Solved exercises for the course of

Foundations of Operations Research

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Reduction to standard form

Reduce the following *LP* problem into standard form.

$$\begin{aligned}
\max f(x) &= 2x_1 - 4x_2 - 7x_3 - x_4 - 5x_5 \\
g_1(x) &= -x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 7 \\
g_2(x) &= -x_1 + 2x_2 + 3x_3 \qquad + x_5 \leq 6 \\
g_3(x) &= -x_1 + x_2 + x_3 + 2x_4 \qquad \geq 4 \\
x_1, x_2, x_3 &\geq 0 \\
x_4 &\in \mathbb{R} \\
x_5 &\leq 3
\end{aligned}$$

Standard form

The *standard form* of a *LP* problem, is an equivalent problem under the form:

$$\begin{aligned}
\min f'(x) &= cx \\
Ax &= b \\
x &\geq 0
\end{aligned}$$

Transformation

Any linear problem can be reduced into standard form applying the following transformations:

1. **Maximization problem:** reverse the sign of the objective function

$$\max f \ \rightarrow \ \min(-f')$$

2. $\leq$ **constraint:** introduce a new variable, denoted as slack *variable*, and assign to it the difference between the right and left side of the inequality, which is necessarily nonnegative

$$g(x) \leq 0 \ \rightarrow \ g(x) + x' = 0, \quad x' \geq 0$$

3. $\geq$ **constraint:** introduce a new variable, denoted as surplus *variable*, and assign to it the difference between the left and right side of the inequality, which is necessarily nonnegative

$$g(x) \geq 0 \ \rightarrow \ g(x) - x' = 0, \quad x' \geq 0$$

4. **Variables with a nonzero lower bound:** replace the variable in the objective and in the constraint functions with a suitably transposed variable, which is necessarily nonnegative

$$x_j \geq l_j \;\rightarrow\; \begin{cases} x_j & = x'_j + l_j \\ x'_j & \geq 0 \end{cases}$$

5. **Variables with an upper bound:** replace the variable in the objective and in the constraint functions with a suitably transposed and reversed variable, which is necessarily nonnegative

$$x_j \leq u_j \;\rightarrow\; \begin{cases} x_j & = u_j - x'_j \\ x'_j & \geq 0 \end{cases}$$

6. **Free variables:** replace the variable in the objective and in the constraint functions with the difference of two nonnegative variables

$$x_j \in \mathbb{R} \;\rightarrow\; x_j = x_j^+ - x_j^-$$

Notice that transformation 1 does not affect the feasibility of the solutions; it affects the objective value, but it does not affect the order of all solutions from the best to the worst. All transformations from 2 to 5 create a one-to-one correspondence between solutions of the original and of the resulting problem, such that each feasible solution of the former corresponds to a feasible solution with the same value of the latter. As for transformation 6, the correspondence is not one-to-one, but one-to-infinite: in fact, the resulting problem has one more variable, and one more degree of freedom. However, the two problems are equivalent because no feasible solution corresponds to an unfeasible one, nor vice versa, and the value of the objective in corresponding solutions is the same.

Another way to deal with free variables If a free variables occurs in an equality constraint, it is always possible to express it in terms of the other variables occuring in the same constraint. Then, it is possible to replace it with the expression in the objective and constraint functions. The resulting problem is equivalent to the original one, because the corresponding solutions keep their feasibility status and objective value.

In general, the $x = x^+ - x^-$ transformation produces a larger problem (one variable more), whereas the derivation and replacement produces a smaller one (one variable and one constraint less). So, the former is simpler, but the latter is in general preferrable.

Solution

Now, we can consider the given problem and reduce it into standard form.

$$\max f(x) = 2x_1 - 4x_2 - 7x_3 - x_4 - 5x_5 \tag{1}$$
$$g_1(x) = -x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 7 \tag{2}$$
$$g_2(x) = -x_1 + 2x_2 + 3x_3 + x_5 \leq 6 \tag{3}$$
$$g_3(x) = -x_1 + x_2 + x_3 + 2x_4 \geq 4 \tag{4}$$
$$x_1, x_2, x_3 \geq 0 \tag{5}$$
$$x_4 \in \mathbb{R} \tag{6}$$
$$x_5 \leq 3 \tag{7}$$

Objective function (1) The problem must become a minimization problem, instead of a maximization one:

$$\min f'(x) = -f(x) = -2x_1 + 4x_2 + 7x_3 + x_4 + 5x_5$$

Constraint (2) This constraint is already an equality: it remains unchanged.

Constraints (3) **and** (4) The former requires a *slack* variable, the latter a *surplus* variable.

$$\begin{aligned} -x_1 + 2x_2 + 3x_3 + x_5 + x_6 &= 6 \\ -x_1 + x_2 + x_3 + 2x_4 - x_7 &= 4 \end{aligned}$$

with $x_6, x_7 \geq 0$.

Free variable x_4 In the whole model, we replace x_4 with $x_4^+ - x_4^-$, with $x_4^+ \geq 0$ and $x_4^- \geq 0$

$$\begin{aligned} \min f'(x) = -2x_1 + 4x_2 + 7x_3 + \left(x_4^+ - x_4^-\right) + 5x_5 & \\ -x_1 + x_2 + 2x_3 + \left(x_4^+ - x_4^-\right) + 2x_5 &= 7 \\ -x_1 + 2x_2 + 3x_3 + x_5 + x_6 &= 6 \\ -x_1 + x_2 + x_3 + 2\left(x_4^+ - x_4^-\right) - x_7 &= 4 \\ x_1, x_2, x_3, x_4^+, x_4^- &\geq 0 \\ x_5 &\leq 3 \end{aligned}$$

from which:

$$\begin{array}{rcl}
\min f'(x) = -2x_1 + 4x_2 + 7x_3 + x_4^+ - x_4^- + 5x_5 & & \\
-x_1 + x_2 + 2x_3 + x_4^+ - x_4^- + 2x_5 & = & 7 \\
-x_1 + 2x_2 + 3x_3 \qquad\qquad + x_5 + x_6 & = & 6 \\
-x_1 + x_2 + x_3 + 2x_4^+ - 2x_4^- \qquad\qquad - x_7 & = & 4 \\
x_1, x_2, x_3, x_4^+, x_4^- & \geq & 0 \\
x_5 & \leq & 3
\end{array}$$

Variable x_5 upper bounded Variable x_5 is upper bounded. One could consider it as a free variable, and the upper bound as a $\leq$ constraint, introducing two auxiliary variables, plus a *slack* variable. A more compact form can be obtained replacing x_5 with $3 - x_5'$ in the whole model, and constraint $x_5 \leq 3$ with the nonnegativity condition $x_5' \geq 0$.

$$\begin{array}{rcl}
\min f'(x) = -2x_1 + 4x_2 + 7x_3 + x_4^+ - x_4^- + 5\,(3 - x_5') & & \\
-x_1 + x_2 + 2x_3 + x_4^+ - x_4^- + 2\,(3 - x_5') & = & 7 \\
-x_1 + 2x_2 + 3x_3 \qquad\qquad + (3 - x_5') + x_6 & = & 6 \\
-x_1 + x_2 + x_3 + 2x_4^+ - 2x_4^- \qquad\qquad - x_7 & = & 4 \\
x_1, x_2, x_3, x_4^+, x_4^-, x_5' & \geq & 0
\end{array}$$

from which:

$$\begin{array}{rcl}
\min f'(x) = -2x_1 + 4x_2 + 7x_3 + x_4^+ - x_4^- - 5x_5' + 15 & & \\
-x_1 + x_2 + 2x_3 + x_4^+ - x_4^- - 2x_5' & = & 1 \\
-x_1 + 2x_2 + 3x_3 \qquad\qquad - x_5' + x_6 & = & 3 \\
-x_1 + x_2 + x_3 + 2x_4^+ - 2x_4^- \qquad\qquad - x_7 & = & 4 \\
x_1, x_2, x_3, x_4^+, x_4^-, x_5' & \geq & 0
\end{array}$$

Free variable x_4 (alternative approach) A more compact standard form can be obtained replacing x_4 with $x_1 - x_2 - 2x_3 - 2x_5 + 7$, from the first constraint.

$$\begin{array}{rcl}
\min f'(x) = -2x_1 + 4x_2 + 7x_3 + (x_1 - x_2 - 2x_3 - 2x_5 + 7) + 5x_5 & & \\
-x_1 + x_2 + 2x_3 + (x_1 - x_2 - 2x_3 - 2x_5 + 7) + 2x_5 & = & 7 \\
-x_1 + 2x_2 + 3x_3 \qquad\qquad + x_5 + x_6 & = & 6 \\
-x_1 + x_2 + x_3 + 2\,(x_1 - x_2 - 2x_3 - 2x_5 + 7) \qquad - x_7 & = & 4 \\
x_1, x_2, x_3 & \geq & 0 \\
x_5 & \leq & 3
\end{array}$$

from which

$$
\begin{array}{rcl}
\min f'(x) = -x_1 + 3x_2 + 5x_3 - 3x_5' & + & 16 \\
-x_1 + 2x_2 + 3x_3 - x_5' + x_6 & = & 3 \\
x_1 - x_2 - 3x_3 + 4x_5' \qquad - x_7 & = & 2 \\
x_1, x_2, x_3, x_5', x_6, x_7 & \geq & 0
\end{array}
$$

Solution retrieval Now assume that, after obtaining the standard form with the first approach, the following feasible solution has been obtained with some solving algorithm: $x_1 = 2$, $x_2 = x_3 = 0$, $x_4^+ = 20$, $x_4^- = 1$ and $x_5' = 8$, with value $f' = -10$.

In order to determine the original solution, one simply applies the inverse transformations:

- $x_4 = x_4^+ - x_4^- = 20 - 1 = 19$
- $x_5 = 3 - x_5' = 3 - 8 = -5$
- $f = -f' = 10$

As a result, the original solution is $x = (2, 0, 0, 19, -5)$ and its value is $f(x) = 10$.

As well, assume that the standard form has been obtained with the alternative approach, and the following feasible solution has been obtained with some solving algorithm: $x_1 = 2$, $x_2 = x_3 = 0$, $x_5' = 8$, with value $f' = -10$.

The original solution can be computed as:

- $x_5 = 3 - x_5' = 3 - 8 = -5$
- $f = -f' = 10$
- $x_4 = x_1 - x_2 - 2x_3 - 2x_5 + 7 = 2 - 0 - 2 \cdot 0 - 2 \cdot (-5) + 7 = 2 + 10 + 7 = 19$

and of course, the final result is the same: $x = (2, 0, 0, 19, -5)$ and its value is $f(x) = 10$.