Foundations of Operations Research

Master of Science in Computer Engineering

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Tuesday 13.15 - 15.15

Thursday 10.15 - 13.15

http://homes.di.unimi.it/~cordone/courses/2014-for/2014-for.html

Lesson 14: Special cases of the simplex method — Como, Fall 2013

The general scheme

Algorithm Simplex(A, b, c)

(A, b) := IntroduceFullRank(A, b);

If $A = \emptyset$ **then Return** Unfeasible; { Inconsistent constraints }

B := FindFeasibleBasis(A, b);

If $B = \emptyset$ **then Return** Unfeasible; { Unfeasible problem }

Optimum := *False*; $\bar{A} := A$; $\bar{b} := b$; $\bar{c} := c$;

While Optimum = *False* **do**

If $\bar{c} \geq 0$ **then** Optimum := *True*;
else $j := \arg\min_{j=1,\ldots,n} \bar{c}_j$; { Select *pivot* column }

If $\bar{A}_j \leq 0$ **then Return** Unbounded; { Unbounded problem }

$i := \arg \min_{i:a_{ij}>0} \frac{\bar{b}_j}{a_{ij}}$; { Select *pivot* row }

$(\bar{A}, \bar{b}, \bar{c}, \bar{d})$:= Pivot$(\bar{A}, \bar{b}, \bar{c}, \bar{d}, i, j)$; { Perform *pivot* operation }
$B := B \cup \{j\} \setminus \{B_i\}$; { Exchange basic and nonbasic column }

Return $(\bar{A}, \bar{b}, \bar{c}, \bar{d})$;

The complexity of a single *pivot* step is $O(mn)$

Unbounded problems

In any step, if a column of the *tableau* has $\bar{c}_j < 0$ and $\overline{A}_j \leq 0$

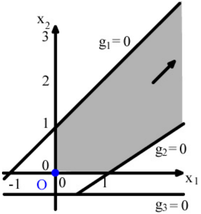
- the nonbasic variable improves the objective entering the basis
- all basic variables keep unchanged or increase their values

there is no limit on the value of the new variable: the problem is unbounded

$$\min f = -x_1 - x_2$$
$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ \frac{1}{2}x_1 - x_2 &\leq \frac{1}{2} \\ -x_2 &\leq \frac{1}{2} \\ x_1, x_2 &\geq 0 \end{aligned}$$

$-d$	x_1	x_2	x_3	x_4	x_5
0	-1	-1	0	0	0
1	-1	1	1	0	0
$\frac{1}{2}$	$\frac{1}{2}$	-1	0	1	0
$\frac{1}{2}$	0	-1	0	0	1

Select (arbitrarily) column 2 and (necessarily) row 1



Unbounded problems

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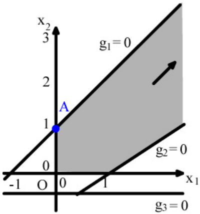
there is no limit on the value of the new variable: the problem is unbounded

$-d$	x_1	x_2	x_3	x_4	x_5
1	-2	0	1	0	0
1	-1	1	1	0	0
$\frac{3}{2}$	$-\frac{1}{2}$	0	1	1	0
$\frac{3}{2}$	-1	0	1	0	1



Variable 1 should enter the basis: it can increase without limit, improving the objective

Degenerate basic solutions

A degenerate basic solution is a basic solution in which one of the basic variables has value zero

A degenerate basic solution corresponds to several bases, because any basis including the nonzero variables yields the same solution

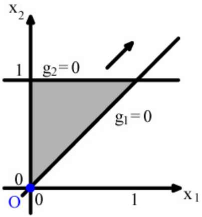
$$\begin{aligned} \min f = -x_1 - x_2 & & \\ x_1 - x_2 & \leq & 0 \\ x_2 & \leq & 1 \\ x_1, x_2 & \geq & 0 \end{aligned}$$

$-d$	x_1	x_2	x_3	x_4
0	-1	-1	0	0
0	1	-1	1	0
1	0	1	0	1

Select (arbitrarily) column 1 and (necessarily) row 1

$-d$	x_1	x_2	x_3	x_4
0	0	-2	1	0
0	1	-1	1	0
1	0	1	0	1

The solution does not change at all



Degeneracy and the simplex algorithm

If a basic variable is 0

- it cannot decrease when leaving the basis
- the new basic variable enters the basis with zero value

so that the new solution is identical to the old one

The simplex method does not guarantee to decrease the objective at each step: it only guarantees not to increase the objective

Only the following steps might increase the zero basic variable and strictly improve the objective

It is possible to cycle through a repeated sequence of degenerate bases associated to the same vertex, which is in general nonoptimal

Fortunately, there exist several anticycling rules, that is rules to choose the *pivot* column and row which guarantee to avoid solution cycles

The basic rule is Bland's rule: always choose the variables of minimum index among the candidate ones; under this rule,

- all steps visit different solutions, so that the number of steps is $\leq \binom{n}{m}$
- in pathological cases, the number of steps is actually exponential
- on average, it is polynomial and very low (linear in m, logarithmic in n)

The first phase of the simplex method

Any LP problem in standard form $Ax = b$ admits an auxiliary LP problem in feasible basic canonical form with $b \geq 0$ and auxiliary variables y

$$\begin{aligned} A\,x + I\,y &= b \\ x \geq 0, y &\geq 0 \end{aligned}$$

The auxiliary problem is not equivalent to the original one, but:

- the basic solutions of the original problem correspond one-to-one to basic solutions with $y = 0$ of the auxiliary problem
- the feasible solutions of the original problem correspond one-to-one to feasible solutions with $y = 0$ of the auxiliary problem

but the auxiliary problem has more feasible solutions; in particular

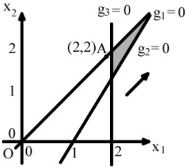
- it always admits the feasible basic solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$

The idea is to move from the trivial solution to other basic solutions minimizing an auxiliary objective function $h(x, y) = \sum_{i=1}^{m} y_i$, that is reducing the unfeasibility with respect to the original problem

- if $h^* = \sum_{i=1}^{m} y_i^* = 0$, x^* provides a feasible solution of the original problem
- if $h^* = \sum_{i=1}^{m} y_i^* > 0$, the original problem has no feasible solution

Example 1

$$\begin{aligned} \min f = -x_1 - x_2 & & \\ -x_1 + x_2 &\leq 0 \\ \frac{3}{2}x_1 - x_2 &\leq \frac{3}{2} \\ -x_1 &\leq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$



1) Add the slack variables
2) $b_3 < 0$: exchange the two sides of constraint (3), add auxiliary variable y_3 and minimize $h = y_3$

$$\begin{aligned} \min h = \quad & y_3 & & \\ -x_1 - x_2 &= f \\ -x_1 + x_2 + x_3 &= 0 \\ \frac{3}{2}x_1 - x_2 + x_4 &= \frac{3}{2} \\ x_1 - x_5 + y_3 &= 2 \\ x_1, \ldots, x_5, y_3 &\geq 0 \end{aligned}$$

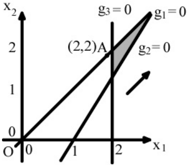
	$-d$	x_1	x_2	x_3	x_4	x_5	y_3
h	0	0	0	0	0	0	1
f	0	-1	-1	0	0	0	0
	0	-1	1	1	0	0	0
	3/2	3/2	-1	0	1	0	0
	2	1	0	0	0	-1	1

3) Put into canonical form for basis (x_3, x_4, y_3): only the auxiliary objective row needs modifying

	$-d$	x_1	x_2	x_3	x_4	x_5	y_3
h	-2	-1	0	0	0	1	0
f	0	-1	-1	0	0	0	0
	0	-1	1	1	0	0	0
	3/2	3/2	-1	0	1	0	0
	2	1	0	0	0	-1	1

Example 1

$$\min f = -x_1 - x_2$$
$$-x_1 + x_2 \leq 0$$
$$\frac{3}{2}x_1 - x_2 \leq \frac{3}{2}$$
$$-x_1 \leq -2$$
$$x_1, x_2 \geq 0$$



4) Improve the auxiliary objective with a *pivot* operation on a_{21}

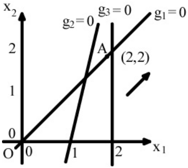
	$-d$	x_1	x_2	x_3	x_4	x_5	y_3
h	-1	0	-2/3	0	2/3	1	$0h$
f	1	0	-5/3	0	2/3	0	$0f$
	1	0	1/3	1	2/3	0	0
	1	1	-2/3	0	2/3	0	0
	1	0	2/3	0	-2/3	-1	1

5) Improve the auxiliary objective with a *pivot* operation on a_{32}

	$-d$	x_1	x_2	x_3	x_4	x_5	y_3
h	0	0	0	0	0	0	1
f	7/2	0	0	0	-1	-5/2	5/2
	1/2	0	0	1	1	1/2	-1/2
	2	1	0	0	0	-1	1
	3/2	0	1	0	1	-3/2	3/2

The current basis (x_3, x_1, x_2) is feasible: we can remove the auxiliary column y_3 and row h and proceed with the regular *pivot* operations to improve the original objective function f

$$\begin{aligned}
\min f &= -x_1 - x_2 \\
-x_1 + x_2 &\leq 0 \\
3x_1 - x_2 &\leq 3 \\
-x_1 &\leq -2 \\
x_1, x_2 &\geq 0
\end{aligned}$$



1) Add the slack variables
2) $b_3 < 0$: exchange the two sides of constraint (3), add auxiliary variable y_3 and minimize $h = y_3$

$$\begin{aligned}
\min h = \quad & y_3 \\
-x_1 - x_2 &= f \\
-x_1 + x_2 + x_3 &= 0 \\
3x_1 - x_2 + x_4 &= 3 \\
x_1 - x_5 + y_3 &= 2 \\
x_1, \dots, x_5, y_3 &\geq 0
\end{aligned}$$

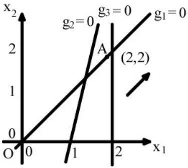
	$-d$	x_1	x_2	x_3	x_4	x_5	y_3
h	0	0	0	0	0	0	1
f	0	-1	-1	0	0	0	0
	0	-1	1	1	0	0	0
	3	3	-1	0	1	0	0
	2	1	0	0	0	-1	1

3) Put into canonical form for basis (x_3, x_4, y_3): only the auxiliary objective row needs modifying

	$-d$	x_1	x_2	x_3	x_4	x_5	y_3
h	-2	-1	0	0	0	1	0
f	0	-1	-1	0	0	0	0
	0	-1	1	1	0	0	0
	3	3	-1	0	1	0	0
	2	1	0	0	0	-1	1

Example 2

$$\min f = -x_1 - x_2$$
$$\begin{aligned} -x_1 + x_2 &\leq 0 \\ 3x_1 - x_2 &\leq 3 \\ -x_1 &\leq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$



4) Improve the auxiliary objective with a *pivot* operation on a_{21}

5) Improve the auxiliary objective with a *pivot* operation on a_{12}

	$-d$	x_1	x_2	x_3	x_4	x_5	y_3	$-d$	x_1	x_2	x_3	x_4	x_5	y_3
h	-1	0	-1/3	0	1/3	1	0h	-1/2	0	0	1/2	1/2	1	0
f	1	0	-4/3	0	1/3	0	0f	3	0	0	2	1	0	0
	1	0	2/3	1	1/3	0	0	3/2	0	1	3/2	1/2	0	0
	1	1	-1/3	0	1/3	0	0	3/2	1	0	1/2	1/2	0	0
	1	0	1/3	0	-1/3	-1	1	1/2	0	0	-1/2	-1/2	-1	1

The current basis (x_2, x_1, y_3) is optimal, but unfeasible for the original problem, since $\phi = y_3 = 1/2$

The original problem has no feasible solution