Solved exercises for the course of

Foundations of Operations Research

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Building the dual problem

Write the dual of the following problem:

$$
\begin{array}{rcl}
\max f(x) & = & 2x_1 - 4x_2 - 7x_3 - x_4 - 5x_5 \\
g_1(x) & = & -x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 7 \\
g_2(x) & = & -x_1 + 2x_2 + 3x_3 \qquad + x_5 \leq 6 \\
g_3(x) & = & -x_1 + x_2 + x_3 + 2x_4 \qquad \geq 4 \\
x_1, x_2, x_3 & \geq & 0 \\
x_4 & & \in \mathbb{R} \\
x_5 & \leq & 0
\end{array}
$$

Definition of the dual problem

In Linear Programming the *dual problem* is the problem built from the given one, denoted as *primal problem*, applying the following rules:

- a maximization problem corresponds to a minimization one,
  a minimization problem corresponds to a maximization one
- each constraint of the primal problem corresponds to a variable of the dual problem; the kind of constraint determines the kind of variable:
  - a constraint that "makes opposition" to the sign of the problem, that is a $\geq$ constraint in a minimization problem or a $\leq$ constraint in a maximization one, corresponds to a nonnegative variable
  - a constraint that "gives support" to the sign of the problem, that is a $\leq$ constraint in a minimization problem or a $\geq$ constraint in a maximization one, corresponds to a nonpositive variable
  - an equality constraint correspond to a free variable
- each variable of the primal problem corresponds to a constraint of the dual problem; the kind of variable determines the kind of problem:
  - a nonnegative variable corresponds to a constraint that "makes opposition" (as defined above)
  - a nonpositive variable corresponds to a constraint that "gives support" (as defined above)
  - a free variable corresponds to an equality
- the right-hand-sides of the primal problem become the objective function coefficients of the dual problem

- the objective function coefficients of the primal problem become the right-hand-sides of the dual problem
- the constraint coefficient matrix is transposed

The three rules concerning the constraints are actually a single rule (the first one). The other two rules can be derived reducing the constraint to the desired form, building the dual and transforming the dual into an equivalent problem. The same occurs for the three rules concerning the variables: only the rule concerning nonnegative variables is necessary.

Solution

Let us consider the given problem:

$$\begin{aligned}
\max f(x) = 2x_1 - 4x_2 - 7x_3 - x_4 - 5x_5 & & & \quad (1)\\
g_1(x) = -x_1 + x_2 + 2x_3 + x_4 + 2x_5 & = & 7 & \quad (2)\\
g_2(x) = -x_1 + 2x_2 + 3x_3 \qquad + x_5 & \leq & 6 & \quad (3)\\
g_3(x) = -x_1 + x_2 + x_3 + 2x_4 \qquad & \geq & 4 & \quad (4)\\
x_1, x_2, x_3 & \geq & 0 & \quad (5)\\
x_4 & & \text{libera} & \quad (6)\\
x_5 & \leq & 0 & \quad (7)
\end{aligned}$$

The primal is a maximization problem; therefore, the dual is a minimization problem.

The three constraints of the primal correspond to three variables in the dual:

- y_1 is free, because the first constraint is an equality;
- y_2 is nonnegative, because the second constraint makes opposition to the sign of the problem ($\leq$ constraint in a maximization problem)
- y_3 is nonpositive, because the third constraint gives support to the sign of the problem ($\geq$ constraint in a maximization problem)

The five variables correspond to five constraints:

- the first three dual constraints are of the $\geq$ kind, because variables x_1, x_2 and x_3 are nonnegative and the dual is a minimization problem;
- the fourth constraint is an equality, because x_4 is a free variable;

- the fifth constraint is of the $\leq$ kind, because x_5 is nonpositive and the dual is a minimization problem.

In conclusion:

$$
\begin{array}{rcl}
\min \phi(y) = 7y_1 + 6y_2 + 4y_3 & & \\
-y_1 - y_2 - y_3 & \geq & 2 \\
y_1 + 2y_2 + y_3 & \geq & -4 \\
2y_1 + 3y_2 + y_3 & \geq & -7 \\
y_1 \qquad + 2y_3 & = & -1 \\
2y_1 + y_2 \qquad & \leq & -5 \\
y_1 & & \text{libera} \\
y_2 & \geq & 0 \\
y_3 & \leq & 0
\end{array}
$$

The most common reason to build the dual problem is to solve it. Since the simplex algorithm required nonnegative variables, it is frequent to put the primal into a suitable form in order to obtain this result before building the dual. This means to turn the constraints into the $\leq$ kind in maximization problems, into the $\geq$ kind in minimization problems). In this example, the third constraint would become

$$x_1 - x_2 - x_3 - 2x_4 \leq -4$$

producing the same dual, with the exception that all coefficients of variable y_3 would have the opposite sign. On the other side, variable y_3 would be nonnegative, instead of nonpositive. So, this is like replacing y_3 with $y_3' = -y_3$. In other words, given two equivalent LP problems, their duals are equivalent, as well.