Solved exercises for the course of

Foundations of Operations Research

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Graphical solution

A farm restaurant is selling to its customers food produced internally and food purchased from neighbour producers. The manager of the farm wants to organize production so as to maximize the resulting profit. The main decision is to determine how much internal and how much external product to sell. The unitary profit is known for both cases, and it is double for the internal product with respect to the external one. The consumption of terrain is zero for the external products and equal to 20 hectares for each weight unit of internal products. The labour required is 3 years-man for each weight unit of internal products and 2 years-man for the external products. The farm owns 100 hectares of land and 18 full-time workers. The law imposes to farm restaurants that at least half of the products sold to the customers should be produced internally.

1. Write a mathematical programming model for this problem.
2. Solve the problem in graphical form.
3. Determine the basic solutions on the graph, indicating their number, distinguishing the natural variables from the possible auxiliary variables (*slack* and *surplus*) and indicating to what they correspond from the graphical point of view, in reference to the fundamental theorem of Linear Programming.

Model

In order to build a mathematical programming model, one must clearly identify its three fundamental elements:

1. the *decision variables*: in this case, the natural variables are the total weight of internal production and the total weight purchased from neighbour producers, both expressed in units of weight per year;
2. the *objective function*: in this case, profit, expressed in Euros/year;
3. the *constraints*: in in this case, the maximum available land, the availability of workforce, the law regulations on production and the nonnegativity of production and purchase levels.

Then, one must formally describe the relation of the objective and the constraints with the decision variables. As for the objective function, the total profit is the sum of the profits deriving from internal and external products. Both are given by the product of the corresponding unitary profit times the production or purchase level. We do not know the unitary profit, but we know that, denoting as P the unitary profit of purchased products, that of internal products is $2P$. Therefore:

$$\max f = 2P\,x_1 + P\,x_2$$

Then, we state that the land consumed by the internal products (the other ones consume no land) does not exceed the available land:

$$g_1(x) = 20\,x_1 \leq 100$$

That the workforce employed by both kinds of products does not exceed the available one:

$$g_2(x) = 3\,x_1 + 2\,x_2 \leq 18$$

That the internal products are at least half of the total, i. e. that the internal products exceed the purchased ones.

$$x_1 \geq \frac{1}{2}\,(x_1 + x_2) \Leftrightarrow x_1 \geq x_2$$

The resulting model is:

$$\begin{aligned} \max f = 2x_1 + x_2 & \\ 20x_1 & \leq 100 \\ 3x_1 + 2x_2 & \leq 18 \\ x_1 - x_2 & \geq 0 \\ x_1, x_2 & \geq 0 \end{aligned}$$

where factor P has been removed because maximizing $(2x_1 + x_2)$ is equivalent to maximizing $(2Px_1 + Px_2)$.

Graphical solution

Since the problem has only two decision variables, it is possible to solve it graphically. This requires to represent in (x_1, x_2) the feasible region and the level lines (or the direction in which the objective function improves, which is the gradient ∇f, given that f should be maximized). In a linear problem, each constraint $g_i\,(x) \leq 0$ corresponds to an affine half-space, identified by a *separating hyperplane* $g_i\,(x) = 0$. The feasible region is therefore the intersection of a finite number of affine half-spaces, that is a *polyedron* (in particular, a *polytope* if bounded and nonempty).

On a two-dimensional space, the half-spaces reduce to half-planes, the separating hyperplanes to lines and the feasible polyhedron to a polygon. To draw it, it is sufficient to:

1. draw the separating line which corresponds to each constraint

2. identify the feasible half-plane between the two thus sketched

The first step is simple. For a linear constraint written as:

$$ax_1 + bx_2 \leq c \quad \text{or} \quad ax_1 + bx_2 \geq c$$

the separating hyperplane is

$$ax_1 + bx_2 = c$$

If $a = 0$ or $b = 0$, the line is parallel to one of the axes and it is trivial to draw it. As well, it is trivial to draw it if $c = 0$, when the line passes through the origin. In general, the line can be derived from its intercepts, i. e. its intersections with the axes: $(c/a, 0)$ and $(0, c/b)$. The second step is trivial: given any point $P = \left(x_1^P, x_2^P\right)$ out of the separating line, evaluate $g_i(P) = ax_1^P + bx_2^P$. If $g_i(P)$ satisfies the constraint, the feasible half-plane is the one including P; otherwise, it is the other one. The origin is the simplest point for this test (unless $c = 0$, of course). In fact, $g_i(O) = c$.

The direction of fastest improvement of the objective is $[2\ 1]^T$: it is growing for both variables, and x_1 weighs double as much as x_2. Therefore, the direction has an x_1 component double than the x_2 component.

Concluding, Figure 1 shows the graphical representation of the problem, from which it is clear that the optimal solution is point $C = (5, 3/2)$, corresponding to solution $x_1 = 5$ and $x_2 = 3/2$.

Discussion

In the graphical representation, each constraint of the problem corresponds to an affine half-plane. The set of points in which a constraint is *active* (i. e. satisfied with equality) is a separating line. Consequently, the feasible region, being an intersection of half-planes, is a polygon.

The points of each separating hyperplane are the points in which the *slack* (or *surplus*) variable associated to the constraint is zero. The axes x_1 and x_2 can be interpreted as separating lines for the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. In the perspective, the natural variables x_1 and x_2 are *surplus* variables for the nonnegativity constraints, and the distinction between natural and auxiliary variables is more apparent than real: each variable is associated to a constraint, each constraint to a variable.

The vertices of the feasible set are the points in which the maximum possible number of separating line intersect. In general, this number is equal to the dimension of the problem (in the plane, two). Equivalently, in those point the maximum possible number of variables (natural, *slack* or *surplus*) have zero value. They are the *basic solutions*, that is the solutions obtained settinng to zero as many variables as the dimension of the problem.

The fundamental theorem of Linear Programming states that:

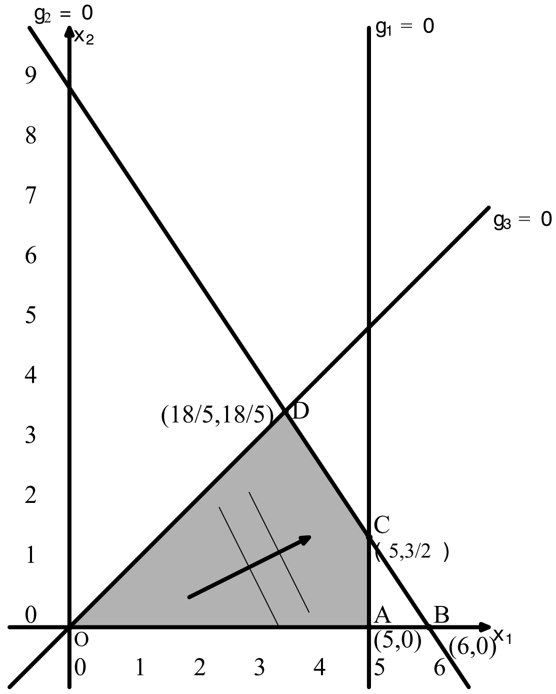

Figura 1: Risoluzione grafica del modello dell'agriturismo

1. if a Linear Programming problem is feasible, then it has basic feasible solutions: in the graphical representation, this means that if there is a feasible polygon, then at least one of its vertices is feasible;

2. if a Linear Programming problem has optimal solutions, then it has optimal basic solutions: in the graphical representation, this means that at least one of the feasible points on the best level line is a vertex; it is intuitive that such a point is either the only feasible point on the line or the feasible points form a whole segment on that line, whose extreme points are two vertices; in a space with more than two dimensions, the feasible points could form a whole face, identified by more than two vertices.

Let us put the problem into standard form:

$$
\begin{array}{lrcl}
\min f = -2x_1 - x_2 & & & \\
& 20x_1 \qquad + x_3 & \leq & 100 \\
& 3x_1 + 2x_2 \qquad + x_4 & \leq & 18 \\
& -x_1 + x_2 \qquad\qquad + x_5 & \leq & 0 \\
& x_1, x_2 & \geq & 0
\end{array}
$$

This problem has $\binom{n+m}{n} = \binom{2+3}{2} = 10$ bases. Let us identify the corresponding basic solutions on the graphical representation. The four feasible basic solutions are obvious: $O = (0,0)$, $A = (5,0)$, $C = (5,3/2)$ and $D = (18/5, 18/5)$. Three unfeasible basic solutions can be found easily: $B = (6,0)$, $(5,5)$ and $(0,9)$. Where are the three missing solutions?

Rigorously speaking they do not exist. In order to find a basic solution, one must remember that they are intersections of $n = 2$ separating lines. Apart from the already identified solutions, three pairs of constraints (and corresponding variables) could correspond to bases. Two of them provide solution $(0,0)$, which has already been identified and is, in fact, degenerate, since three separating lines intersect in it, that is to say three pairs of separating lines.

The last "missing solution" should be the intersection of equalities $x_1 = 0$ and $x_1 = 5$, that is it should set to zero variables x_1 and x_3, corresponding to basis (x_2, x_4, x_5). However, columns 2, 4 and 5 in the coefficient matrix are linearly dependent, so that they do not provide a basis. Correspondingly, the two separating lines are parallel, and do not intersect.

In a way, one could think that the origin correspond to three coincident basis solutions and that the last missing solution reside at infinity. More rigorously, $\binom{n+m}{n}$ is an upper bound on the number of basis, correct only when all n-tuples of columns are linearly independent, and that the number of basis is an upper bound on the number of basic solutions, correct only when no basis is degenerate.