Foundations of Operations Research

Master of Science in Computer Engineering

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A diet model

In 1945, George J. Stigler studied the problem of determining the minimum cost diet for the soldiers of the United States Army.

Given a set of 77 *available foods, he wanted to compute the yearly amount of each food which would have provided the adequate quantity of* 9 *basic nutrients (calories, proteins, minerals and vitamins).*

Among the feasible solutions, he wanted to compute one with the minimum total cost.

At the time, no algorithm existed for this problem.

He devised a heuristic method, which required 10 *individuals working for* 12 *days on hand-operated desk calculators.*

They found a solution costing 39.93 *Dollars/year (the optimum is* 39.69*).*

The general diet problem

In all diet problems, we have:

- a set of foods F
- a set of nutrients N
- a unitary cost c_j for each food $j \in F$
- a required amount r_i for each nutrient $i \in N$
- a content a_{ij} of each nutrient $i \in N$ in each food $j \in F$

The natural decision variables are the amounts x_j for each food $j \in F$

The general model for the diet problem is therefore:

$$\min f = \sum_{j \in F} c_j \, x_j$$

$$\sum_{j \in F} a_{ij} \, x_j \geq r_i \qquad i \in N$$

$$x_j \geq 0 \qquad j \in F$$

A transportation model

A freight transporation company must carry the production of a set M of factories to a set N of customers.

The productive capacity p_i of each factory $i \in M$ and the demand d_j of each customer $j \in N$ are known.

A maximum amount q_{ij} can be transported from each factory $i \in M$ to each customer $j \in N$ at a unitary cost equal to c_{ij} Euros/unit.

We adopt as decision variables x_{ij} the number of product units transported from factory $i \in M$ to customer $j \in N$

The general model for the transportation problem is:

$$\min f = \sum_{i \in M} \sum_{j \in N} c_{ij}\, x_{ij}$$

$$\sum_{j \in N} x_{ij} \leq p_i \qquad i \in M$$

$$\sum_{i \in M} x_{ij} \geq d_j \qquad j \in N$$

$$0 \leq x_j \leq q_{ij} \qquad i \in M, j \in N$$

Linear Programming: fundamental assumptions

All these models fall into the framework of Linear Programming (*LP*)

They can be immediately recognized from the following assumptions:

1. divisibility: the decision variables can assume any fractional value
2. linearity: the objective and the constraint functions satisfy
   - proportionality: each decision variable x_j gives a contribution proportional to its value: $c_j x_j$
   - additivity: the contributions of the decision variables are summed

$$\sum_{j \in N} c_j x_j$$

   except for a suitable constant term (all functions involved are affine)

$$\sum_{j \in N} c_j x_j + d$$

Algebraic and matrix form

Since

- the objective function $f(x)$ is affine: $f(x) = \sum_{j=1}^{n} c_j\, x_j + d$
- the constraint functions $g_i(x)$ are affine: $g_i(x) = \sum_{j=1}^{n} a_{ij}\, x_j + b_i$

the general mathematical programming form reduces to:

$$\min f = \sum_{j=1}^{n} c_j\, x_j + d$$

$$\sum_{j=1}^{n} a_{ij}\, x_j + b_i \leq 0 \qquad\qquad i = 1, \ldots, m$$

or to the compact matrix form with a decision variable vector $x \in \mathbb{R}^n$

$$\min f = c^T x + d$$

$$A\, x + b \leq 0$$

where $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$

Standard form

However, LP problems are commonly represented in the standard form:

$$
\begin{aligned}
\min f = \sum_{j \in N} c_j \, x_j & \\
\sum_{j \in N} a_{ij} \, x_j = b_i & \qquad i \in N \\
x_j \geq 0 & \qquad j \in N
\end{aligned}
$$

where

- the objective function f is minimized
- all constraints are equalities
- all decision variables are nonnegative

What if the problem has not this form?

Reduction into standard form

Any LP problem can be reduced to an equivalent problem in standard form applying the following transformations

1. maximization is turned into minimization reverting the objective's sign

$$\max f(x) = c^T x \quad \Leftrightarrow \quad \min \hat{f}(x) = \hat{c}^T x \text{ with } \hat{c} = -c$$

2. inequalities are turned into equalities introducing nonnegative variables
   - a slack variable s_i for each $\leq$ inequality

$$a_i^T x \leq b_i \quad \Leftrightarrow \quad \begin{cases} a_i^T x + s_i = b_i \\ s_i \geq 0 \end{cases}$$

   - a surplus variable s_i for each $\geq$ inequality

$$a_i^T x \geq b_i \quad \Leftrightarrow \quad \begin{cases} a_i^T x - s_i = b_i \\ s_i \geq 0 \end{cases}$$

3. unrestricted variables are replaced by nonnegative ones
   (each unrestricted variable is the difference of two nonnegative variables)

$$x_j \in \mathbb{R} \quad \Leftrightarrow \quad x_j \to x_j^+ - x_j^- \text{ with } x_j^+ \geq 0 \text{ and } x_j^- \geq 0$$

   (*alternatively, use an equality constraint to determine the expression of* x_j *in terms of the other variables and replace* x_j *with that expression*)

Example

Reduce the following *LP* problem into standard form:

$$\begin{aligned} \max f &= 2x_1 - 3x_2 \\ 4x_1 - 7x_2 &\leq 5 \\ 6x_1 - 2x_2 &\geq 4 \\ x_1 &\geq 0 \\ x_2 &\in \mathbb{R} \end{aligned}$$

1) Turn maximization into minimization: $\hat{f} = -f = -2x_1 + 3x_2$

$$\begin{aligned} \min \hat{f} &= {\color{red}-2x_1 + 3x_2} \\ 4x_1 - 7x_2 &\leq 5 \\ 6x_1 - 2x_2 &\geq 4 \\ x_1 &\geq 0 \\ x_2 &\in \mathbb{R} \end{aligned}$$

Example

$$\begin{aligned} \min \hat{f} &= -2x_1 + 3x_2 \\ 4x_1 - 7x_2 &\leq 5 \\ 6x_1 - 2x_2 &\geq 4 \\ x_1 &\geq 0 \\ x_2 &\in \mathbb{R} \end{aligned}$$

2) Turn the inequalities into equalities introducing a new slack variable s_1 (x_3) and a new surplus variable s_2 (x_4)

$$\begin{aligned} \min \hat{f} &= -2x_1 + 3x_2 \\ 4x_1 - 7x_2 + s_1 &= 5 \\ 6x_1 - 2x_2 - s_2 &= 4 \\ x_1 &\geq 0 \\ x_2 &\in \mathbb{R} \\ s_1 &\geq 0 \\ s_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} \min \hat{f} &= -2x_1 + 3x_2 \\ 4x_1 - 7x_2 + x_3 &= 5 \\ 6x_1 - 2x_2 - x_4 &= 4 \\ x_1, x_3, x_4 &\geq 0 \\ x_2 &\in \mathbb{R} \end{aligned}$$

Example

$$\begin{aligned} \min \hat{f} &= -2x_1 + 3x_2 \\ 4x_1 - 7x_2 + x_3 &= 5 \\ 6x_1 - 2x_2 + x_4 &= 4 \\ x_1, x_3, x_4 &\geq 0 \\ x_2 &\in \mathbb{R} \end{aligned}$$

3a) Turn the unrestricted variable into a difference of nonnegative variables: $x_2 = x_2^+ - x_2^- = x_5 - x_6$ ($x_5 = x_2^+$ and $x_6 = x_2^-$)

$$\begin{aligned} \min \hat{f} &= -2x_1 + 3\left(x_2^+ - x_2^-\right) \\ 4x_1 - 7\left(x_2^+ - x_2^-\right) + x_3 &= 5 \\ 6x_1 - 2\left(x_2^+ - x_2^-\right) - x_4 &= 4 \\ x_1, x_3, x_4 &\geq 0 \\ x_2^+, x_2^- &\geq 0 \end{aligned}$$

$$\begin{aligned} \min \hat{f} &= -2x_1 + 3x_5 - 3x_6 \\ 4x_1 + x_3 - 7x_5 + 7x_6 &= 5 \\ 6x_1 - x_4 - 2x_5 + 2x_6 &= 4 \\ x_1, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Example

$$\begin{aligned} \min \hat{f} = -2x_1 + 3x_2 & \\ 4x_1 - 7x_2 + x_3 &= 5 \\ 6x_1 - 2x_2 + x_4 &= 4 \\ x_1, x_3, x_4 &\geq 0 \\ x_2 &\in \mathbb{R} \end{aligned}$$

3b) Alternatively, replace the unrestricted variables with an expression derived from an equality (for example, the second: $x_2 = 3x_1 - x_4/2 - 2$):

$$\begin{aligned} \min \hat{f} = -2x_1 + 3 \cdot (3x_1 - x_4/2 - 2) & \\ 4x_1 - 7 \cdot (3x_1 - x_4/2 - 2) + x_3 &= 5 \\ x_1, x_3, x_4 &\geq 0 \end{aligned}$$

$$\begin{aligned} \min \hat{f} = 7x_1 - 3/2\, x_4 - 6 & \\ -17x_1 + x_3 + 7/2\, x_4 &= -9 \\ x_1, x_3, x_4 &\geq 0 \end{aligned}$$

The first equality provides a different, but equivalent, standard form

Other transformations

We shall see that any *LP* problem also admits other special forms, which allow only $\leq$ inequalities or only $\geq$ inequalities

They can be obtained, as well, by simple algebraic transformations

$\leq$ form	$=$ form	$\geq$ form
$ax \leq b$	$\begin{cases} ax + s = b \\ s \geq 0 \end{cases}$	$-ax \geq -b$
$\begin{cases} ax \leq b \\ -ax \leq -b \end{cases}$	$ax = b$	$\begin{cases} ax \geq b \\ -ax \geq -b \end{cases}$
$-ax \leq -b$	$\begin{cases} ax - s = b \\ s \geq 0 \end{cases}$	$ax \geq b$

Two *LP* problems are equivalent when corresponding solutions have

- the same feasibility status: both feasible or both unfeasible
- the same objective value, possibly except for a constant offset