Solved exercises for the course of

Foundations of Operations Research

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Shortest path problem

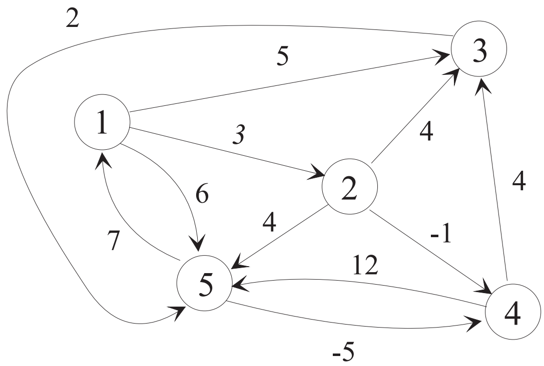
Given a graph with the following cost matrix

	1	2	3	4	5
1	0	3	5	∞	6
2	∞	0	4	-1	4
3	∞	∞	0	∞	2
4	∞	∞	4	0	12
5	7	∞	∞	-5	0

determine the shortest paths from node 1 to all other nodes with the proper algorithm, justifying the choice.

Solution

First of all, let us represent the graph corresponding to the cost matrix.



We recall that

- in general, the shortest path problem is ill-posed, because the existence of negative cost circuits makes the solution unbounded or requires additional constraints on the number of visits of each arc or node
- if such constraints are introduced, the shortest path problem becomes $\mathcal{NP}$-hard
- if no negative cost circuits exist, the shortest path problem can be solved with *Floyd-Warshall's algorithm* (which otherwise detects negative cost cycles)
- if all arc costs are nonnegative, the shortest path problem ($c_{ij} \geq 0$ for all $(i, j) \in A$) can be solved with *Dijkstra's algorithm*
- if the graph has no circuits, the shortest path problem can be solved with a simple dynamic programming algorithm

In the present case, the proper algorithm is Floyd-Warshall's algorithm, which provides more than what is required, since it provides a shortest path from each node to every other node, instead of simply from node 1 to every other node.

If the problem admits an optimal solution, the set of all shortest paths from a node to all other ones yields an *arborescence* rooted in the starting node. In fact, in the opposite case some nodes would have indegree different from 1, i. e. nonreachable from the starting node (if the indegree is zero) or reachable through different paths (if the indegree is larger than 1). On the contrary, the solution reports a single optimal path for each node.

Floyd-Warshall's algorithm

Floyd-Warshall(N, A, c)

For each $i \in N$ **do**

    For each $j \in N$ **do**

        $\ell_{ij} := c_{ij}$; $\pi_j := i$;

For $h := 1$ **to** n **do**

    For each $i \in N \setminus \{h\}$ **do**

        For each $j \in N \setminus \{h\}$ **do**

            If $\ell_{ij} > \ell_{ih} + \ell_{hj}$ **then**

                $\ell_{ij} := \ell_{ih} + \ell_{hj}$; $\pi_{ij} := \pi_{hj}$;

If $\exists i \in N : \ell_{ii} < 0$

    then Return "Negative circuit detected";

    else Return (ℓ, π);

Initialization Matrices ℓ and π at first represent the paths consisting of a single arc.

Distance matrix

ℓ	1	2	3	4	5
1	0	3	5	$+\infty$	6
2	$+\infty$	0	4	-1	4
3	$+\infty$	$+\infty$	0	$+\infty$	2
4	$+\infty$	$+\infty$	4	0	12
5	7	$+\infty$	$+\infty$	-5	0

Predecessor matrix

π	1	2	3	4	5
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5

First iteration Try and improve each path visiting node 1. Of course, the paths which starts and end in node 1 cannot be improved. As for the other ones:

- $\ell_{22} = 0$ is not improved by $\ell_{21} + \ell_{12} = \infty + 3$
- $\ell_{23} = 4$ is not improved by $\ell_{21} + \ell_{13} = \infty + 5$
- $\ell_{24} = -1$ is not improved by $\ell_{21} + \ell_{14} = \infty + \infty$
- $\ell_{25} = 4$ is not improved by $\ell_{21} + \ell_{15} = \infty + 6$
- $\ell_{32} = +\infty$ is not improved by $\ell_{31} + \ell_{12} = \infty + 3$
- $\ell_{33} = 0$ is not improved by $\ell_{31} + \ell_{13} = \infty + 5$
- $\ell_{34} = +\infty$ is not improved by $\ell_{31} + \ell_{14} = \infty + \infty$
- $\ell_{35} = 2$ is not improved by $\ell_{31} + \ell_{15} = \infty + 6$
- $\ell_{42} = +\infty$ is not improved by $\ell_{41} + \ell_{12} = \infty + 3$
- $\ell_{43} = 4$ is not improved by $\ell_{41} + \ell_{13} = \infty + 5$
- $\ell_{44} = 0$ is not improved by $\ell_{41} + \ell_{14} = \infty + \infty$
- $\ell_{45} = 12$ is not improved by $\ell_{41} + \ell_{15} = \infty + 6$
- $\ell_{52} = +\infty$ is improved by $\ell_{51} + \ell_{12} = 7 + 3 = 10$
- $\ell_{53} = +\infty$ is improved by $\ell_{51} + \ell_{13} = 7 + 5 = 12$
- $\ell_{54} = -5$ is not improved by $\ell_{51} + \ell_{14} = 7 + \infty$
- $\ell_{55} = 0$ is not improved by $\ell_{51} + \ell_{15} = 7 + 6$

Distance matrix

ℓ	1	2	3	4	5
1	0	3	5	$+\infty$	6
2	$+\infty$	0	4	-1	4
3	$+\infty$	$+\infty$	0	$+\infty$	2
4	$+\infty$	$+\infty$	4	0	12
5	7	10	12	-5	0

Predecessor matrix

π	1	2	3	4	5
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	1	1	5	5

Second iteration Try and improve each path visiting node 2. Of course, the paths which starts and end in node 2 cannot be improved. As for the other ones:

- $\ell_{11} = 0$ is not improved by $\ell_{12} + \ell_{21} = 3 + \infty$
- $\ell_{13} = 5$ is not improved by $\ell_{12} + \ell_{23} = 3 + 4$
- $\ell_{14} = +\infty$ is improved by $\ell_{12} + \ell_{24} = 3 - 1 = 2$
- $\ell_{15} = 6$ is not improved by $\ell_{12} + \ell_{25} = 3 + 4$
- $\ell_{31} = +\infty$ is not improved by $\ell_{32} + \ell_{21} = \infty + \infty$
- $\ell_{33} = 0$ is not improved by $\ell_{32} + \ell_{23} = \infty + 4$
- $\ell_{34} = +\infty$ is improved by $\ell_{32} + \ell_{24} = \infty - 1$
- $\ell_{35} = 2$ is not improved by $\ell_{32} + \ell_{25} = \infty + 4$
- $\ell_{41} = +\infty$ is not improved by $\ell_{42} + \ell_{21} = \infty + \infty$
- $\ell_{43} = 4$ is not improved by $\ell_{42} + \ell_{23} = \infty + 4$
- $\ell_{44} = 0$ is not improved by $\ell_{42} + \ell_{24} = \infty - 1$
- $\ell_{45} = 12$ is not improved by $\ell_{42} + \ell_{25} = \infty + 4$
- $\ell_{51} = 7$ is not improved by $\ell_{52} + \ell_{21} = 10 + \infty$
- $\ell_{53} = 12$ is not improved by $\ell_{52} + \ell_{23} = 10 + 4$
- $\ell_{54} = -5$ is not improved by $\ell_{52} + \ell_{24} = 10 - 1$
- $\ell_{55} = 0$ is not improved by $\ell_{52} + \ell_{25} = 10 + 4$

Distance matrix

ℓ	1	2	3	4	5
1	0	3	5	2	6
2	$+\infty$	0	4	-1	4
3	$+\infty$	$+\infty$	0	$+\infty$	2
4	$+\infty$	$+\infty$	4	0	12
5	7	10	12	-5	0

Predecessor matrix

π	1	2	3	4	5
1	1	1	1	2	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	1	1	5	5

Third iteration Try and improve each path visiting node 3. Of course, the paths which start and end in node 3 cannot be improved. As for the other ones:

- $\ell_{11} = 0$ is not improved by $\ell_{13} + \ell_{31} = 5 + \infty$
- $\ell_{12} = 3$ is not improved by $\ell_{13} + \ell_{32} = 5 + \infty$
- $\ell_{14} = 2$ is not improved by $\ell_{13} + \ell_{34} = 5 + \infty$
- $\ell_{15} = 6$ is not improved by $\ell_{13} + \ell_{35} = 5 + 2$
- $\ell_{21} = +\infty$ is not improved by $\ell_{23} + \ell_{31} = 4 + \infty$
- $\ell_{22} = 0$ is not improved by $\ell_{23} + \ell_{32} = 4 + \infty$
- $\ell_{24} = -1$ is not improved by $\ell_{23} + \ell_{34} = 4 + \infty$
- $\ell_{25} = 4$ is not improved by $\ell_{23} + \ell_{35} = 4 + 2$
- $\ell_{41} = +\infty$ is not improved by $\ell_{43} + \ell_{31} = 4 + \infty$
- $\ell_{42} = +\infty$ is not improved by $\ell_{43} + \ell_{32} = 4 + \infty$
- $\ell_{44} = 0$ is not improved by $\ell_{43} + \ell_{34} = 4 + \infty$
- $\ell_{45} = 12$ is improved by $\ell_{43} + \ell_{35} = 4 + 2 = 6$
- $\ell_{51} = 7$ is not improved by $\ell_{53} + \ell_{31} = 12 + \infty$
- $\ell_{52} = 10$ is not improved by $\ell_{53} + \ell_{32} = 12 + \infty$
- $\ell_{54} = -5$ is not improved by $\ell_{53} + \ell_{34} = 12 + \infty$
- $\ell_{55} = 0$ is not improved by $\ell_{53} + \ell_{35} = 12 + 2$

Distance matrix

ℓ	1	2	3	4	5
1	0	3	5	2	6
2	$+\infty$	0	4	-1	4
3	$+\infty$	$+\infty$	0	$+\infty$	2
4	$+\infty$	$+\infty$	4	0	6
5	7	10	12	-5	0

Predecessor matrix

π	1	2	3	4	5
1	1	1	1	2	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	3
5	5	1	1	5	5

Fourth iteration Try and improve each path visiting node 4. Of course, the paths which start and end in node 4 cannot be improved. As for the other ones:

- $\ell_{11} = 0$ is not improved by $\ell_{14} + \ell_{41} = 2 + \infty$
- $\ell_{12} = 3$ is not improved by $\ell_{14} + \ell_{42} = 2 + \infty$
- $\ell_{13} = 5$ is not improved by $\ell_{14} + \ell_{43} = 2 + 4$
- $\ell_{15} = 6$ is not improved by $\ell_{14} + \ell_{45} = 2 + 6$
- $\ell_{21} = +\infty$ is improved by $\ell_{24} + \ell_{41} = -1 + \infty$
- $\ell_{22} = 0$ is not improved by $\ell_{24} + \ell_{42} = -1 + \infty$
- $\ell_{23} = 4$ is improved by $\ell_{24} + \ell_{43} = -1 + 4 = 3$
- $\ell_{25} = 4$ is not improved by $\ell_{24} + \ell_{45} = -1 + 6$
- $\ell_{31} = +\infty$ is not improved by $\ell_{34} + \ell_{41} = \infty + \infty$
- $\ell_{32} = +\infty$ is not improved by $\ell_{34} + \ell_{42} = \infty + \infty$
- $\ell_{33} = 0$ is not improved by $\ell_{34} + \ell_{43} = \infty + 4$
- $\ell_{35} = 2$ is not improved by $\ell_{34} + \ell_{45} = \infty + 6$
- $\ell_{51} = 7$ is not improved by $\ell_{54} + \ell_{41} = -5 + \infty$
- $\ell_{52} = 10$ is not improved by $\ell_{54} + \ell_{42} = -5 + \infty$
- $\ell_{53} = 12$ is improved by $\ell_{54} + \ell_{43} = -5 + 4 = -1$
- $\ell_{55} = 0$ is not improved by $\ell_{54} + \ell_{45} = -5 + 6$

Distance matrix

ℓ	1	2	3	4	5
1	0	3	5	2	6
2	$+\infty$	0	3	-1	4
3	$+\infty$	$+\infty$	0	$+\infty$	2
4	$+\infty$	$+\infty$	4	0	6
5	7	10	-1	-5	0

Predecessor matrix

π	1	2	3	4	5
1	1	1	1	2	1
2	3	2	4	2	2
3	3	3	3	3	3
4	4	4	4	4	3
5	5	1	4	5	5

Fifth iteration Try and improve each path visiting node 5. Of course, the paths which start and end in node 5 cannot be improved. As for the other ones:

- $\ell_{11} = 0$ is not improved by $\ell_{15} + \ell_{51} = 6 + 7$
- $\ell_{12} = 3$ is not improved by $\ell_{15} + \ell_{52} = 6 + 10$
- $\ell_{13} = 5$ is not improved by $\ell_{15} + \ell_{53} = 6 - 1$
- $\ell_{14} = 2$ is improved by $\ell_{15} + \ell_{54} = 6 - 5 = 1$
- $\ell_{21} = +\infty$ is improved by $\ell_{25} + \ell_{51} = 4 + 7 = 11$
- $\ell_{22} = 0$ is not improved by $\ell_{25} + \ell_{52} = 4 + 10$
- $\ell_{23} = 3$ is not improved by $\ell_{25} + \ell_{53} = 4 - 1$
- $\ell_{24} = -1$ is not improved by $\ell_{25} + \ell_{54} = 4 - 5$
- $\ell_{31} = +\infty$ is improved by $\ell_{35} + \ell_{51} = 2 + 7 = 9$
- $\ell_{32} = +\infty$ is improved by $\ell_{35} + \ell_{52} = 2 + 10 = 12$
- $\ell_{33} = 0$ is not improved by $\ell_{35} + \ell_{53} = 2 - 1$
- $\ell_{34} = +\infty$ is improved by $\ell_{35} + \ell_{54} = 2 - 5 = -3$
- $\ell_{41} = +\infty$ is improved by $\ell_{45} + \ell_{51} = 6 + 7 = 13$
- $\ell_{42} = +\infty$ is improved by $\ell_{45} + \ell_{52} = 6 + 10 = 16$
- $\ell_{43} = 4$ is not improved by $\ell_{45} + \ell_{53} = 6 - 1$
- $\ell_{44} = 0$ is not improved by $\ell_{45} + \ell_{54} = 6 - 5$

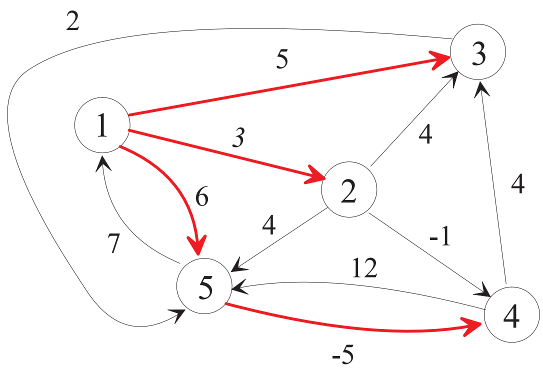
Distance matrix

ℓ	1	2	3	4	5
1	0	3	5	1	6
2	11	0	3	-1	4
3	9	12	0	-3	2
4	13	16	4	0	6
5	7	10	-1	-5	0

Predecessor matrix

π	1	2	3	4	5
1	1	1	1	5	1
2	5	2	4	2	2
3	5	1	3	5	3
4	5	1	4	4	3
5	5	1	4	5	5

Conclusion The two matrices represent the following paths on the original graph.



Dijkstra's algorithm

We here apply the efficient $O(n^2)$ version of the algorithm. This version maintains a vector π of candidate paths and a vector ℓ of corresponding costs; at each step, one of the paths is definitively marked as shortest and used to possibly update the other paths.

Dijkstra(N, A, c, s)

$L_s := 0$; $P_s := s$;

For each $j \in N \setminus \{s\}$ **do**

  If $(i, j) \in A$

    then $L_j := c_{ij}$; $P_j := i$;

    else $L_j := +\infty$; $P_j := -$;

  EndIf;

EndFor

$i^* := s$;

$T := N \setminus \{s\}$;

While $i^* \neq s$ **do**

  If $L_j > L_{i^*} + c_{i^*j}$ **then**

    $L_j := L_{i^*} + c_{i^*j}$;

    $P_j := i^*$;

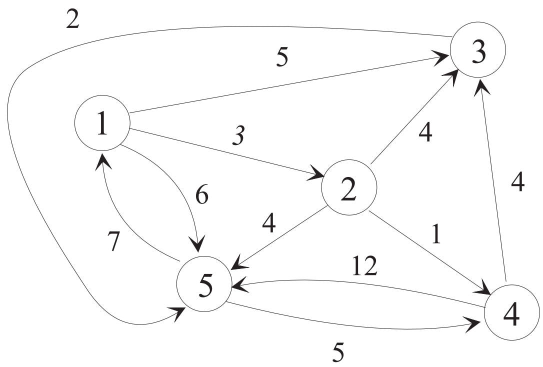
    $i^* := \arg\min_{i \in T} L_i$;

  EndIf;

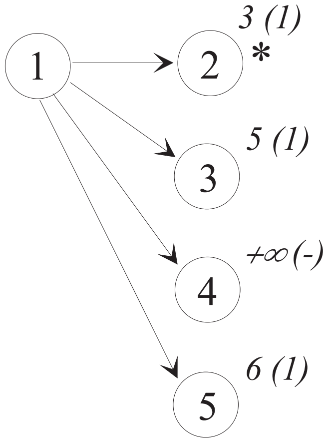
EndWhile

Return (L, P)

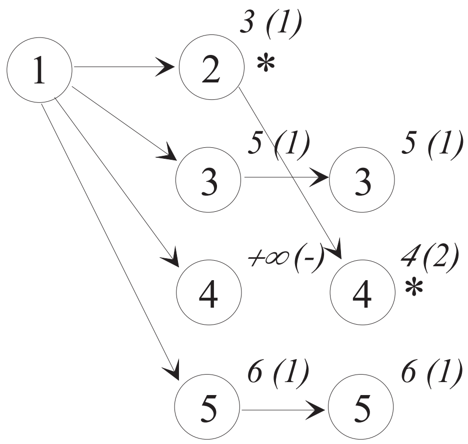
Dijkstra's algorithm cannot be applied to the given problem. It can be applied to a modified problem, obtained reverting the sign of all negative costs.



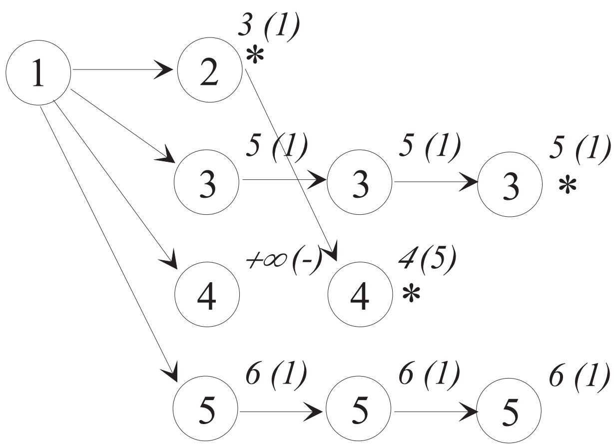
First iteration First consider the arcs going out of node 1. Nodes 2, 3 and 5 admit a direct arc, so that their cost label ℓ_i is given by the cost of the arc c_{si}; the cost label of node 4, on the contrary, is $+\infty$. For all nodes, the predecessor label π_i is given by the label of the starting node 1. In the following figures, the first number aside each node j is ℓ_j, the second one (in round parenthesis) is π_j. The node with the minimum cost label is $i^* = 2$, which is definitively marked.



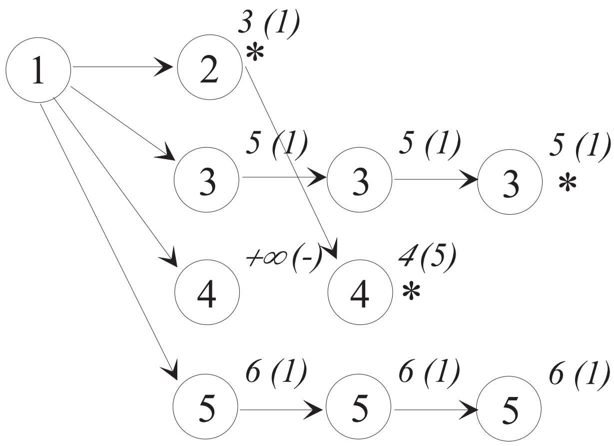
Second iteration For each arc (i^*, j) going out of the last marked node, the algorithm evaluates whether reaching node j through it is more profitable than using the previous candidate path. For example, arc $(2, 3)$ allows to reach node 3 with a path costing $\ell_2 + c_{23} = 3 + 4 = 7$, which is worse than the known one $(\ell_3 = 5)$, so that the label of node 3 does not change. The label of node 4 changes, because arc $(2, 4)$ allows to reach it with a path of cost $\ell_2 + c_{24} = 3 + 1 = 2$, which is better than $\ell_4 = +\infty$. Arc $(2, 5)$, in the end, does not improve the label of node 5. Now, the algorithm marks node 4, which has the smallest cost label.



Third iteration For each arc (i^*, j) going out of the last marked node $(i^* = 4)$, the algorithm evaluates whether it is profitable to reach node j through it. No label is updated. Node 3 is marked, because it has the smallest label.



Fourth iteration For each arc (i^*, j) going out of the last marked node $(i^* = 3)$, the algorithm evaluates whether it is profitable to reach node j through it. No label is updated. Node 5 is marked, because it has the smallest label.



Conclusion Now all nodes are marked: the set of all shortest paths yields an arborescence rooted in node 1.