Solved exercises for the course of

Foundations of Operations Research

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The simplex method

Given the *LP* problem

$$\begin{aligned} \max z = 2x_1 + x_2 & \\ 20x_1 \quad\quad & \leq 100 \\ 3x_1 + 2x_2 & \leq 18 \\ x_1 - x_2 & \geq 0 \\ x_1, x_2 & \geq 0 \end{aligned}$$

1. build the *tableau* for the problem and illustrate the passage from the basic canonical form associated to basis (x_3, x_4, x_5) and that associated to basis (x_3, x_1, x_5). Show the associated move on the graphical representation of the problem.

2. Solve the problem with the simplex method, showing its steps on the *tableau*.

Solution

The *tableau* is a table representation of a *LP* problem. Let us assume that the problem, once in standard form, has m constraints and $n + m$ variables, n of which are natural, while the other m are *slack* variables. The *tableau* has $m + 1$ rows, numbered from 0 to m, and $n + m + 1$ columns, numbered from 0 to $n + m$. The rows correspond to the objective function (row 0) and to the constraints (from row 1 to row m), the columns to the right-hand-side vector (column 0) and to the coefficients of the variables (from column 1 to column $n + m$). Let a_{ij} be the generic element of the *tableau*:

- element $(0, 0)$ is the opposite of the constant term in the objective function
- the other elements of row 0 are the costs of the variables, column by column
- the other elements of column 0 are the right-hand-sides of the constraints, row by row
- the other elements are the coefficients of the column variable in the row constraint

The *tableau* associated to the given problem, when put into standard form, is

0	-2	-1	0	0	0
100	20	0	1	0	0
18	3	2	0	1	0
0	-1	1	0	0	1

and it is already in basic canonical form with respect to basis (x_3, x_4, x_5). The value of the basic variables can be read in column 0: $x_3 = b_1 = 100$ (as the unitary coefficient of column 3 is in row 1), $x_4 = b_2 = 18$ and $x_5 = b_3 = 0$, while the nonbasic variables x_1 and x_2 are null. The corresponding basic solution is the origin of plane (x_1, x_2). The first two constraints are nonactive, while the third one is active (satisfied to equality). The basic solution is degenerate (one of the basic variables has zero value, so that some variables can get into or out of the basis without modifying the solution).

It is possible to move from one basis to another through algebraic transformation, premultiplying the system of equalities by B^{-1}, where B is the submatrix formed by the columns which are required to get into the basis. However, this requires that the desired basis be known and it requires to invert a matrix, a polynomial, but not trivial, operation. The *tableau* allows to simplify this operation, dividing it into a sequence of elementary steps. These are called *pivot operations* and allow to move from a basic solution to an *adjacent* basic solution, that is a solution in which a single variable has left the basis and has been replaced by a previously nonbasic variable.

Algebraically, this corresponds to determining the variable which must be added to the basis in terms of the other nonbasic variables and of the variable which must be removed from the basis. Replacing the expression obtained, both in the objective function and in the other constraints, one obtains an equivalent problem, which is the canonical form associated to the new basis. For example, let us move from basis (x_3, x_4, x_5) to basis (x_3, x_1, x_5): x_4 must get out of the basis, while x_1 must get in. So, we determine x_1 from the second constraint, that is the only one in which x_4 occurs:

$$3x_1 + 2x_2 + x_4 = 18 \Leftrightarrow x_1 = -\frac{2}{3}x_2 - \frac{1}{3}x_4 + 6$$

and we replace it in the objective function and in the other constraints.

$$\begin{array}{rcl}
\min z = \frac{1}{3}x_2 \quad + \frac{2}{3}x_4 & & \\
-\frac{4}{3}x_2 + x_3 - \frac{20}{3}x_4 & \leq & -20 \\
x_1 + \frac{2}{3}x_2 + \quad + \frac{1}{3}x_4 & \leq & 6 \\
+\frac{5}{3}x_2 \quad + \frac{1}{3}x_4 + x_5 & \leq & 6 \\
x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}$$

On the *tableau* the operation is very simple:

1. select the *pivot element* $a_{i^*j^*}$ (in this case a_{21}), which is
   - in the column associated to the variable entering the basis (column 1)

- in the row containing the unitary coefficient of the column going out of the basis (x_4 goes out and $a_{24} = 1$, hence row 2)

2. divide the entire *pivot* row by the element itself. This corresponds to determining x_1 as a function of the other variables occurring in the constraint.

3. sum to every other row i of the *tableau* the *pivot* row i^* multiplied by $-a_{ij^*}/a_{i^*j^*}$, so as to cancel element a_{ij^*}. This corresponds to replacing the expression of x_1 in the other rows.

It is possible to let every variable in or out of the basis, provided that the *pivot* element is not zero. In that case, in fact, we would be trying to build a submatrix made of linearly dependent columns, that is not a basis.

In the example:

0	-2	-1	0	0	0
100	20	0	1	0	0
18	③	2	0	1	0
0	-1	1	0	0	1

first becomes

0	-2	-1	0	0	0
100	20	0	1	0	0
6	①	2/3	0	1/3	0
0	-1	1	0	0	1

and then

12	0	1/3	0	2/3	0
-20	0	-40/3	1	-20/3	0
6	1	2/3	0	1/3	0
6	0	5/3	0	1/3	1

The resulting solution has $x_2 = x_4 = 0$ as they are nonbasic, while $x_1 = 6$ (the only unitary coefficient of column 1 is in row 2, whose right-hand-side is $b_2 = 6$), $x_3 = -20$ and $x_5 = 6$. It is a basis solution, but unfeasible: point $(6, 0)$. The violated constraint is the first one, which corresponds to *slack* variable x_3; this is, in fact, negative.

Graphically, the *pivot* operation consists in moving from the intersection of $n - m = 2$ constraints to the intersection of other two constraints, keeping on the separating line associated to the variable x_2, which remains nonbasic, and therefore null.

Simplex algorithm and *tableau*

The *tableau* allows to perform the simplex algorithm in an elementary way: each move corresponds, in fact, to a *pivot* operation. Moreover, the conditions for the selection of the variable which will get into the basis and of the one which will get out of it, as well as the condition for the termination of the algorithm, can be read very clearly on the *tableau* itself.

Let us start from the initial basic canonical form.

0	-2	-1	0	0	0
100	20	0	1	0	0
18	3	2	0	1	0
0	-1	1	0	0	1

The cost coefficient, denoted as *reduced costs*, are represented as $\tilde{c}$, to distinguish them from the original costs. They allow to determine which of the basic solutions adjacent to the current one are better and which are worse. Let us assume that we want to increase variable x_{j^*} by δx_{j^*}, keeping all other nonbasic variables out of the basis (in this case only x_2). Correspondingly, the objective function changes by

$$\delta z = \sum_{j \in N} c_j \delta x_j + \sum_{j \in B} c_j \delta x_j$$

where B and N are, respectively, the sets of the indices of the basic and nonbasic variables. The costs of the basic variables are null (from the definition of basic canonical form: $\tilde{c}_j = 0, \forall j \in B$), while the nonbasic variables different from x_{j^*} keep unchanged by assumption ($\delta x_j = 0, \forall j \in N \setminus \{j^*\}$). Consequently, $\delta z = \tilde{c}_i \, \delta x_{j^*}$. This holds in general: the reduced cost of a nonbasic variable provides the unitary variation of the objective function when the variable enters the basis. In order to improve the objective, only variables with negative reduced cost must enter the basis.

This is a *necessary* condition to improve the solution. It is not sufficient in general, because it is possible that, if the current solution is degenerate, the new variable enter the basis with a zero value, so that $\delta z = \tilde{c} \, \delta x = 0$ even if $\tilde{c} < 0$. This happens in the example if x_2 gets into and x_5 out of the basis.

The choice among the variables of negative reduced cost is arbitrary: all of the bring an advantage, or at least bring no disadvantage. It is usual to select:

- the variable with the minimum reduced cost $\tilde{c}$;
- the variable which, entering the basis, provides the strongest reduction in the objective function.

The first rule is more miopic, but only requires to compare the reduced costs, while the second requires to perform the *pivot* operation for all candidate variables, so as to compare their effects. In the example, both rules suggest that x_1

should enter the basis, but if $c_2 = -3$ the first rule would have suggested x_3 and the second rule would have suggested x_1.

Now let us consider the choice of the variable which must get out of the basis. We already know the constraints which remain active, so that we know the direction towards which the current solution moves in the graphical representation of the problem. It remains to determine where the solution stops, and specifically in which other intersection. It is easy to see that only two intersections along the line (the current one and a single other one) are feasible. So, the first intersection met is the good one. In algebraic terms, this corresponds to increasing the new basic variable (in the example, x_1) keeping all other nonbasic variables to zero and modifying the basic variables so as to keep all constraints in the basic canonical form satisfied. For example, in the second constraint ($3x_1 + 2x_2 + x_4 = 18$), if x_1 increases and x_2 remains zero, x_4 must decrease. Thanks to the basic canonical form, each constraint includes a single different basic variable, so that the effect of the new basic variable on each constraint is exactly compensated by the variation of the corresponding basic variable.

All depends on the coefficients of the new basic column j, row by row:

- If $a_{ij} = 0$, variable j has no effect on constraint i, so that the basic variable does not change (movement parallel to the separating line)

- If $a_{ij} < 0$, variable j decreases the left-hand-side of constraint i, so that the basic variable must increase. There is no limit to the increase, so that these constraints can be ignored (the movement gets farther from the separating line).

- If $a_{ij} > 0$, variable j increases the left-hand-side of constraint i, so that the basic variable must decrease. Since at first its value is b_i and it must remain nonnegative, the maximum increase of x_j is $\max \delta x_j = b_i/a_{ij}$ (the movement gets the solution closer and finally crosses the separating line).

Since all constraints must be respected, the most restrictive condition dominates: x_j assumes the minimum value $\min\limits_{i:a_{ij}>0} b_i/a_{ij}$. This condition in general identifies a single row, and correspondingly a single basic variable which must get out of the basis. In particular cases, it could identify more than one row: then, select one at random; the *pivot* operation will produce a degenerate solution, in which more than one basic variable will assume zero value, even if only one officially gets out of the basis.

Notice that, if all elements of a column of negative reduced cost are ≤ 0, this indicates that the corresponding variable can increase *ad libitum* without producing any unfeasibility, and therefore the problem is unbounded.

Let us apply the method to the given problem.

0	-2	-1	0	0	0
100	(20)	0	1	0	0
18	3	2	0	1	0
0	-1	1	0	0	1

The minimum reduced cost column is the first one. Element a_{31} is negative, so it must be ignored. For the other two elements, we compute the ratio b_i/a_{i1}: the first one is smaller ($100/20 < 18/3$). Hence, the *pivot* element is a_{11}.

10	0	-1	1/10	0	0
5	1	0	1/20	0	0
3	0	(2)	-3/20	1	0
5	0	1	1/20	0	1

Now, variable x_2 has a negative reduced cost. The *pivot* element is a_{22}, because the first coefficient is zero and the last one has a larger ratio b_i/a_{i2} ($5/1 < 3/2$).

23/2	0	0	1/40	1/2	0
5	1	0	1/20	0	0
3/2	0	1	-3/40	1/2	0
7/2	0	0	5/40	-1/2	1

from which the basic solution is $x_1 = 5$, $x_2 = 3/2$, $x_3 = x_4 = 0$, $x_5 = 7/2$, since the basis is (x_1, x_2, x_5). This solution is feasible (all right-hand-sides are nonnegative) and optimal (all reduced costs are nonnegative). This is confirmed by the analysis of the graph. Moreover, the value of the objective function is $f^* = -a_{00} = -23/2$.