Solved exercises for the course of

Foundations of Operations Research

Roberto Cordone

The two phase method

Apply the simplex algorithm with the *tableau* representation to the following problem, and solve it for every value of parameter $\alpha > 0$.

$$\max f = x_1 + x_2$$
$$\begin{array}{rcl} x_1 & \geq & x_2 \\ x_2 & \geq & \alpha\,(x_1 - 1) \\ x_1 & \geq & 2 \\ x_2 & \geq & 0 \end{array}$$

The simplex algorithm moves from a basic solution to an adjacent basic solution. It consists of two phases

1. the former starts from a generic basic solution and finds a feasible one, solving an optimization problem, with the same constraints and an auxiliary objective function;

2. the latter starts from the feasible basic solution found in the former phase and move step by step through feasible improving (or at least nonworsening) basic solutions, until an optimal solution has been found.

With suitable conditions, the methods selects at each step a nonbasic variable to introduce into the basis, so as to improve the current objective, and a basic variable to remove from the basis, so as to maintain feasibility.

Obtaining a basic solution is easy: each *pivot* operation isolates the variable associated to the *pivot* column in the constraint associated to the *pivot* row and removes it from the other constraints and from the objective function. In the *tableau* representation, this operation produces a column made of zero elements, except for a single unitary element, that is a valid column to be part of a basis. After m operations, the coefficient matrix contains an identity submatrix of order m and the corresponding reduced costs are zero. The problem thus obtained is denoted as *basic canonical form* associated to the considered basis. In general, the associated basic solution is unfeasible.

Let us consider the i-th constraint:

$$\sum_{j \in N} a_{ij} x_j + x_{B_i} = b_i < 0$$

where N is the set of the indices of the nonbasic variables, and x_{B_i} is the i-th basic variable. Let us exchange the sides of the constraints, so as to make the right-hand-side positive.

$$\sum_{j \in N} -a_{ij} x_j - x_{B_i} = -b_i$$

where $-b_i > 0$. However, the problem is no longer in basic canonical form.

The idea is to *relax* the constraint, allowing to violate it, stating that we accept not only the original feasible solutions, but also those for which

$$\sum_{j \in N} -a_{ij} x_j - x_{B_i} \leq -b_i$$

which can be put into standard form, as usual, introducing an auxiliary *slack* variable $\hat{x}_i$.

$$\sum_{j \in N} -a_{ij} x_j - x_{B_i} + \hat{x}_i = -b_i \qquad \hat{x}_i \geq 0$$

This is a different problem, but it includes all the solutions of the original one. In particular, the solutions of the original problem correspond one-to-one to the solutions with $\hat{x}_i = 0$ of the new one. The new problem also admits solutions (those with $\hat{x}_i > 0$) that have no corresponding solution in the original problem.

Moreover, after this operation the problem is again in basic canonical form, with $\hat{x}_i$ in the basis and x_{B_i} is out of it, and the basic solution obtained is feasible ($\hat{x}_i = -b_i > 0$).

The first phase of the simplex method aims to obtain a feasible solution of the original problem. It does so minimizing $\hat{x}_i$:

$$\min f = \hat{x}_i$$

If the optimum of this problem is $f^* = \hat{x}_i^* = 0$ (it cannot be negative), the other variables provide a feasible (nonoptimal, in general) solution for the original problem, and we can start from there with the simplex algorithm (second phase). Otherwise, it $f^* = \hat{x}_i^* > 0$, then the auxiliary problem admits no solution with $\hat{x}_i = 0$, and the original problem admits no feasible solution at all.

In general, the starting basis includes more than one negative variable. For each of them, we introduce an auxiliary variable, and the original problem minimizes their sum

$$\min f = \sum_{i : b_i < 0} \hat{x}_i$$

The second phase of the simplex method starts from the basis obtained in this way. In order to build the *tableau*, we remove the artificial columns and replace row 0 with the original one. In this way, the *tableau* contains an identity submatrix, but the cost vector is not in canonical form. To obtain this form, we perform m *pivot* operations on the elements of the identity submatrix. These operations are very simple, because they actually modify only row 0.

Solution

First, let us put the problem into standard form. This requires to introduce *surplus* variables. These variables form a trivial basis, but the resulting *tableau* contains the opposite of an identity submatrix $(-I)$ instead of I. Some *pivot* operations on these m coefficients would be necessary to obtain a basic canonical form. It is equivalent (and faster) to first replace the $\geq$ constraints with $\leq$ constraints (exchanging their sides) and then introduce *slack* variables:

$$\begin{array}{rcl}
\min z = -x_1 - x_2 & & \\
-x_1 + x_2 + x_3 & = & 0 \\
\alpha x_1 - x_2 \qquad + x_4 & = & \alpha \\
-x_2 \qquad\qquad + x_5 & = & -2 \\
x_1, x_2 & \geq & 0
\end{array}$$

This yields directly a a basic canonical form with respect to variables x_3, x_4 and x_5. Notice that variable x_2 was a free variable, but the third constraint $(x_1 \geq 2)$ implies that it is actually nonnegative. We have therefore introduced a redundant nonnegativity constraint $x_1 \geq 0$. The aim is to avoid the useless complication implied by dealing with x_2 as with a free variable.

The problem in basic canonical form has the following *tableau*:

0	-1	-1	0	0	0
0	-1	1	1	0	0
α	α	-1	0	1	0
-2	0	-1	0	0	1

The corresponding basic solution is unfeasible: $x_1 = x_2 = x_3 = 0$, $x_4 = \alpha > 0$, but $x_5 = -2$. Figures 1, 2 and 3 show that point $(0, 0)$ is unfeasible for all values of $\alpha > 0$.

Let us introduce an artificial variable for each negative basic variable: in this case, a single variable, associated to the third constraint. Moreover, let us assume as objective the minimization of this variable. The auxiliary problem is

$$\begin{array}{rcl}
\min z = \qquad\qquad\qquad + x_6 & & \\
-x_1 + x_2 + x_3 & = & 0 \\
\alpha x_1 - x_2 \qquad + x_4 & = & \alpha \\
x_2 \qquad\qquad - x_5 + x_6 & = & 2 \\
x_1, x_2 & \geq & 0
\end{array}$$

and its *tableau* is

0	0	0	0	0	0	1
0	-1	1	1	0	0	0
α	α	-1	0	1	0	0
2	0	1	0	0	-1	1

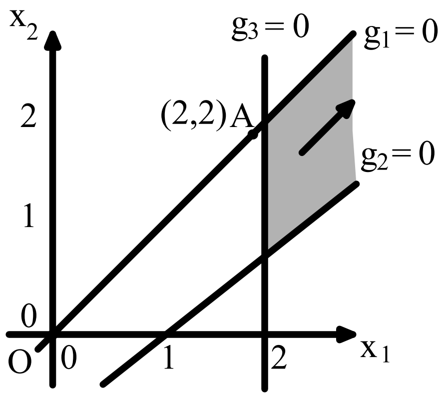

Figura 1: Graphical solution when $\alpha \leq 1$

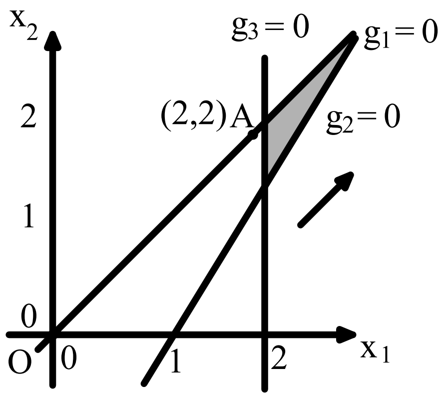

Figura 2: Graphical solution when $1 < \alpha \leq 2$

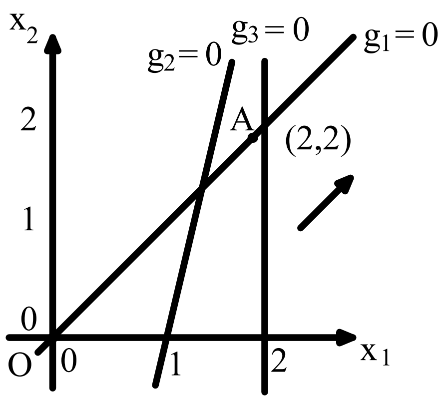

Figura 3: Graphical solution when $\alpha > 2$

This must be put into basic canonical form with respect to (x_3, x_4, x_6) setting the reduced costs of the basic variables to zero (here, $c_6 = 1$). A single *pivot* operation is enough: the *pivot* element is a_{36}, and the operation only subtracts row 3 from row 0:

-2	0	-1	0	0	1	0
0	-1	1	1	0	0	0
α	α	-1	0	1	0	0
2	0	1	0	0	-1	1

Now $x_6 = 2$ and the value of the objective is $-a_{00} = 2$. Indeed point $(0, 0)$ has a positive unfeasibility.

A single variable can improve the objective: x_2. The *pivot* operation is performed in the positive element of column 2 with the minimum ratio b_i/a_{i2}, that is element $a_{12} = 1$. This means that the current *slack* variable of constraint 1, that is x_3, will get out of the basis and will become zero.

-2	-1	0	1	0	1	0
0	-1	1	1	0	0	0
α	α-1	0	1	1	0	0
2	1	0	-1	0	-1	1

Notice that x_6 is still basic. Given that the aim of this procedure is to remove x_6, one could wonder why we do not directly remove x_6 from the basis. The answer it that introducing x_2 and removing x_6 constraint 1 would be violated, and the method would not work. By contrast, the simplex algorithm guarantees to minimize x_6 without introducing any other unfeasibility.

In the second step the *pivot* element depends on the value of α. Let us distinguish two cases:

$$\begin{cases} \frac{\alpha}{\alpha-1} < 2 \\ \frac{\alpha}{\alpha-1} \geq 2 \end{cases} \Leftrightarrow \begin{cases} \alpha > 2 \\ \alpha \leq 2 \end{cases}$$

First case: $\alpha > 2$

The *pivot* element is a_{21}.

-2 + $\frac{\alpha}{\alpha-1}$	0	0	$1 + \frac{1}{\alpha-1}$	$1 + \frac{1}{\alpha-1}$	1	0
$\frac{\alpha}{\alpha-1}$	0	1	$1 + \frac{1}{\alpha-1}$	$\frac{1}{\alpha-1}$	0	0
$\frac{\alpha}{\alpha-1}$	1	0	$\frac{1}{\alpha-1}$	$\frac{1}{\alpha-1}$	0	0
2 - $\frac{\alpha}{\alpha-1}$	0	0	-1 - $\frac{1}{\alpha-1}$	- $\frac{1}{\alpha-1}$	-1	1

Notice that this basic solution is feasible and optimal, as the right-hand-sides and the reduced costs are all nonnegative. Hence, it is the solution with minimal unfeasibility. However, the objective value is $f = -a_{00} = 2 - \frac{\alpha}{\alpha-1} > 0$. So, the unfeasibility cannot be completely removed: for $\alpha > 2$ the original problem admits no feasible solution. It is easy to see it in Figure 3.

Second case: $\alpha \leq 2$

The *pivot* element is a_{31}.

0	0	0	0	0	0	1
2	0	1	0	0	-1	1
2-α	0	0	α	1	α-1	1-α
2	1	0	-1	0	-1	1

Notice that this basic solution ($x_1 = x_2 = 2$, $x_3 = x_5 = x_6 = 0$, $x_4 = 2 - \alpha$) is feasible, since $2 - \alpha \geq 0$. Since all the reduced costs are nonnegative, it is also optimal with respect to the auxiliary objective, that is, it minimizes unfeasibility. Moreover, the objective value is $f = -a_{00} = 0$, so that the solution found corresponds to a feasible solution of the original problem. Graphically, it corresponds to point $A = (2, 2)$, which is the starting solution of the second phase.

Second phase Once the first phase has ended, we remove the auxiliary column from the *tableau*: the artificial variable, in fact, is zero (out of the basis) and it will remain such. Then, we replace row 0 with the original one.

0	-1	-1	0	0	0
2	0	1	0	0	-1
2-α	0	0	α	1	α-1
2	1	0	-1	0	-1

This is not a basic canonical form, because row 0 has not been submitted to the same *pivot* operations of the other rows. However, the rest of the *tableau* suggests quite clearly to adopt basis (x_2, x_4, x_1). In order to set the reduced costs of these three variables to zero, we combine row 0 with the other ones. In particular, we sum rows 1 and 3 to row 0. These are actually three *pivot* operations (one of them with a zero multiplier, and two with unitary multipliers). Since they are very simple, we perform them all together.

4	0	0	α-1	1	-2
2	0	1	0	0	-1
2-α	0	0	α	1	α-1
2	1	0	-1	0	-1

The reduced cost of variable x_5 is the most negative one for all values of α. Anyway, the only necessary condition is that it is negative, thus making profitable to include x_5 into the basis. In order to determine the variable which will get out of the basis, we analyze the elements of column 5. There are two cases: $\alpha \leq 1$ and $1 < \alpha \leq 2$.

If $\alpha \leq 1$, all the elements of the column are negative or zero: increasing variable x_5, the basic variables suffer on effect or are forced to increase. Thus, the nonnegativity constraints never intervene, and the solution can be improved without any limit. There is no optimal solution, and the problem is unbounded. Graphically, this corresponds to the case in which the separating line of the second constraint is parallel to that of the first one, or forms an open angle with it, and the feasible region becomes an unbounded polyhedron (as in Figure 1).

If $1 < \alpha \leq 2$, we perform a *pivot* operation on element a_{25}.

-4+2$\frac{2-\alpha}{\alpha-1}$	0	0	$\frac{2\alpha}{\alpha-1} - 1$	$\frac{2}{\alpha-1}$	0
2+2$\frac{2-\alpha}{\alpha-1}$	0	1	$\frac{\alpha}{\alpha-1}$	$\frac{1}{\alpha-1}$	0
$\frac{2-\alpha}{\alpha-1}$	0	0	$\frac{\alpha}{\alpha-1}$	$\frac{1}{\alpha-1}$	1
2+2$\frac{2-\alpha}{\alpha-1}$	1	0	$\frac{\alpha}{\alpha-1}$-1	$\frac{1}{\alpha-1}$	0

This solution is feasible and optimal for all values $\alpha \in (1; 2]$. In fact, the reduced costs are all nonnegative (in particular, $\tilde{c}_3 = 2\alpha/(\alpha - 1) - 1 = (\alpha + 1)/(\alpha - 1) > 0$) and the right-hand-sides are nonnegative, as well. This case is reported in Figure 2.