Foundations of Operations Research

Master of Science in Computer Engineering

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Fundamental theorem of Linear Programming

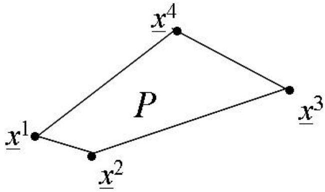
Let $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \neq \emptyset$ be a nonempty polyhedron.
The *LP* problem

$$\min_{x \in P} c^T x$$

either is unbounded or has at least one optimal vertex solution

Any *LP* problem can be solved considering only the vertices of P

- their number is finite
- their number can be exponential with respect to the number of variables and constraints



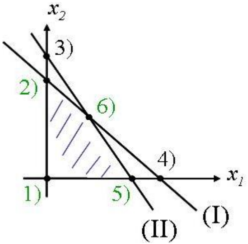
How can the vertices be found algorithmically?

Vertices and inequalities

Consider a LP problem with only $\leq$ inequalities (always possible)

$$P = \left\{ x \in \mathbb{R}^{n'} : Ax \leq b, x \geq 0 \right\}$$

$$\begin{aligned} \min f &= -x_1 - 3\,x_2 \\ x_1 + x_2 &\leq 6 \quad (I) \\ 2\,x_1 + x_2 &\leq 8 \quad (II) \\ x_1, x_2 &\geq 0 \end{aligned}$$



- All vertices are feasible intersections of $n' = 2$ separating hyperplanes
  - $p_1 = (0, 0)$ is the intersection of $x_1 = 0$ and $x_2 = 0$
  - $p_2 = (0, 6)$ is the intersection of $x_1 = 0$ and $x_1 + x_2 = 6$
  - $p_5 = (4, 0)$ is the intersection of $2x_1 + x_2 = 8$ and $x_2 = 0$
  - $p_6 = (2, 4)$ is the intersection of $x_1 + x_2 = 6$ and $2x_1 + x_2 = 8$
- The unfeasible intersections are not vertices
  - $p_3 = (0, 8)$ is the (unfeasible) intersection of $x_1 = 0$ and $2x_1 + x_2 = 8$
  - $p_4 = (0, 6)$ is the (unfeasible) intersection of $x_1 + x_2 = 6$ and $x_2 = 0$

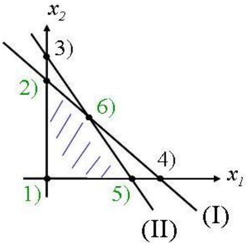
Vertices and equalities

In order to obtain the standard form, simply add m slack variables

$$P = \{(x, s) \in \mathbb{R}^n : Ax + s = b, x \geq 0, s \geq 0\}$$

Notice that now $n = n' + m = 4$!

$$\begin{array}{rcll} \min f & = & -x_1 - 3\,x_2 & \\ x_1 + x_2 + s_1 \quad & = & 6 & (I) \\ 2\,x_1 + x_2 \quad + s_2 & = & 8 & (II) \\ x_1, x_2, s_1, s_2 & \geq & 0 & \end{array}$$



- All vertices are feasible solutions with $n - m = 2$ variables set to zero
  - $p_1 = (0, 0)$ has $x_1 = 0$ and $x_2 = 0$ (feasible: $s_1 = 6$ and $s_2 = 8$)
  - $p_2 = (0, 6)$ has $x_1 = 0$ and $s_1 = 0$ (feasible: $x_2 = 6$ and $s_2 = 2$)
  - $p_5 = (4, 0)$ has $s_2 = 0$ and $x_2 = 0$ (feasible: $x_1 = 4$ and $s_1 = 2$)
  - $p_6 = (2, 4)$ has $s_1 = 0$ and $s_2 = 0$ (feasible: $x_1 = 2$ and $x_2 = 4$)
- Only the feasible solutions of this type are vertices
  - $p_3 = (0, 8)$ has $x_1 = 0$ and $s_2 = 0$ (unfeasible: $s_1 = -2$)
  - $p_4 = (0, 6)$ has $s_1 = 0$ and $x_2 = 0$ (unfeasible: $s_2 = -2$)

Vertices and equalities

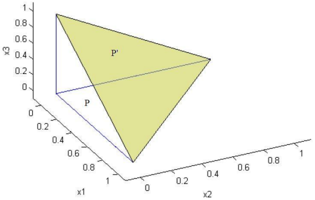
Given the feasible sets of an LP problem and of its standard form

$$P = \left\{x \in \mathbb{R}^{n'} : Ax \leq b, x \geq 0\right\} \qquad P' = \{(x, s) \in \mathbb{R}^{n} : Ax + s = b, x \geq 0, s \geq 0\}$$

- a facet is obtained setting one variable to 0 in the standard form
- a vertex is obtained setting $n - m$ variables to 0 in the standard form

$$P = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1, \; x_1, x_2 \geq 0\}$$

$$P' = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1, \; x_1, x_2, x_3 \geq 0\}$$



- Three facets (edges) are obtained setting $x_i = 0$ $(i = 1, \ldots, 3)$
- Three vertices are obtained setting $n - m = 3 - 1 = 2$ variables to 0

$$V(P') = \{(0,0,1)\ (0,1,0)\ (1,0,0)\} \rightarrow V(P) = \{(0,0)\ (0,1)\ (1,0)\}$$

Full rank assumption

Given a *LP* problem in standard form, defined on a polyedron

$$P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

if P is nonempty, it is always possible to assume that $\mathrm{rank}(A) = m \leq n$

Proof: *By definition,* $\mathrm{rank}(A) \leq \min(m, n)$. *Assume that* $\mathrm{rank}(A) < m$. *Then, at least one row* a_i^T *is a linear combination of the other ones:*

- *if the right-hand-side* b_i *is the same linear combination of the other right-hand-sides, the constraint is redundant*
- *if it is not, the problem has no feasible solution*

$$\begin{array}{rcll} x_1 + x_2 & = & 1 & (I) \\ x_1 \quad + x_3 & = & 1 & (II) \\ 2x_1 + x_2 + x_3 & = & 2 & (III) \\ x_1, x_2, x_3 & \geq & 0 & \end{array}$$

Redundant
($a_3 = a_1 + a_2$ and $b_3 = b_1 + b_2$)

$$\begin{array}{rcll} x_1 + x_2 & = & 1 & (I) \\ x_1 \quad + x_3 & = & 1 & (II) \\ 2x_1 + x_2 + x_3 & = & 3 & (III) \\ x_1, x_2, x_3 & \geq & 0 & \end{array}$$

Unfeasible
($a_3 = a_1 + a_2$, but $b_3 \neq b_1 + b_2$)

Bases and basic solutions

Under the full rank assumption

- the set of all m rows of A is linearly independent
- at least one subset of m columns of A is linearly independent

A basis is any subset of m linearly independent columns of A

Permuting the columns (*rearranging the variables*) so that the basic columns are the first ones, A is partitioned into two submatrices

1. the basic matrix B
2. the nonbasic matrix N

$$A = [\underbrace{B}_{m} \,\vdots\, \underbrace{N}_{n\text{-}m}]$$

The cost vector c and the variable vector x are permuted and partitioned

- $c^T = [c_B^T \mid c_N^T]$ (basic and nonbasic cost coefficients)
- $x^T = [x_B^T \mid x_N^T]$ (basic and nonbasic variables)

Basic solutions

Under the full rank assumption, hence, the system can be rewritten as

$$\begin{aligned} \min f = c_B^T x_B + c_N^T x_N \quad &+ \quad d \\ B\, x_B + N\, x_N \quad &= \quad b \\ x_B, x_N \quad &\geq \quad 0 \end{aligned}$$

If the $n - m$ nonbasic variables are fixed arbitrarily,
the m basic variables are uniquely determined

$$B\, x_B + N\, x_N = b \Rightarrow x_B = B^{-1}\, b - B^{-1}\, N\, x_N$$

and the value of the objective function is

$$f(x) = c_B\, x_B + c_N\, x_N + d = \left(c_N - B^{-1}\, N \right) x_N + c_B\, B^{-1}\, b + d$$

A basic solution is a solution obtained setting to 0 the nonbasic variables

$$\begin{cases} B\, x_B + N\, x_N = b \\ x_N = 0 \end{cases} \Rightarrow x_B = B^{-1} b \Rightarrow x = \begin{bmatrix} B^{-1} b \\ 0 \end{bmatrix} \text{ and } f(x) = c_B\, B^{-1}\, b + d$$

The basic solution is feasible if $x_B = B^{-1}\, b \geq 0$, unfeasible otherwise

Example

$$c^T = \begin{bmatrix} 2 & 1 & 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
\min f &= 2x_1 + x_2 + 5x_3 \\
x_1 + x_2 + x_3 + x_4 &= 4 \\
x_1 \qquad + x_5 &= 2 \\
+ x_3 \qquad + x_6 &= 3 \\
3x_2 + x_3 \qquad + x_7 &= 6 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0
\end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

Columns 4, 5, 6 and 7 form a basis, with

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } c_B^T = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding basic solution is $(0, 0, 0, 4, 2, 3, 6)$ and is feasible

Its cost is $f(x) = c_B B^{-1} b + d = 0$

Example

$$c^T = \begin{bmatrix} 2 & 1 & 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{rcl} \min f = 2x_1 + x_2 + 5x_3 & & \\ x_1 + x_2 + x_3 + x_4 & = & 4 \\ x_1 \qquad\qquad + x_5 & = & 2 \\ + x_3 \qquad + x_6 & = & 3 \\ 3x_2 + x_3 \qquad\qquad + x_7 & = & 6 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 & \geq & 0 \end{array}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

Columns 2, 5, 6 and 7 form a basis, with

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}$$

$$c_B^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow c_B B^{-1} b = 4$$

The corresponding basic solution is $(0, 4, 0, 0, 2, 3, -6)$ and is unfeasible

Its cost is $f(x) = c_B B^{-1} b + d = 4$ (*useless, since the solution is unfeasible*)

Counting the basic solutions

The number of subset of $n - m$ columns out of n is

$$C_m^n = \binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

This is an overestimate of the number of vertices because

- not all subsets of columns correspond to a basis
  (*some subsets could be linearly dependent*)
- different bases can correspond to the same solution
  (*when some basic variable is zero, the solution obtained exchanging those basic variables with nonbasic ones is the same*)
- not all basic solutions are feasible
  (*some solutions have negative basic components*)

number of vertices $=$ number of feasible basic solutions $\leq$ number of basic solutions $\leq$

$\leq$ number of bases $\leq$ $C_m^n = \dfrac{n!}{m!\,(n-m)!}$

An algorithmic idea

A *LP* problem, therefore, could be solved

1. introducing the full rank assumption
   (if not possible, the problem is unfeasible)
2. enumerating the subsets of $n - m$ columns of A
3. for each subset B
   1. if it is linearly independent, set $x_N := 0$
   2. compute $x_B = B^{-1} b$
   3. if $x_B \geq 0$, compute $f(x) = c_B B^{-1} b + d$ and save its minimum

The complexity of such an algorithm is clearly exponential as it is proportional to C_m^n

The simplex method (Dantzig, 1947) explores the vertices in a smart way

- in the average case, it explores a very limited subset
- but in the worst case, it explores all vertices

Luckily, *the worst case is very rare*

Polynomial algorithms have been introduced from 1979
Presently, in the average case they are slower than the simplex method

A sample problem

Try and find the optimal solution of this problem

$$\begin{aligned} \min f = 5x_1 + 4x_2 + 3x_3 & & \\ 2x_1 + 3x_2 + \ x_3 &\leq& 5 \\ 4x_1 + \ x_2 + 2x_3 &\leq& 11 \\ 3x_1 + 4x_2 + 2x_3 &\leq& 8 \\ x_1, x_2, x_3 &\geq& 0 \end{aligned}$$

There are $C_3^6 = 20$ subsets of $n - m = 3$ columns

Each one corresponds to a candidate solution

But the optimal solution is absolutely obvious...