Foundations of Operations Research

Master of Science in Computer Engineering

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Tuesday 13.15 - 15.15

Thursday 10.15 - 13.15

http://homes.di.unimi.it/~cordone/courses/2014-for/2014-for.html

Lesson 7: Project planning — Como, Fall 2013

Project planning

A project consists of a set $N = \{a_1, \ldots, a_n\}$ of activities

A duration $d : N \to \mathbb{R}^+$ is associated to each activity

A precedence relation connects some pairs of activities:
$a_i \prec a_j$ means that activity a_j can start only after activity a_i has ended

A project can be modelled by an arc-weighted directed graph $G = (N, A)$

- nodes stand for activity start events, plus
  a fictitious node s for the beginning of the project and
  a fictitious node t for the end of the project
- arcs stand for precedence constraints: the given ones plus the fact that s precedes all activities and all activities precede t
- the arc cost $c_{ij} = d_i$ stands for the activity duration: the starting times of activities a_i and a_j must differ at least by d_i

The graph representing a project is acyclic (*otherwise, the start event of an activity should follow its end event: a logical paradox, or a deadlock*)

The problem

Find for each activity $a_i \in N$ a start time x_i respecting the precedence constraints so as to minimize the overall duration $f = x_t - x_s$
(i. e. the time needed to complete the whole project)

We conventionally set $x_s = 0$ and define as

- $\tau_i^{\min}$ the earliest start time for activity $a_i \in N$ in any feasible solution
- $\tau_i^{\max}$ the latest start time for activity $a_i \in N$ in any optimal solution
  All solutions with $\tau_i^{\min} \leq x_i \leq \tau_i^{\max}$ for all $a_i \in N$ are optimal
- slack $\sigma_i = \tau_i^{\max} - \tau_i^{\min}$ the amount of time by which an activity can be postponed without affecting the overall duration
- critical activity an activity with identical earliest and latest time ($\sigma_i = 0$)
  A critical activity cannot be postponed without delaying the whole project

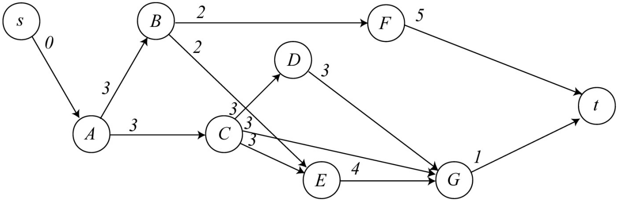
Property: The cost of each directed path from s to t is a lower bound on the overall duration f

$$f \geq c_{P_{st}} \qquad \text{for all } P_{st} \text{ path from } s \text{ to } t$$

Proof: P_{st} *is a sequence of activities that must be performed in the same order in which they are visited in the path*

Example

Activity	Duration	Predecessors
A	3	-
B	2	A
C	3	A
D	3	C
E	4	B,C
F	5	B
G	1	C,D,E



Why does s reach only A? Why is t reached only by F and G?

The Critical Path Method (*CPM*)

The critical path method (CPM) finds a solution $x^* : N \to \mathbb{R}$ such that

- all constraints are respected
- the overall duration equals the maximum cost of a path from s to t

$$f\left(x_t^*\right) = \max\left\{c_{P_{st}} : P_{st} \text{ is a path from } s \text{ to } t \text{in } G\right\}$$

If the cost of a feasible solution equals a lower bound on the optimum the solution is optimal (*by contradiction: if it were not optimal, the optimum would be smaller than the lower bound*)

The CPM finds $\tau_i^{\min}$, $\tau_i^{\max}$ and the slack σ_i for each activity $a_i \in N$ applying dynamic programming

Remember that any solution with $\tau_i^{\min} \leq x_i \leq \tau_i^{\max}$ is optimal

The Critical Path Method (*CPM*)

1. Sort the nodes topologically (*the graph is acyclic*); by definition, $s = n_0$ and $t = n_n$ (*s precedes and t follows all the other nodes*)

In order to find an optimal solution

2. Set $\tau_0^{\min} = 0$
3. Compute the earliest time $\tau_h^{\min}$ of each node $n_h \in N$ for h

$$\tau_h^{\min} = \max_{i:(n_i,n_h)\in\Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right) \qquad h \text{ increasing from } 1 \text{ to } n$$

(*computing the minimum cost paths with inverted sign costs*)

The optimal solution is $x_i^* = \tau_i^{\min}$ and its cost is $f(x^*) = \tau_n^{\min}$

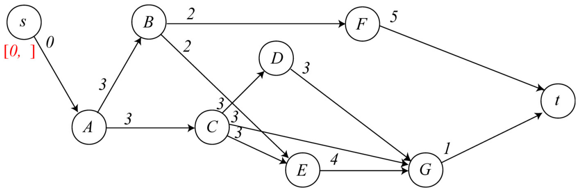
In order to find all optimal solutions

4. set $\tau_n^{\max} = f(x^*)$
5. Compute the latest time $\tau_h^{\max}$ of each node $n_h \in N$

$$\tau_h^{\max} = \min_{i:(n_h,n_i)\in\Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right) \qquad h \text{ decreasing from } n-1 \text{ to } 0$$

(*computing backward minimum cost paths with inverted sign costs*)

Application of the *CPM* (1)



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

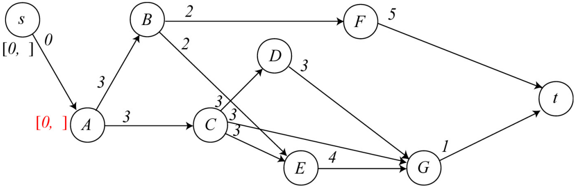
For $h := n - 1$ **downto** 0 **do**

$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$\tau_0^{\min} := 0$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n-1$ **downto** 0 **do**

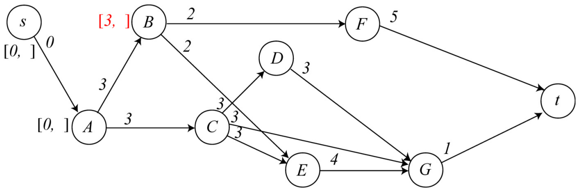
$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 1$

$\tau_1^{\min} := \max\left(\tau_0^{\min} + d_{n_0}\right) = 0$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n - 1$ **downto** 0 **do**

$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

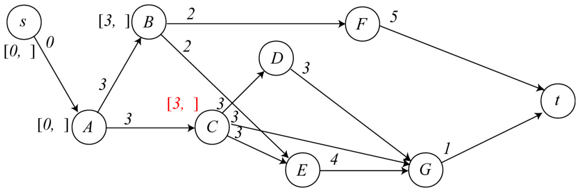
Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 2$

$\tau_2^{\min} := \max\left(\tau_1^{\min} + d_{n_1}\right) = 3$

Application of the *CPM* (4)



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta_{n_h}^-} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n - 1$ **downto** 0 **do**

$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta_{n_h}^+} \left(\tau_i^{\max} - d_h\right);$$

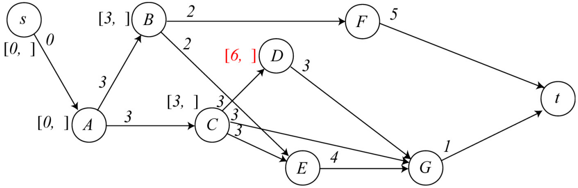
Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 3$

$\tau_3^{\min} := \max\left(\tau_1^{\min} + d_{n_1}\right) = 3$

Application of the *CPM* (5)



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n - 1$ **downto** 0 **do**

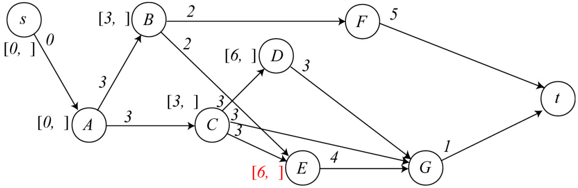
$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 4$

$\tau_4^{\min} := \max\left(\tau_3^{\min} + d_{n_3}\right) = 6$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N:(i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n-1$ **downto** 0 **do**

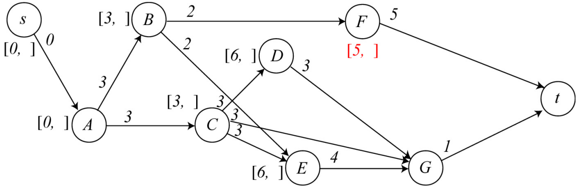
$$\tau_h^{\max} := \min_{i \in N:(n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 5$

$$\tau_5^{\min} := \max\left(\tau_2^{\min} + d_{n_2}, \tau_3^{\min} + d_{n_3}\right) = \\ = 6$$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta_{n_h}^-} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n-1$ **downto** 0 **do**

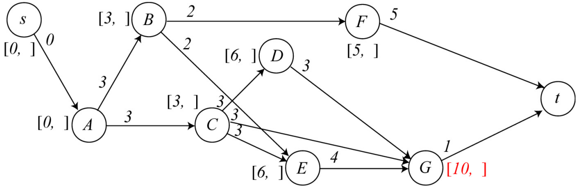
$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta_{n_h}^+} \left(\tau_i^{\max} - d_h\right);$$

Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 6$

$\tau_6^{\min} := \max\left(\tau_2^{\min} + d_{n_2}\right) = 5$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N:(i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n-1$ **downto** 0 **do**

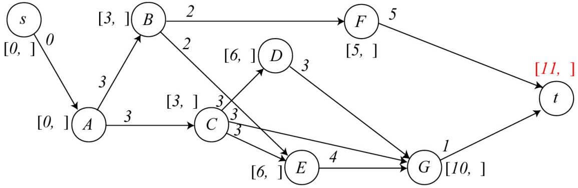
$$\tau_h^{\max} := \min_{i \in N:(n_h,n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 7$

$$\tau_7^{\min} := \max(\tau_3^{\min} + d_3, \tau_4^{\min} + d_{n_4}, \tau_{n_5}^{\min} + d_5) = 10$$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n-1$ **downto** 0 **do**

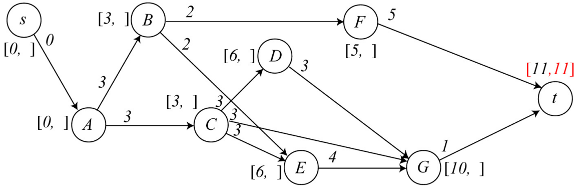
$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 8$

$$\tau_8^{\min} := \max\left(\tau_6^{\min} + d_{n_6}, \tau_7^{\min} + d_{n_7}\right) =$$
$$= 11$$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N:(i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

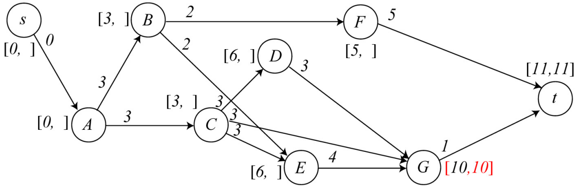
For $h := n-1$ **downto** 0 **do**

$$\tau_h^{\max} := \min_{i \in N:(n_h,n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $(\tau^{\min}, \tau^{\max})$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$\tau_8^{\max} := 11$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n - 1$ **downto** 0 **do**

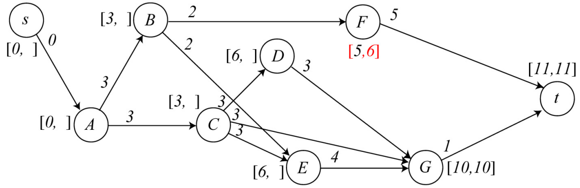
$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $(\tau^{\min}, \tau^{\max})$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 7$

$\tau_7^{\max} := \min\left(\tau_8^{\max} - d_{n_7}\right) = 10$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n - 1$ **downto** 0 **do**

$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

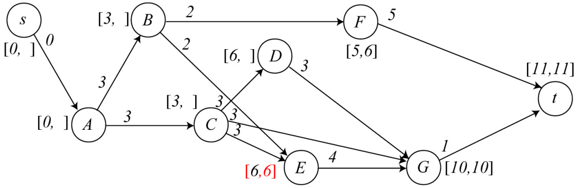
Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 6$

$\tau_6^{\max} := \min\left(\tau_8^{\max} - d_{n_6}\right) = 6$

Application of the *CPM* (13)



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N:(i,h) \in \Delta_{n_h}^-} (\tau_i^{\min} + d_i);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n - 1$ **downto** 0 **do**

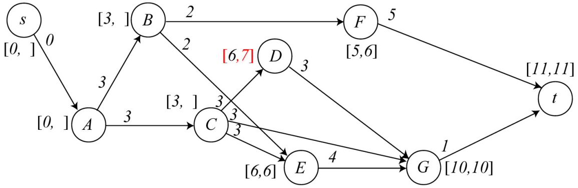
$$\tau_h^{\max} := \min_{i \in N:(n_h,n_i) \in \Delta_{n_h}^+} (\tau_i^{\max} - d_h);$$

Return $(\tau^{\min}, \tau^{\max})$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 5$

$\tau_5^{\max} := \min\left(\tau_7^{\max} - d_{n_5}\right) = 6$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N:(i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n - 1$ **downto** 0 **do**

$$\tau_h^{\max} := \min_{i \in N:(n_h,n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

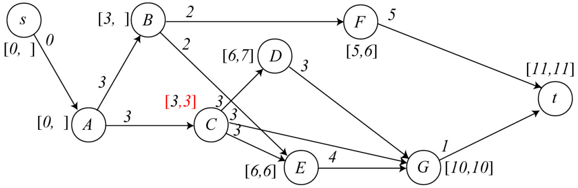
Return $\left(\tau^{\min}, \tau^{\max}\right)$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 4$

$\tau_4^{\max} := \min\left(\tau_7^{\max} - d_{n_4}\right) = 7$

Application of the *CPM* (15)



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n-1$ **downto** 0 **do**

$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

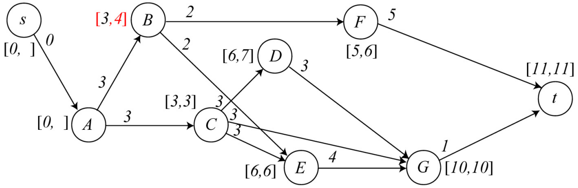
Return $(\tau^{\min}, \tau^{\max})$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 3$

$$\tau_3^{\max} := \min(\tau_4^{\max} - d_{n_3}, \tau_5^{\max} - d_{n_3}, \tau_7^{\max} - d_{n_3}) = 3$$

Application of the *CPM* (16)



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N : (i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n-1$ **downto** 0 **do**

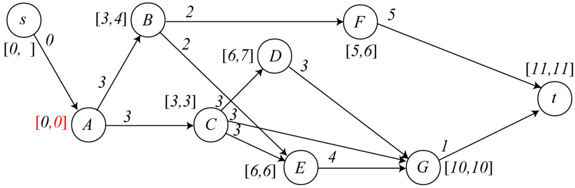
$$\tau_h^{\max} := \min_{i \in N : (n_h, n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $(\tau^{\min}, \tau^{\max})$;

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 2$

$$\tau_2^{\min} := \max\left(\tau_5^{\min} - d_{n_2}, \tau_6^{\min} - d_{n_2}\right) = \\ = 4$$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0;$

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N:(i,h)\in\Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min};$

For $h := n-1$ **downto** 0 **do**

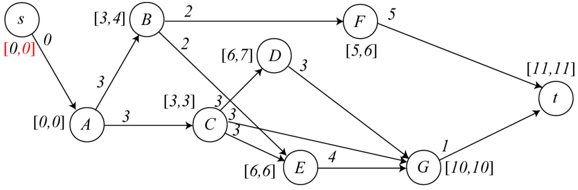
$$\tau_h^{\max} := \min_{i \in N:(n_h,n_i)\in\Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $(\tau^{\min}, \tau^{\max});$

n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 1$

$$\tau_1^{\min} := \max\left(\tau_2^{\min} - d_{n_1}, \tau_3^{\min} - d_{n_1}\right) = $$
$$= 0$$



CPM(N, A, d)

TopologicalSort(N);

$\tau_0^{\min} := 0$;

For $h := 1$ **to** n **do**

$$\tau_h^{\min} := \max_{i \in N:(i,h) \in \Delta^-_{n_h}} \left(\tau_i^{\min} + d_i\right);$$

$\tau_n^{\max} := \tau_n^{\min}$;

For $h := n-1$ **downto** 0 **do**

$$\tau_h^{\max} := \min_{i \in N:(n_h,n_i) \in \Delta^+_{n_h}} \left(\tau_i^{\max} - d_h\right);$$

Return $\left(\tau^{\min}, \tau^{\max}\right)$;

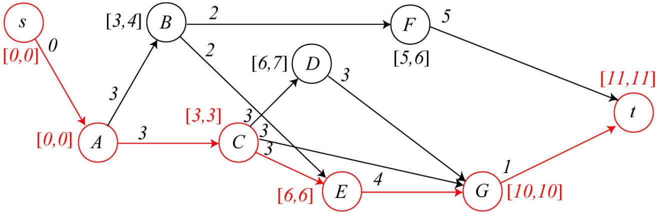
n_0	n_1	n_2	n_3	n_4
s	A	B	C	D
n_5	n_6	n_7	n_8	
E	F	G	t	

$h = 0$

$\tau_0^{\max} := \min\left(\tau_1^{\max} - d_{n_0}\right) = 0$

Critical path

An optimal solution always has at least one critical path, that is a path from s to t whose nodes correspond to critical activities

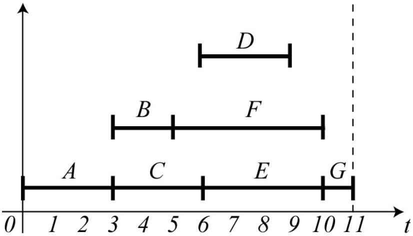


- (s, A, C, E, G, t) is a critical path
- A, C, E and G are critical activities
- B, D and F are noncritical activities:
  their starting time can be delayed by the same slack ($\sigma_2 = \sigma_4 = \sigma_6 = 1$) without affecting the overall shortest duration ($f^* = 11$)

Gantt diagram (at earliest)

The Gantt diagram is a temporal representation of a project

- the horizontal axis represent time
- the activities are distributed along the vertical axis
- activity schedules are horizontal segments
  1. from $\tau_i^{\min}$ to $\tau_i^{\min} + d_i$ in the diagram at earliest
  2. from $\tau_i^{\max} - d_i$ to $\tau_i^{\max}$ in the diagram at latest

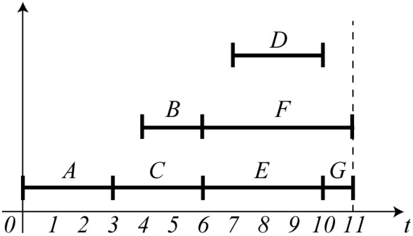


i	d	$\tau_i^{\min}$	$\tau_i^{\min} + d_i$
A	3	0	3
B	2	3	5
C	3	3	6
D	3	6	9
E	4	6	10
F	5	5	10
G	1	10	11

Gantt diagram (at latest)

The Gantt diagram is a temporal representation of a project

- the horizontal axis represent time
- the activities are distributed along the vertical axis
- activity schedules are horizontal segments
  1. from $\tau_i^{\min}$ to $\tau_i^{\min} + d_i$ in the diagram at earliest
  2. from $\tau_i^{\max} - d_i$ to $\tau_i^{\max}$ in the diagram at latest



i	d	$\tau_i^{\max}$	$\tau_i^{\max} + d_i$
A	3	0	3
B	2	4	6
C	3	3	6
D	3	7	10
E	4	6	10
F	5	6	11
G	1	10	11