Solved exercises for the course of

# Foundations of Operations Research

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# Complementary slackness

Given the following $LP$ problem:

$$\begin{aligned} \min z = 3x_1 + 4x_2 + 2x_3 & \\ x_1 + 4x_2 + x_3 &\geq 3 \\ 3x_1 - x_2 + 2x_3 &\geq 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

1. write the dual problem;
2. solve the primal problem to optimality exploiting the dual problem, the graphical method and the complementary slackness conditions.
3. repeat the process with the following right-hand-side vectors: $b' = [\ 1\ ;\ 3\ ]'$ and $b'' = [\ 1\ ;\ 2\ ]'$, and comment on the results.

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# Building the dual problem

The dual problem is

$$\begin{aligned} \max w = 3y_1 + 2y_2 & \\ y_1 + 3y_2 &\leq 3 \\ 4y_1 - y_2 &\leq 4 \\ y_1 + 2y_2 &\leq 2 \\ y_1, y_2 &\geq 0 \end{aligned}$$

The application of the rules is quite simple, since the primal is a minimization problem, all its constraints are of the $\geq$ kind and all its variables are nonnegative. The dual problem has only two variables, so that it can be solved graphically. This allows to compute the optimal dual solution and to derive from it the optimal primal solution.

# Graphical solution of the dual problem

Figure 1 reports the graphical representation of the dual problem, which has an optimal solution in $A = (10/9, 4/9)$.

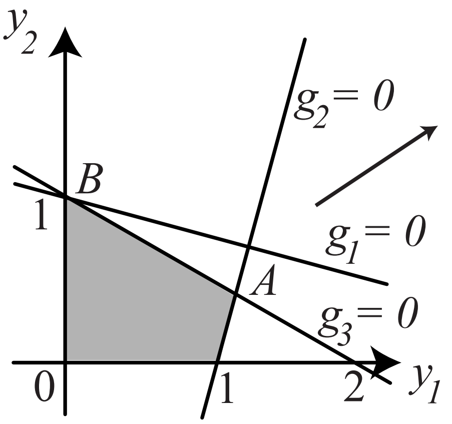


Figura 1: Graphical resolution of the dual problem

## Solving the primal problem through the dual and the complementary slackness conditions

The complementary slackness theorem states that two vectors $x^*$ and $y^*$ are optimal solutions, respectively, of the primal and the dual problem if and only if each of them is a feasible solution of one of the two and the following *complementary slackness conditions* hold:

$$x_j^* \left( c_j - \sum_i a_{ij} y_i^* \right) = 0 \qquad j = 1, \ldots, n$$

$$\left( \sum_j a_{ij} x_j^* - b_i \right) y_i^* = 0 \qquad i = 1, \ldots, m$$

that is, that the products of the variables of a problem times the corresponding *slacks* of the constraints of the other problem are null. Reminding that the slacks coincide with the *slack* or *surplus* variables, one can also say that each variable (natural or *slack*) of either problem corresponds to a variable of the other problem and that at optimality the product of the corresponding variables in the two problems are zero. This condition allows to determine the optimal solution of a problem, given the optimal solution of the other one.

In our case, we have an optimal solution of the dual problem ($y_1^* = 10/9$, $y_2^* = 4/9$) and we are looking for the optimal solution $x^*$ of the primal. Necessarily, it must satisfy the following conditions:

$$\begin{aligned} x_1^* \,(3 - y_1^* - 3y_2^*) &= 0 \\ x_2^* \,(4 - 4y_1^* + y_2^*) &= 0 \\ x_3^* \,(2 - y_1^* - 2y_2^*) &= 0 \\ (x_1^* + 4x_2^* + x_3^* - 3)\, y_1^* &= 0 \\ (3x_1^* - x_2^* + 2x_3^* - 2)\, y_2^* &= 0 \end{aligned}$$

The second and third of the five conditions are satisfied because the optimal solution of the dual lies on the separating line of the second and third constraint, so that the corresponding *slack* variables have zero value. This is a general situation: there is always a basic optimal solution of a *LP* problem, and it belongs to the intersection of a number of constraints. Therefore, the complementary slackness conditions related to the nonbasic variables (which are equal to zero) are trivially satisfied.

In the example, there are still three equations in three unknown variables:

$$\begin{aligned} x_1^* \left(3 - \frac{10}{9} - 3\,\frac{4}{9}\right) &= 0 \\ (x_1^* + 4x_2^* + x_3^* - 3)\, \frac{10}{9} &= 0 \\ (3x_1^* - x_2^* + 2x_3^* - 2)\, \frac{4}{9} &= 0 \end{aligned}$$

from which

$$\begin{aligned} x_1^* &= 0 \\ 4x_2^* + x_3^* &= 3 \\ 3x_1^* - x_2^* &= 2 \end{aligned}$$

and therefore $x_1^* = 0$, $x_2^* = 4/9$ and $x_3^* = 11/9$.

## The case $\max w = y_1 + 3y_2$

When $\max w = y_1 + 3y_2$, the optimal dual solution is degenerate, and corresponds to point $B = (0, 1)$. The primal problem has infinite optimal solutions. In fact, the complementary slackness conditions:

$$\begin{aligned} x_1^* \,(3 - y_1^* - 3y_2^*) &= 0 && (1) \\ x_2^* \,(4 - 4y_1^* + y_2^*) &= 0 && (2) \\ x_3^* \,(2 - y_1^* - 2y_2^*) &= 0 && (3) \\ (x_1^* + 4x_2^* + x_3^* - 1)\, y_1^* &= 0 && (4) \\ (3x_1^* - x_2^* + 2x_3^* - 3)\, y_2^* &= 0 && (5) \end{aligned}$$

become, if $y_1^* = 0$ and $y_2^* = 1$:

$$\begin{aligned} 0x_1^* &= 0 \quad (6)\\ 5x_2^* &= 0 \quad (7)\\ 0x_3^* &= 0 \quad (8)\\ (x_1^* + 4x_2^* + x_3^* - 1)\, 0 &= 0 \quad (9)\\ 3x_1^* - x_2^* + 2x_3^* - 3 &= 0 \quad (10) \end{aligned}$$

so that the first, third and fourth are trivially satisfied. Two equations in three variables are still given:

$$\begin{aligned} x_2^* &= 0 \quad (11)\\ 3x_1^* + 2x_3^* &= 3 \quad (12) \end{aligned}$$

providing the equation of a line in space $(x_1, x_2, x_3)$.

The optimal solutions are not all the points of this line, but only the feasible ones for constraints $x_1 + 4x_2 + x_3 \geq 3$, $3x_1 - x_2 + 2x_3 \geq 2$, plus of course the nonnegativity constraints $x_1 \geq 0$, $x_2 \geq 0$ e $x_3 \geq 0$. In general, they are a segment[1]. For problems with more dimensions, the set of the optimal solutions could be a whole face of the feasible polyhedron.

## The case $\max w = y_1 + 2y_2$

When $\max w = y_1 + 2y_2$, the dual problem has infinitely many optimal solutions (the whole segment $\overline{AB}$, that is $y_1^* = 2 - 2y_2^*$). Let us express the complementary slackness conditions with respect to the two optimal basic solutions, $A$ and $B$. With respect to $A$, they become:

$$\begin{aligned} x_1^*\,(3 - y_1^* - 3y_2^*) &= 0\\ x_2^*\,(4 - 4y_1^* + y_2^*) &= 0\\ x_3^*\,(2 - y_1^* - 2y_2^*) &= 0\\ (x_1^* + 4x_2^* + x_3^* - 1)\, y_1^* &= 0\\ (3x_1^* - x_2^* + 2x_3^* - 2)\, y_2^* &= 0 \end{aligned}$$

from which, since the second and third one are satisfied thanks to $y^*$,

$$\begin{aligned} x_1^* &= 0\\ x_1^* + 4x_2^* + x_3^* &= 1\\ 3x_1^* - x_2^* + 2x_3^* &= 2 \end{aligned}$$

[1]One can also draw this segment, since condition $x_2 = 0$ allows to limit the study to plane $(x_1, x_3)$

and therefore $x^* = (0,0,1)$. Since the slacks of the constraints are zero, the optimal solution has a zero basic variable, and is therefore degenerate.

If, on the contrary, we express the conditions with respect to $B$

$$\begin{aligned} x_1^*\,(3 - y_1^* - 3y_2^*) &= 0 \\ x_2^*\,(4 - 4y_1^* + y_2^*) &= 0 \\ x_3^*\,(2 - y_1^* - 2y_2^*) &= 0 \\ (x_1^* + 4x_2^* + x_3^* - 1)\,y_1^* &= 0 \\ (3x_1^* - x_2^* + 2x_3^* - 2)\,y_2^* &= 0 \end{aligned}$$

they become, since the first, third and fourth are satisfied thanks to $y^*$,

$$\begin{aligned} 5x_2^* &= 0 \\ 3x_1^* - x_2^* + 2x_3^* &= 2 \end{aligned}$$

Apparently, they are infinite solutions. In practice, drawing their graph on plane $x_2 = 0$ and imposing $x_1 \geq 0$ and $x_3 \geq 0$, one finds out that only a single solution is optimal, once again $x^* = (0,0,1)$. They seem at first to be infinitely many because point $B$ is degenerate.

Summing up, degenerate optimal solutions in the primal problem correspond to infinitely many optimal solutions in the dual problem; infinite optimal solutions in the primal to degenerate optimal solutions in the dual. This derives from the exchange of the right-hand-side $b$ (which is the value of a basic variable in the optimal solution) and of the reduced cost $c$ (which is the variation of the objective function related to a nonbasic variable in the optimal solution).