Foundations of Operations Research

Master of Science in Computer Engineering

Roberto Cordone
roberto.cordone@unimi.it

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Thursday 10.15 - 13.15

http://homes.di.unimi.it/~cordone/courses/2014-for/2014-for.html

A sample problem

$$
\begin{aligned}
\min f = 5x_1 + 4x_2 + 3x_3 & & \\
2x_1 + 3x_2 + x_3 + x_4 & = & 5 \\
4x_1 + x_2 + 2x_3 \qquad + x_5 & = & 11 \\
3x_1 + 4x_2 + 2x_3 \qquad\qquad + x_6 & = & 8 \\
x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0
\end{aligned}
$$

There are 20 basic solutions, but it is easy to find the optimal one

1. matrix A shows an obvious basic submatrix $B = I$ (columns 4, 5 and 6) and the basic cost subvector is null: $c_B = 0$:
   - the corresponding basic solution is $\bar{x}_N = 0 \Rightarrow x_B = B^{-1} b = b$
   - the value of the objective function is $f(\bar{x}) = c_B \bar{x}_B + c_N \bar{x}_N + d = d$
2. the right-hand-side vector b is nonnegative: the obvious basic solution is feasible
3. the nonbasic cost subvector c_N is nonnegative: $c_N \geq 0$ a generic feasible solution x costs $f(x) = c_N^T x_N + d \geq d$

Therefore, the obvious basic solution $\bar{x} = \begin{bmatrix} b \\ 0 \end{bmatrix}$ is optimal

Not all *LP* problems enjoy such properties, but all feasible and bounded *LP* problems admit an equivalent problem that enjoys them

Basic canonical form

A basic canonical form is a special standard form in which

- the coefficient matrix A includes a submatrix $B = I$
- the cost vector c includes a subvector $c_B = 0$
- B and c_B correspond to the same variables

This is the first of the three conditions mentioned above

An LP problem admits a basic canonical form for each of its bases
A direct way to obtain it is to multiply the equality constraints by B^{-1}

$$B^{-1} A x = B^{-1} [B \,|\, N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = B^{-1} b \Rightarrow x_B + B^{-1} N x_N = B^{-1} b$$

and then replace x_B in the expression of the objective function

$$f(x) = c^T x = \begin{bmatrix} c_B^T \,|\, c_N^T \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \left(c_N^T - c_B^T B^{-1} N \right) x_N + c_B^T B^{-1} b + d$$

From the basic canonical form, it is easy to obtain

- the associated basic solution: $\bar{x} = \bar{b} = \begin{bmatrix} B^{-1} b \\ 0 \end{bmatrix}$
- the associated objective value: $f(\bar{x}) = \bar{d} = c_B^T B^{-1} b + d$
- the associated reduced cost vector: $\bar{c}^T = \left[0 \,|\, c_N^T - c_B^T B^{-1} N \right]$

An algorithmic idea

Given an *LP* problem in standard form

1. put it into a basic canonical form (*How? Premultiplying by B^{-1}?*)
2. if $\bar{b} \geq 0$, the associated basic solution $\bar{x}$ is feasible;
   otherwise, put the problem into another basic canonical form
   and go back to step 1 (*How? What form? Is it always possible?*)
3. if $\bar{c}_N \geq 0$, the associated feasible basic solution $\bar{x}$ is also optimal;
   otherwise, put the problem into another basic canonical form
   and go back to step 2 (*How? What form? Is it always possible?*)

The simplex method examines a sequence of basic canonical forms (solutions) in two phases

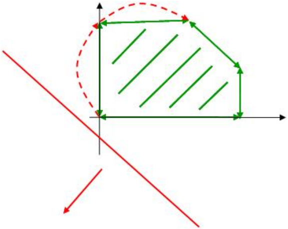
1. achieving a basic feasible solution by subsequent reductions of the unfeasibility (suitable conditions reveal whether the problem is unfeasible)
2. achieving a basic optimal solution by subsequent reductions of the objective (suitable conditions reveal whether the problem is unbounded)

The simplex method: general idea

The idea is to move from a basic canonical form to an "adjacent" one

1. a single nonbasic column gets into the basis; correspondingly, a single nonbasic variable changes from zero to nonnegative
2. a single basic column gets out of the basis; correspondingly, a single basic variable changes from nonnegative to zero

From a geometrical point of view, the idea is that
the current solution moves from a vertex to an "adjacent" vertex



The simplex method consists of two phases:

1. the current basic solution gets closer to the feasible region
2. the basic solution keeps feasible, and the objective function improves

Obtaining a basic canonical form

The following problem is in standard form, but not in any basic canonical form

$$
\begin{aligned}
\min f = -6\,x_1 - 5\,x_3 - \frac{7}{3}\,x_4 + \frac{8}{3}\,x_5 \quad &+ \quad \frac{26}{3} \\
3\,x_1 + 3\,x_3 + x_4 - 2\,x_5 \quad &= \quad 2 \\
3\,x_1 + 3\,x_2 + x_4 + x_5 \quad &= \quad 8 \\
6\,x_3 + x_4 - 5\,x_5 + 3\,x_6 \quad &= \quad 1 \\
x_1, \dots, x_6 \quad &\geq \quad 0
\end{aligned}
$$

Arbitrarily selecting columns $(1, 2, 6)$, one obtains a basis B:

$$
B = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow B^{-1} = \frac{1}{27} \cdot \begin{bmatrix} 9 & 0 & 0 \\ -9 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}
$$

The corresponding canonical form can be easily built by simple matrix operations

$$
\bar{A} = B^{-1} \cdot A = \begin{bmatrix} 1 & 0 & 1 & 1/3 & 2/3 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1/3 & -5/3 & 1 \end{bmatrix} \qquad \bar{b} = B^{-1} b = \begin{bmatrix} 2/3 \\ 2 \\ 1/3 \end{bmatrix}
$$

The corresponding reduced cost vector and constant term are

$$
\bar{c}^T = \begin{bmatrix} 0 & 0 & 1 & -1/3 & -4/3 & 0 \end{bmatrix} \qquad \bar{d} = c_B^T B^{-1} b + d = \frac{14}{3}
$$

However, there is a simpler and more efficient way to proceed

Equivalent equalities

Any *LP* problem in standard form can be transformed into an equivalent one by

1. multiplying both sides of an equality by a constant coefficient

$$3\,x_1 + 3\,x_3 + x_4 - 2\,x_5 = 2 \quad \Leftrightarrow \quad 1\,x_1 + 1\,x_3 + \frac{1}{3}\,x_4 - \frac{2}{3}\,x_5 = \frac{2}{3}$$

2. using an equality to determine a variable in terms of the other ones and replacing the variable everywhere else

$$\begin{aligned}
\min f = -6\,x_1 - 5\,x_3 - \frac{7}{3}\,x_4 + \frac{8}{3}\,x_5 \quad &+ \quad \frac{26}{3}\\
3\,x_1 + 3\,x_3 + x_4 - 2\,x_5 \quad &= \quad 2\\
3\,x_1 + 3\,x_2 + x_4 + x_5 \quad &= \quad 8\\
6\,x_3 + x_4 - 5\,x_5 + 3\,x_6 \quad &= \quad 1\\
x_1, \dots, x_6 \quad &\geq \quad 0
\end{aligned}$$

Use the first constraint to determine $x_1 = \dfrac{2 - 3\,x_3 - x_4 + 2\,x_5}{3}$

$$\begin{aligned}
\min f = \qquad x_3 - \frac{1}{3}\,x_4 - \frac{4}{3}\,x_5 \quad &+ \quad \frac{14}{3}\\
x_1 + x_3 + \frac{1}{3}\,x_4 - \frac{2}{3}\,x_5 \quad &= \quad \frac{2}{3}\\
3\,x_2 - 3\,x_3 + 3\,x_5 \quad &= \quad 6\\
6\,x_3 + x_4 - 5\,x_5 + 3\,x_6 \quad &= \quad 1\\
x_1, \dots, x_6 \quad &\geq \quad 0
\end{aligned}$$

Equivalent equalities

The same transformation can be done

1. multiplying both sides of an equality by a constant coefficient

$$3\,x_1 + 3\,x_3 + x_4 - 2\,x_5 = 2 \quad \Leftrightarrow \quad 1\,x_1 + 1\,x_3 + \frac{1}{3}x_4 - \frac{2}{3}x_5 = \frac{2}{3}$$

2. subtracting from an equality another equality multiplied by a suitable constant

$$\begin{array}{rcl}
\min f = -6\,x_1 - 5\,x_3 - \frac{7}{3}x_4 + \frac{8}{3}x_5 & + & \frac{26}{3} \\
3\,x_1 + 3\,x_3 + x_4 - 2\,x_5 & = & 2 \\
3\,x_1 + 3\,x_2 + x_4 + x_5 & = & 8 \\
6\,x_3 + x_4 - 5\,x_5 + 3\,x_6 & = & 1 \\
x_1, \ldots, x_6 & \geq & 0
\end{array}$$

$$\begin{array}{rcl}
\min f = x_3 - \frac{1}{3}x_4 - \frac{4}{3}x_5 & + & \frac{14}{3} \\
x_1 + x_3 + \frac{1}{3}x_4 - \frac{2}{3}x_5 & = & \frac{2}{3} \\
3\,x_2 - 3\,x_3 + 3\,x_5 & = & 6 \\
6\,x_3 + x_4 - 5\,x_5 + 3\,x_6 & = & 1 \\
x_1, \ldots, x_6 & \geq & 0
\end{array}$$

Multiply the first constraint by $1/3$

Subtract the first constraint

- multiplied by 3
  from the second constraint
- multiplied by 0
  (x_1 *is already absent*)
  from the third constraint

Consider the objective function as an equality: $f - d = c^T x$ and subtract from it the first constraint multiplied by -6

The *tableau* representation

A common representation of an *LP* problem in standard form is the so called *tableau*, that is a matrix with $m+1$ rows and $n+1$ columns

- row 0 corresponds to the objective function: $a_{0j} := c_j$ $(j = 1, \dots, n)$
  rows $1, \dots, m$ correspond to the constraints
- column 0 corresponds to the right-hand sides: $a_{i0} := b_i$ $(i = 1, \dots, m)$
  columns $1, \dots, n$ correspond to the variables

Since $c^T x = f - d$, we set $a_{00} := -d$

$$\begin{aligned} \min f = c^T x &\;+\; d \\ A x &\;=\; b \\ x &\;\geq\; 0 \end{aligned}$$

$-d$	c^T
b	A

$$\begin{aligned} \min f = -6\,x_1 - 5\,x_3 - \frac{7}{3}\,x_4 + \frac{8}{3}\,x_5 &\;+\; \frac{26}{3} \\ 3\,x_1 + 3\,x_3 + x_4 - 2\,x_5 &\;=\; 2 \\ 3\,x_1 + 3\,x_2 + x_4 + x_5 &\;=\; 8 \\ 6\,x_3 + x_4 - 5\,x_5 + 3\,x_6 &\;=\; 1 \\ x_1, \dots, x_6 &\;\geq\; 0 \end{aligned}$$

$-\frac{26}{3}$	-6	0	-5	$-\frac{7}{3}$	$\frac{8}{3}$	0
2	3	0	3	1	-2	0
8	3	3	0	1	1	0
1	0	0	6	1	-5	3

The *pivot* operation

The *pivot* operation represents on the *tableau* the substitution of a variable
It is based on

- a *pivot* column j: the variable x_j expressed in terms of the others
- a *pivot* row i: the constraint $\left(a_i^T, b_i\right)$ used to determine the variable

The *pivot* is the element a_{ij} identified by the *pivot* row and column

The *pivot* operation consists in:

- dividing the *pivot* row $\left[\, b_i \mid a_i^T \,\right]$ by the *pivot* element a_{ij} $(a_{ij} \neq 0!)$

$$\bar{a}_i^T := \frac{a_i^T}{a_{ij}} \qquad (note\text{: } b_i = a_{i0} \Rightarrow \bar{b}_i := \frac{b_i}{a_{ij}})$$

- subtracting the *pivot* row multiplied by a suitable value from all other rows

$$\bar{a}_h := a_h - \frac{a_{hj}}{a_{ij}} a_i \quad \forall h \neq i$$

$$(note\text{: } c_j = a_{0j} \Rightarrow \bar{c} := c - \frac{c_j}{a_{ij}} a_i \text{ and } d = a_{00} \Rightarrow \bar{d} := d - \frac{c_j}{a_{ij}} b_i)$$

For the *pivot* column j, the operation yields $\bar{c}_j = \bar{a}_{0j} := 0$ and $\bar{a}_{hj} := 0$

The effect of a *pivot* operation is that the *pivot* column gets into the basis

Obtaining a basic canonical form

Given an LP problem and a basis, one obtains the associated basic canonical form applying the transformation to each column of the basis; for each basic variable:

- the coefficient in the objective function becomes $\bar{c}_j := 0$
- the *pivot* element becomes $\bar{a}_{ij} := 1$
- the coefficients of the *pivot* column in the nonpivot rows become $\bar{a}_{hj} := 0, \forall h \neq i$

$\Rightarrow \bar{B} = I$ and $\bar{c}_B = 0$

This takes one step for each column: $m = 3$ steps on elements $(1,1)$, $(2,2)$ and $(3,6)$

$$
\begin{aligned}
\min f = -6\,x_1 - 5\,x_3 - \frac{7}{3}\,x_4 + \frac{8}{3}\,x_5 \quad & + \quad \frac{26}{3} \\
3\,x_1 + 3\,x_3 + x_4 - 2\,x_5 \quad & = \quad 2 \\
3\,x_1 + 3\,x_2 + x_4 + x_5 \quad & = \quad 8 \\
6\,x_3 + x_4 - 5\,x_5 + 3\,x_6 \quad & = \quad 1 \\
x_1, \ldots, x_6 \quad & \geq \quad 0
\end{aligned}
$$

$-\frac{26}{3}$	-6	0	-5	$-\frac{7}{3}$	$\frac{8}{3}$	0
2	3	0	3	1	-2	0
8	3	3	0	1	1	0
1	0	0	6	1	-5	3

$$
\begin{aligned}
\min f = x_3 - \frac{1}{3}\,x_4 - \frac{4}{3}\,x_5 \quad & + \quad \frac{14}{3} \\
x_1 \quad + \quad x_3 + \frac{1}{3}\,x_4 - \frac{2}{3}\,x_5 \quad & = \quad \frac{2}{3} \\
- \quad x_3 \quad + \quad x_5 \quad & = \quad 2 \\
2\,x_3 + \frac{1}{3}\,x_4 - \frac{5}{3}\,x_5 + x_6 \quad & = \quad \frac{1}{3} \\
x_1, \ldots, x_6 \quad & \geq \quad 0
\end{aligned}
$$

$\frac{14}{3}$	0	0	1	$-\frac{1}{3}$	$-\frac{4}{3}$	0
$\frac{2}{3}$	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0
2	0	1	-1	0	1	0
$\frac{1}{3}$	0	0	2	$\frac{1}{3}$	$-\frac{5}{3}$	1

Graphical representation

$$\min f = -x_1 - x_2$$
$$\begin{aligned} 6x_1 + 4x_2 + x_3 &= 24 \\ 3x_1 - 2x_2 + x_4 &= 6 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$-d$	x_1	x_2	x_3	x_4
0	-1	-1	0	0
24	6	4	1	0
6	3	-2	0	1

The basis is $(3, 4)$ and the basic solution $(0, 0, 24, 6)$
The basic reduced costs are zero, and the objective value is 0

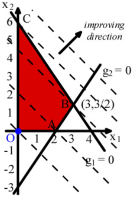
Perform *pivot* operations on $(1, 1)$ and $(2, 2)$ to obtain basis $(1, 2)$

$-d$	x_1	x_2	x_3	x_4
$\frac{9}{2}$	0	0	$\frac{5}{24}$	$-\frac{1}{12}$
3	1	0	$\frac{1}{12}$	$\frac{1}{6}$
$\frac{3}{2}$	0	1	$\frac{1}{8}$	$-\frac{1}{4}$

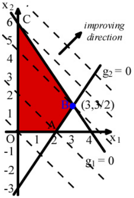
The basis is $(1, 2)$ and the basic solution is $(3, 3/2, 0, 0)$

The basic reduced costs are zero, and the objective value is $-\frac{9}{2}$

$$\min f = -x_1 - x_2$$
$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \\ 3x_1 - 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$



The current vertex is $(0, 0)$



The current vertex is $(3, 3/2)$

Moving from a basis to another basis

When the problem is already in basic canonical form
a *pivot* operation on element a_{ij} represents a basis change

- the *pivot* column j enters the basis
- the basic column h with coefficient $a_{ih} = 1$ in the *pivot* row i exits

Therefore, the *pivot* element determines the resulting basis $\bar{B} = B \cup \{j\} \setminus \{h\}$

We will follow the evolution of the algorithm from three parallel points of view

1. algebraic representation: variables and equations
2. programming representation: *tableau*
3. graphical representation: possible only for 2 natural variables

A wrong choice of the *pivot* element

For example, we can move from basis $(3,4)$ to basis $(2,3)$:

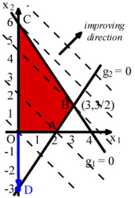
- x_2 gets into the basis: the *pivot* column is 2
- x_4 gets out of the basis: the *pivot* row is 2 (only nonzero coefficient)

the *pivot* element is $a_{22} = -1$

This choice of the *pivot* element is completely wrong:

- it produces an unfeasible solution:
  if x_4 goes to zero, x_1 is unchanged and
  $3x_1 - 2x_2 + x_4 = 6$, then x_2 decreases
  $\bar{x}_2 = \bar{b}_2 := b_2/a_{22} = 6/-2 = -3 < 0$
- it produced a worse solution:
  if x_1 is unchanged, x_2 decreases and
  $f(x) = -x_1 - x_2$, then the cost increases
  $-\bar{d}_j := -d + (c_j/a_{ij})\, b_i = 0 + (-1/-2)\,6 = 3$

See also the graphic



The current vertex is $(0,-3)$

How to choose the *pivot* element

At first, we introduce a fundamental assumption: let the current basic solution be feasible ($b_i \geq 0$ for all i) (*we will see later how to obtain it*)

The choice of the new basis, that is of the *pivot* element, must guarantee that

1. the new basic variable remains nonnegative
   - algebraic perspective: if the old basic variable decreases to zero, the new one must increase
   - *tableau* perspective: $\bar{b}_i := \frac{b_i}{a_{ij}} \geq 0$
   - graphical perspective: we must move towards the feasible region

   Therefore, choose a positive *pivot* element: $a_{ij} > 0$
2. the objective function improves
   - algebraic perspective: when the new basic variable increases, the total cost must decrease
   - *tableau* perspective: $-\bar{d} = -d - \frac{c_j}{a_{ij}} b_i > -d$
   - graphical perspective: we must move consistently with the most improving direction

   Therefore, choose a *pivot* column with negative reduced cost: $c_j < 0$

How to choose the *pivot* element

The choice of the new basis, that is of the *pivot* element, must guarantee that

3. the other basic variables remain feasible (one decreases to zero)
   - algebraic perspective: if x_j increases by ϵ, and the nonbasic variables keep zero, each old basic variable changes by $-\epsilon\, a_{hj}$ $(h = 1, \dots, m)$; they keep feasible as long as

$$x_{\bar{B}_h} = b_h - \epsilon\, a_{hj} \geq 0 \Leftrightarrow \epsilon\, a_{hj} \leq b_h \text{ for all } h \neq i$$

   - *tableau* perspective:

$$\bar{b}_h := b_h - \frac{a_{hj}}{a_{ij}}\, b_i \geq 0 \Leftrightarrow \begin{cases} \dfrac{b_i}{a_{ij}} \leq \dfrac{b_h}{a_{hj}} & \text{for } a_{hj} > 0 \\ \dfrac{b_i}{a_{ij}} \geq 0 & \text{for } a_{hj} = 0 \text{ (trivial)} \\ \dfrac{b_i}{a_{ij}} \geq \dfrac{b_h}{a_{hj}} & \text{for } a_{hj} < 0 \text{ (trivial)} \end{cases}$$

   - graphical perspective:
     - if $a_{hj} < 0$, we move away from the separating hyperplane $a_h^T x = b_h$
     - if $a_{hj} = 0$, we move parallel to the separating hyperplane $a_h^T x = b_h$
     - if $a_{hj} > 0$, we move closer to the separating hyperplane $a_h^T x = b_h$

Therefore, choose the *pivot* row with minimum b_h / a_{hj} among those with $a_{hj} > 0$

$$i := \arg \min_{i : a_{hj} > 0} \frac{b_h}{a_{hj}}$$

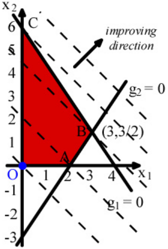
Example (1)

$$\begin{aligned} \min f &= -x_1 - x_2 \\ 6x_1 + 4x_2 + x_3 \quad &= 24 \\ 3x_1 - 2x_2 \quad + x_4 &= 6 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$-d$	x_1	x_2	x_3	x_4
0	-1	-1	0	0
24	6	4	1	0
6	3	-2	0	1

The basis is $(3, 4)$
The basic solution $(0, 0, 24, 6)$
The objective value is 0
The basic reduced costs are zero

$$\begin{aligned} \min f &= -x_1 - x_2 \\ 6x_1 + 4x_2 &\leq 24 \\ 3x_1 - 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$



The current vertex is $(0, 0)$

Two nonbasic variables have the same negative reduced cost: $c_1 = c_2 = -1$

Variable x_1 enters the basis (x_2 *would be better, but the selection rule is heuristic*)

The ratio b_i / a_{ij} is minimum for $i = 2$: variable x_4 leaves the basis; *pivot* element a_{21}

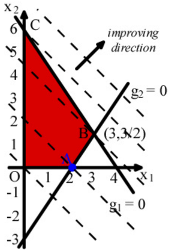
Example (2)

$$\begin{aligned} \min f &= -x_1 - x_2 \\ 6x_1 + 4x_2 + x_3 \quad &= 24 \\ 3x_1 - 2x_2 \quad + x_4 &= 6 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$-d$	x_1	x_2	x_3	x_4
2	0	$-\frac{5}{3}$	0	$\frac{1}{3}$
12	0	8	1	-2
2	1	$-\frac{2}{3}$	0	$\frac{1}{3}$

The basis is $(1, 3)$
The basic solution $(2, 0, 12, 0)$
The objective value is -2
The basic reduced costs are zero

$$\begin{aligned} \min f &= -x_1 - x_2 \\ 6x_1 + 4x_2 &\leq 24 \\ 3x_1 - 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$



The current vertex is $(2, 0)$

Only one nonbasic variable has a negative reduced cost: $c_2 = -5/3$

Variable x_2 enters the basis
The ratio b_i/a_{ij} is positive only for row $i = 1$: variable x_3 leaves the basis;

pivot element a_{12}

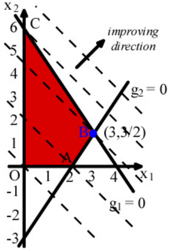
Example (3)

$$\min f = -x_1 - x_2$$
$$6x_1 + 4x_2 + x_3 = 24$$
$$3x_1 - 2x_2 + x_4 = 6$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$-d$	x_1	x_2	x_3	x_4
$\frac{9}{2}$	0	0	$\frac{5}{24}$	$-\frac{1}{12}$
$\frac{3}{2}$	0	1	$\frac{1}{8}$	$-\frac{1}{4}$
3	1	0	$\frac{1}{12}$	$\frac{1}{6}$

The basis is $(1, 2)$
The basic solution $(3, 3/2, 0, 0)$
The objective value is $-9/2$
The basic reduced costs are zero

$$\min f = -x_1 - x_2$$
$$6x_1 + 4x_2 \leq 24$$
$$3x_1 - 2x_2 \leq 6$$
$$x_1, x_2 \geq 0$$



The current vertex is $(3, 3/2)$

Only one nonbasic variable has a negative reduced cost: $c_4 = -1/12$

Variable x_4 enters the basis
The ratio b_i/a_{ij} is positive only for row $i = 2$: variable x_1 leaves the basis;
pivot element a_{24}

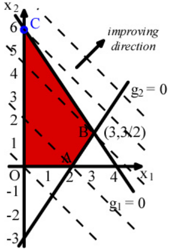
Example (4)

$$\begin{array}{rcl} \min f = -x_1 - x_2 & & \\ 6x_1 + 4x_2 + x_3 & = & 24 \\ 3x_1 - 2x_2 + x_4 & = & 6 \\ x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

$-d$	x_1	x_2	x_3	x_4
6	$\frac{1}{2}$	0	$\frac{1}{4}$	0
6	$\frac{3}{2}$	1	$\frac{1}{4}$	0
18	6	0	$\frac{1}{2}$	1

The basis is $(2, 4)$
The basic solution $(0, 6, 0, 18)$
The objective value is -6
The basic reduced costs are zero

$$\begin{array}{rcl} \min f = -x_1 - x_2 & & \\ 6x_1 + 4x_2 & \leq & 24 \\ 3x_1 - 2x_2 & \leq & 6 \\ x_1, x_2 & \geq & 0 \end{array}$$



The current vertex is $(0, 6)$

All reduced costs are nonnegative: the current solution is optimal