantity of product transported from $i$ to $j$. We consider the following formulation

$$
\begin{array}{lll}
(\mathbf{P_1}) & \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & \\
& -\sum_{j=1}^{n} x_{ij} \geq -b_i & i \in \{1, \ldots, m\} \\
& \sum_{i=1}^{m} x_{ij} \geq d_j & j \in \{1, \ldots, n\} \\
& x_{ij} \geq 0 & i \in \{1, \ldots, m\} \quad j \in \{1, \ldots, n\}
\end{array}
$$

Determine the linear programming dual of $P_1$, providing an economical interpretation.

## 4.6 Complementary slackness

Given the problem

$$
(\mathbf{P_2}) \qquad \begin{array}{rrcrl}
\max & 2x_1 & + & x_2 & \\
 & x_1 & + & 2x_2 & \leq 14 \\
 & 2x_1 & - & x_2 & \leq 10 \\
 & x_1 & - & x_2 & \leq 3
\end{array}
$$

$$x_1, x_2 \geq 0$$

a) write its dual

b) show that $\underline{\bar{x}} = (\frac{20}{3}, \frac{11}{3})$ is a feasible solution

c) show that $\underline{\bar{x}}$ is also an optimal solution, by means of the complementary slackness theorem

d) determine an optimal solution to its dual.

## SOLUTION

**4.4 Linear programming duality**

The dual reads

$$
\begin{array}{rrcrcrcr}
\max & 2y_1 & + & 4y_2 & + & y_3 & & \\
 & y_1 & & & + & 2y_3 & \leq & 0 \\
 & -y_1 & + & 2y_2 & & & \leq & 2 \\
 & & & y_2 & - & y_3 & = & 1 \\
 & 2y_1 & & & + & y_3 & = & -3
\end{array}
$$

$$
\begin{array}{c}
y_1 \geq 0 \\
y_2 \quad \text{unrestricted} \\
y_3 \leq 0
\end{array}
$$

**4.5 Dual of the transportation problem**

Let $u_i, v_j$ be the dual variables of the two classes of constraints. Since for each column of the the constraint matrix of the primal problem only two entries are nonzero, taking value -1 and +1 for, respectively, rows corresponding to the fist and second class of constraints, the dual reads

$$
\begin{array}{lll}
(\mathbf{D_1}) \quad \max -\sum_{i=1}^{m} b_i u_i + \sum_{j=1}^{n} d_j v_j & & \\
\qquad v_j - u_i \leq c_{ij} & i \in \{1,\ldots,m\} & j \in \{1,\ldots,n\} \\
\qquad u_i \geq 0, v_j \geq 0 & i \in \{1,\ldots,m\} & j \in \{1,\ldots,n\}
\end{array}
$$

Economical interpretation. We suppose that the production company (company A) hires a logistics company to handle the transportation business (company B). This company buys all the products from the different factories, paying a unit cost of $u_i$ Euro for factory $i$. Then, it sells the products to the warehouses at a unit cost of $v_j$ Euro, per warehouse $j$. The objective of company B amounts to maximizing the profit, i.e.,

$$\max -\sum_{i=1}^{m} b_i u_i + \sum_{j=1}^{n} d_j v_j$$

Company B has to decide the prices $u_i, v_j$ such that it is not cheaper, for company A, to handle the transportation business without involving B. Suppose that, for a pair $i, j$, the prices are such that $v_j - u_i > c_{ij}$. In such case, company A would prefer to transport the product by itself, being cheaper than hiring company B. Therefore, for each pair $i, j$, the prices must satisfy the condition $v_j - u_i \leq c_{ij}$.

### 4.6 Complementary slackness

a) The dual reads

$$(\mathbf{D_2}) \qquad \begin{array}{lrcrcrl} \min & 14y_1 & + & 10y_2 & + & 3y_3 & \\ & y_1 & + & 2y_2 & + & y_3 & \geq 2 \\ & 2y_1 & - & y_2 & - & y_3 & \geq 1 \end{array}$$

$$y_1, y_2, y_3 \geq 0$$

b) $\bar{x} = (\frac{20}{3}, \frac{11}{3})$ is feasible, as it satisfies the constrains of $(P_2)$.

c-d) By the complementary slackness theorem, if $\bar{x} = (x_1, x_2)$ is feasible for the primal and $\bar{y} = (y_1, y_2, y_3)$ is feasible for the dual, and they satisfy

$$y_i(a_i^T \underline{x} - b_i) = 0 \qquad \forall i$$

$$(c_j - \underline{y}^T A_j)x_j = 0 \qquad \forall j$$

then they are optimal for the respective problems. The complementary slackness conditions for the problem at hand read

$$y_1(x_1 + 2x_2 - 14) = 0$$

$$y_2(2x_1 - x_2 - 10) = 0$$

$$y_3(x_1 - x_2 - 3) = 0$$

$$x_1(y_1 + 2y_2 + y_3 - 2) = 0$$

$$x_2(2y_1 - y_2 - y_3 - 1) = 0$$

We obtain $\bar{y}$, by substituting for $\bar{x}$. Since $\bar{x} = (\frac{20}{3}, \frac{11}{3})$ satisfies as equations the first and third constraints, but not the second one, we deduce $y_2 = 0$. Since $x_1 > 0$ and $x_2 > 0$, we obtain

$$y_1 + 2y_2 + y_3 - 2 = 0$$

$$2y_1 - y_2 - y_3 - 1 = 0$$

$$y_2 = 0$$

By solving the system, we obtain $\bar{y} = (1, 0, 1)$, which satisfies the dual constraints of $D_2$. Since $\bar{x}$ is primal feasible and $\bar{y}$ is dual feasible, the primal/dual pair $(\bar{x}, \bar{y})$ satisfies the complementary slackness constraints and, therefore, $\bar{x}$ is an optimal solution to the primal and $\bar{y}$ is an optimal solution to the dual.

To double check, verify that the two solutions have the same objective function value in the respective problems, i.e., verify that $\underline{c}^T \underline{x} = \underline{y}^T \underline{b}$.