Foundations of Operations Research

Master of Science in Computer Engineering

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http://homes.di.unimi.it/~cordone/courses/2014-for/2014-for.html

Geometry of Linear Programming

A hyperplane is the set of points satisfying an affine equality

$$\left\{x \in \mathbb{R}^n : a^T \cdot x = b\right\}$$

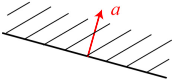
An affine half-space is the set of points satisfying an affine inequality

$$\left\{x \in \mathbb{R}^n : a^T \cdot x \leq b\right\}$$

Vector a is orthogonal to the separating hyperplane and points outwards

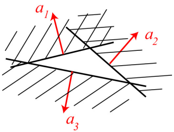
When $n = 2$

- a hyperplane reduces to a line
- an affine half-space reduces to an affine half-plane;



A polyhedron is the intersection of a finite number m of affine half-spaces

$$\left\{x \in \mathbb{R}^n : a_i^T \cdot x \leq b \quad i = 1, \ldots, m\right\}$$



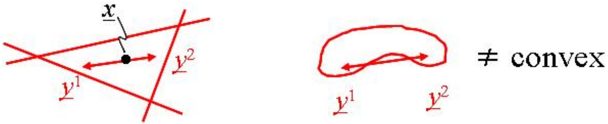
The feasible set of a LP problem is a polyhedron: it can be empty or unbounded

Convexity

A convex combination of r vectors $y_1, \ldots, y_r$ is a linear combination of the vectors with nonnegative coefficients summing up to 1

$$\sum_{i=1}^{r} \alpha_i y_i \text{ with } \sum_{i=1}^{r} \alpha_i = 1 \text{ and } \alpha_i \geq 0 \text{ for } i = 1, \ldots, r$$

A convex set is a set containing every convex combination of its elements



- The affine half-spaces are convex sets (*easy to prove*)
- Intersections of convex sets are convex sets (*easy to prove*)

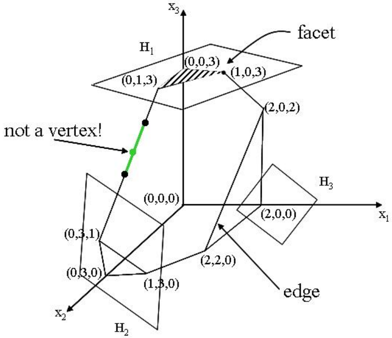
Therefore, polyhedra are convex sets

Vertices

A vertex of a polyhedron P is any point $x \in P$ which cannot be expressed as a convex combination of two different points of P

$$\nexists y, z \in P \text{ and } \lambda \in (0;1) : x = \lambda y + (1 - \lambda) z$$

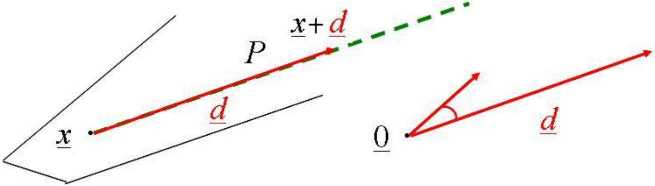
A polyhedron has a finite number of vertices



Rays

A ray of a polyhedron P is any vector $d \in \mathbb{R} \setminus \{0\}$ such that

$$(x + \lambda d) \in P \text{ for all } x \in P, \lambda \geq 0$$



A polyhedron admits rays if and only if it is unbounded

A polytope is a bounded polyhedron (therefore, with no rays)

Gradient and level curves

Given a function $f(x)$

- the gradient vector ∇f is the vector whose components are the partial derivatives of f with respect to the variables x_i
- the level curve of value z is the set of points where $f(x) = z$

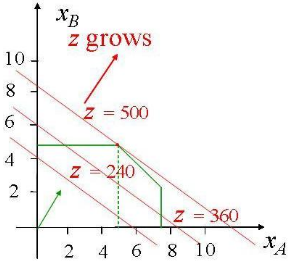
The gradient vector points in the direction of fastest growth of f and is orthogonal to all level curves

For affine functions $f(x) = c^T x$

- the gradient is the objective coefficient vector ($\nabla f(x) = c$)
- the level curves are parallel lines, orthogonal to c

$$\max f(x) = c^T x = 0.04\, x_A + 0.06\, x_B$$

$$\nabla f(x) = c = \begin{bmatrix} 0.04 \\ 0.06 \end{bmatrix}$$



Example: capital investment

You have a capital of 10 000 *Euros to invest.*

Two investments are available, denoted as A and B: the former has an expected return of 4%, *the latter of* 6%

In order to diversify the investment, reducing risk, at most 75% *of the capital can be invested in A, and at most* 50% *in B.*

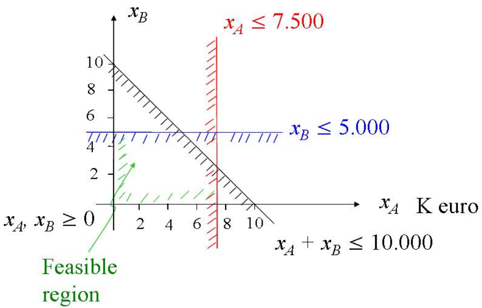
Determine a portfolio that maximizes the expected return, while respecting the diversification constraints.

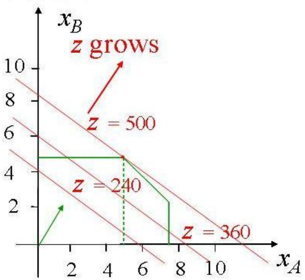
The natural decision variables are

- x_A = capital invested in A, measured in euros
- x_B = capital invested in B, measured in euros

$$\begin{aligned} \max f = 0.04\,x_A + 0.06\,x_B \\ x_A + x_B \leq 10\,000 \\ x_A \leq 7\,500 \\ x_B \leq 5\,000 \\ x_A, x_B \geq 0 \end{aligned}$$

$$\begin{aligned} \max f &= 0.04x_A + 0.06x_B \\ x_A + x_B &\leq 10\,000 \\ x_A &\leq 7\,500 \\ x_B &\leq 5\,000 \\ x_A, x_B &\geq 0 \end{aligned}$$





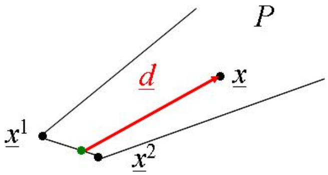
Hyperplane and vertex representation

A polyhedron is defined as an intersection of half-spaces (inequalities), but it admits an equivalent representation based on vertices and rays

Minkowski-Weyl theorem (simplified):
Every point x of a polyhedron P is a convex combination of the vertices $y_1, \ldots, y_r$ of P plus a ray d of P (if rays exist)

$$x = \sum_{i=1}^{r} \alpha_i y_i + d \text{ for all } x \in P \quad \text{with } \sum_{i=1}^{r} \alpha_i = 1 \text{ and } \alpha_i \geq 0 \ (i = 1, \ldots, r)$$

(*of course, α_i and d depend on x*)



If P is a polytope, x is a convex combination of $y_1, \ldots, y_r$

Fundamental theorem of Linear Programming

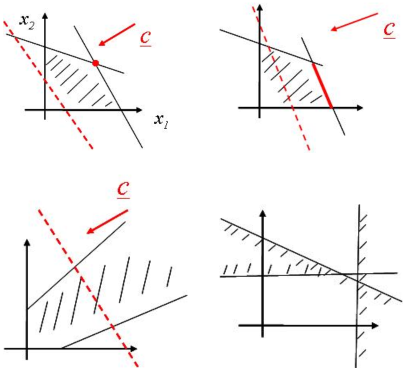
Let $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \neq \emptyset$ be a nonempty polyhedron.
The *LP* problem

$$\min_{x \in P} c^T x$$

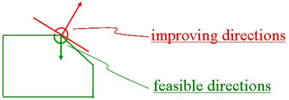
either is unbounded or has at least one optimal vertex solution

This implies four alternative cases:

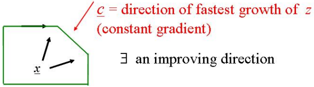
1. unique optimal vertex solution
2. infinite set of optimal solutions, convex combinations of a finite number of optimal vertex solutions
3. unbounded objective value and unbounded polyhedron
4. unfeasible problem and empty polyhedron



1. in an optimal solution all improving directions ("opposite" to c) are unfeasible (lead to unfeasible solution with any step), and vice versa



2. an interior point cannot be an optimal solution



Any *LP* problem could be solved

- enumerating its vertices
- computing and comparing the values of the objective in the vertices

The number of solutions is infinite (and continuous), but the number of relevant solutions is finite, as in Combinatorial Optimization

Linear Programming and Combinatorial Optimization are strictly related: each vertex is the intersection of some separating hyperplanes;
LP problems can be seen as the search for a subset of hyperplanes

Proof of the fundamental theorem

Let $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \neq \emptyset$ be a nonempty polyhedron.
The LP problem

$$\min_{x \in P} c^T x$$

either is unbounded or has at least one optimal vertex solution

Consider the following two cases:

1. *P is nonempty and has a ray d such that $c^T d < 0$*

$$x \in P \Rightarrow (x + \lambda d) \in P \quad \forall \lambda > 0$$

   Now, $f(x + \lambda d) = f(x) + \lambda c^T d < f(x)$.
   Increasing λ, the value of $f(x + \lambda d)$ decreases without bounds

2. *all rays d of P satisfy $c^T d \geq 0$, or P has no ray*

$$x = \sum_{i=1}^{r} \alpha_i y_i + d \qquad \text{such that}$$

- $\alpha_i \geq 0$
- $\sum_{i=1}^{r} \alpha_i = 1$
- $(x + \lambda d) \in P$ and $c^T d \geq 0$

Proof of the fundamental theorem

For a generic solution $x \in P$, there exist d and α_i $(i = 1, \ldots, r)$ such that

- $x = \sum_{i=1}^{r} \alpha_i y_i + d$
- $\alpha_i \geq 0$
- $\sum_{i=1}^{r} \alpha_i = 1$
- $(x + \lambda d) \in P$ and $c^T d \geq 0$

Now consider the value $f(x)$ of the generic solution x

$$f(x) = c^T x = c^T \left(\sum_{i=1}^{r} \alpha_i y_i + d \right) = \sum_{i=1}^{r} \alpha_i c^T y_i + c^T d \geq \sum_{i=1}^{r} \alpha_i c^T y_i$$

Let y_{i^*} be the best vertex: $y_{i^*} = \arg \min_{i=1,\ldots,r} c^T y_i$

$$c^T y_i \geq c^T y_{i^*} \text{ for all } i = 1, \ldots, r$$

which implies that $f(x) \geq \sum_{i=1}^{r} \alpha_i c^T y_{i^*} = c^T y_{i^*} = f(y_{i^*})$

Since y_{i^*} is better than x for all $x \in P$: y_{i^*} is an optimal solution